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J-Prime submodules and some related concepts

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Abstract

Suppose R has been an identity-preserving commutative ring, and suppose V has been a legitimate submodule of R-module W. A submodule V has been J-Prime Occasionally as well as occasionally based on what's needed, it has been acceptable: $x \in V + J(W)$ according to some of that $r \in R$, $x \in W$ and $J(W)$ an interpretation of the Jacobson radical of W, which $x \in V$ or $r \in [V: W] = \{s \in R; sW \subseteq V\}$. To that end, we investigate the notion of J-Prime submodules and characterize some of the attributes of has been classification of submodules.

Subject Classification: *16P70, 11A41.*

Keywords: *Prime submodules, J-Prime submodules.*

1. Introduction

During the entirety of the work, each ring will be shown to be associative to maintain identity, and each module will be demonstrated to be unite. V has been a legitimate submodule of R-module W. A submodule V has been J-prime occasionally as well as occasionally based on what's needed: $r \in R$ and $x \in W$ that should be said $rx \in V$, indicates in has been implies $x \in V$ or $r \in [V:W] =\{ s \in R; sW \subseteq V\}$, [4]. Here, we'll explore an intriguing new category of submodules called J-Prime submodules, in which submodule V of an R-module W has been said to be J-Prime only in this case does it constitute a valid submodule to stay W, if pretty much immediately $rx \in V + J(W)$; $r \in R$ and $x \in W$, indicates that or $x \in V$ or $r \in [V:W]$; J(W) defines the confluence of all maximal submodules of W, and consequently has been the Jacobson radical of W [3]. Here, we describe and characterize a new submodule categorization dubbed J-Prime slide-modules.

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2. J-Prime Sub module s

Sometimes an R-module W has an appropriate sub-module V, we say that V has been prime only in has been case V has been instantaneously$rx \in V$; $r \in R$ and $x \in W$ indicates in has been implies$x \in V$ or $r \in [V: W]$, [4]. A perfect I of R has been defined as a prime only in has been case $r_1, r_2 \in R$ that should be said if $r_1, r_2 \in I$ indicates in has been implies $r_1 \in I$ or $r_2 \in I$, [1].

We present the following explanation:

Definition 2.1: A module R- as to stay the correct V slide-modules the J-Prime sub-module of W has been referred to as W, if pretty much immediately $rm \in V + J(W)$; $m \in W$ and $r \in R$, imply in has been $m \in V$ or $r \in [V: W]$.

Remarks and examples 2.2:

(1) Each sub-module of a J-prime has been its Prime. Although the inverse has been not always the case, has been has been not a universal truth. Suppose $W = Z_2 \oplus Z_4$ join the ranks of modules Z and $V = \{(0,0), (1,0)\}$ has been interesting a submodule of W. Presently $V + J(W) = \{(0,0), (1,0), (0,2), (1,2)\}$, afterward $9(0,2) \in V + J(W)$ Nevertheless $3(0,2) \notin V$ because V has been interesting a semi-prime submodule nevertheless not a J-semiprime.

(2) If P has been prime, afterward there has been no J-prime sub module of Z_{P^∞} across Z as there has been no prime slide-modules of Z_{P^∞}as a Z-module. It follows that there has been no J-prime sub-module of Z in Z_{P^∞}.

(3) A J-prime submodule has been interesting a maximum submodule of an R-module W.

Proof: Given an R-module W, the ratio $[V: W]$ has been interesting a prime ideal of R only in has been case V has been interesting a maximum submodule of W [4]. Suppose assume that $rm \in V + J(W)$, subsequently $rm \in V$, to stay $r = R$,$m = W$. To stay example, if V has been the largest submodule, afterward has been the prime submodule [4]. That leaves mV or $r \in [V: W]$. That V has been interesting a submodule of W that has been J-prime has been proven by has been fact.

More specifically, the sole J-prime submodule of module Q over Z has been 0.

4) Imagine that W has been an interesting module in R. Here has been how we define the submodule of W presently as Rad (W): In specifically, the sole J-prime submodule of the module Q over Z has been 0.$Rad(W)$ $= \cap \{P: P$ has been interesting a prime submodule $of\ W\}$ [3].

If $J(W) =$ a particular prime submodule $of\ W\ is\ a\ J -$prime Only in has been case $Rad(W)\ holds.$

Proof: In has been special instance, V has been of interest in a prime submodule of W, making [V:W] the prime ideal of R, an R-module. So, at has been present moment, $r \in R, m \in W$ that should be said $rm \in V + J(W)$, consequently $rm \in V$, Nevertheless V has been interesting a prime submodule of W, Consequently perhaps $m \in V$ or $r \in [V: W]$. Consequently, has been proves that V has been interesting a J-prime.

The following statements presently outline some of the characterhas beentics of J-Prime submodules:

Proposition 2.3: If V and T are J-Prime submodules of an R-module W, Consequently V $V \cap T$ T has been interesting in a J-Prime submodule of W if $[V: W] = [T: W]$

Proof: If $r \in R$, $x \in W$, and $rx \in (V \cap T) + J(W)$, afterward $rx \in V + J(W)$and $rx \in T + J(W)$ Even beto stay e V has been interesting a J-Prime submodule of W, perhaps $x \in V \; or \; r \in [V: W]$ and that perhaps $x \in T \; or \; r \in [T: W]$ hold, though whereas $[V: W]$= $[T: W]$. Has been implies that $x \in V \cap T$ or $r \in [(V \cap T): W]$ presupposes $V \cap T$, and so VT has been interesting a J-Prime submodule of W.

Proposition 2.4: $V \cap T$ has been interesting a J-Prime submodule of T only in has been case $T \nsubseteq V$ has been interesting a J-Prime submodule of an R-module W and T has been interesting a particular appropriate submodule of W.

Proof: As a corollary, it stands to reason that$V \cap T$ has been interesting a legitimate sub-module of T. Suppose Presently that $rx \in (V \cap T) + J(T)$ though where $r \in R$ and $x \in T\,.Consequently\, rx \in (V \cap T) + J(W)$. If $x \notin V \cap T$, afterward $x \notin V$, Nevertheless V has been interesting a J-Prime submodule of W, subsequently $r \in [V: W]$, Consequently, $rT \subseteq V$, What has been points to has been $rT \subseteq V \cap T$ subsequently $r \in [V \cap T \; : \; T]$. Has been concludes the argument.

The following to stay mutation allows us to describe a J-Prime ideal I of a ring R (2.1): a proper ideal I that sathas beenfies instantaneously $a.b \in I + J(R)$ to stay $a,b \in R$ afterward perhaps $a \in I$ or $b \in I$, though where $J(R)$ represents R's Jacobson extreme.

If an R-module W has a suitable submodule, then W has a prime submodule if and only if it has been presently $[V\!: W]$ has been interesting a prime ideal of R, [4]. An example of a J-Prime sub module has been interesting as follows:

Proposition 2.5: If V has been intriguing, then W has been fascinating and [V:W] has been noteworthy if and only if V has been a J-Prime submodule of R.

Proof: Even beto stay e V has been interesting a correct submodule of W, [V: W] has been interesting a valid ideal of R. Presently, Suppose $r_1.r_2 \in [\,V : W\,] + J(R)$ to stay $r_1, r_2 \in R$. Afterward $r_1 r_2 W \subseteq [V: W\,]W + J(R)W$, Nevertheless $J(R)W \subseteq J(W)$ from [3, P234]. Consequently $r_1 r_2 W \subseteq V + J(W)$. What has been points to has been $r_1(r_2 m) \in V + J(W)$ to stay everyone $m \in W$. Even beto stay e V has been interested in a J-Prime sub module, Thus perhaps $r_2 m \in V$or $r_1 \in [V: W]$. Has been concludes in has been implies$r_2 \in [V: W]$ or $r_1 \in [V: W]$. Afterward $[V: W]$ has been interested in a J-Prime ideal of R.

In most cases, Contrary to expectation, statement (2.5) has been not true. Considering the following illustration:

Suppose $W = Z \oplus Z$ as Z-module, if $V =< (4,0) >$, afterward $[V: W] = 0$ has been the J-prime ideal Nevertheless V has been not a J-Prime submodule of W, though where $J(W) =< (0,0) >$.

The following statement specifies the circumstances in which statement (2.5) becomes false.

Proposition 2.6: So long as V has been interested in a correct submodule of R-module W, we can write

(1) $[T: W] \nsubseteq [V + J(W): W]$ to stay every W submodule T that appropriately incorporates V.

(2) $[V + J(W): W] = [V: W]$.

Afterward, V has been interesting a J-Prime submodule of W, Sometimes and only Sometimes $[V: W]$ has been interesting a J-Prime ideal of R.

Proof: Let us assume that the J-Prime ideal of R [V:W] has been of relevance to us. Let's pretend rxV+ J(W) keeps some of the fact that $r \in R$ and $x \in W$. Assume that $x \notin V$. Presently $L = V+< x >$ has been interesting a submodule of W includes V properly. By assumption $[L : W] \nsubseteq [V + J(W): W]$, consequently there has beents $s \in [L : W]$ and $s \notin [V + J(W): W]$. Presently $sW \subseteq L$ and $sW \nsubseteq V + J(W)$.Afterwards $rsW \subseteq V + J(W)$ and consequently $rs \in [V + J(W): W] = [V: W]$.Subsequently $rs \in [V: W] + J(R)$. Nevertheless $[V: W]$ has been interesting a J-Prime ideal of R and $s \notin [V: W]$, Consequently $r \in [V: W]$. What has been means has been that the module underneath V has been J-Prime.

Suppose we give the following:

Proposition 2.7: If R-module W has a submodule V that has been J-Prime, then:

(1) $[V + J(W): T]$ has been intriguing for a J-prime R ideal. Simply in has been instance does a submodule occur T of W that correctly contains $V + J(W)$.

(2) $[V + J(W):< c >]$ Including all $c \in W$ and $c \notin V + J(W)$has been interesting a J-Prime ideal (W)

Proof:

(1) Whether $[V + J(W): T]$ has been interesting a valid ideal of R has been self-evident as well. Presently, Suppose $rs \in [V + J(W): T] + J(R)$;$r, s \in R$. Afterwards $rs = w + j$; $w \in [V + J(W): T]$ and $j \in J(R)$. To stay everyone

$k \in T$, we possess $rsk = wk + jk \in V + J(W) + J(R)\,T \subseteq V + J(W)$. Consequently $r(sk) \in V + J(W)$. Nevertheless, V has been interesting a J-Prime submodule of W, afterward perhaps $rk \in V$ or $s \in [V{:}W]$, and consequently $rT \subseteq V$, afterward $r \in [V{:}T] \subseteq [V + J(W){:}T]$or $s \in [V{:}W]$, afterward, $sW \subseteq V$, What has been points to has been $sT \subseteq V \subseteq V + J(W)$, thus signifying $s \in [V + J(W){:}T]$ which indicates that $[V + J(W){:}T]$ has been interesting a J-prime ideal of R to stay everyone T as to stay $V + J(W) \subsetneq T$.

(2) $[V + J(W){:}< c >]$ has been interesting a proper ideal of R $\forall$ $c \in W$ as to stay $c \notin V + J(W)$. Presently Suppose $r, s \in R$ that should be said $rs \in [V + J(W){:}< c >] + J(R)$. Subsequently $rs = t + j$, $t \in [V + J(W){:}< c >]$ and $j \in J(R)$. As a corollary, it stands to reason that$rs \in [V + J(W){:}V + J(W)+< c >] + J(R)$ and $V + J(W)+< c >$ includes $V + J(W)$ properly, from (1), $[V + J(W){:}V + J(W)+< c >]$ has been interesting a J-Prime ideal of R. Consequently perhaps $r \in [V + J(W){:}V + J(W) + < c >]$. Subsequently r ($V + J(W) + < c >) \subseteq V + J(W)$, afterward $r < c > \subseteq V + J(W)$ and, Consequently $r \in [V + J(W){:}< c >]$, or $s \in [\,V + J(W){:}V + J(W)+ < c >]$, and consequently $s(V + J(W)+< c >) \subseteq V + J(W)$. Afterward $s < c > \subseteq V + J(W)$. What has been points to has been $s \in [V + J(W){:}< c >]$, Consequently $s \in [V + J(W){:}< c >]$ and subsequently $[V + J(W){:}< c >]$ has been interesting a J-Prime ideal of R.

Remark 2.8: [5] J(W) has been compact in W only in has been case W has been interesting in an inherently produced R-module that is not infinite.
Presently, the accompa particular ing corollary can be derived from proposition 2.7 and Remark 2.8.

Corollary 2.9: *If W has been interesting a non-empty R-module created infinitely ma particular ways, then Only in has been case a proper submodule V of W has been J-Prime, afterward* $[V + J(W){:}W]$ *has been interesting a J-Prime ideal of R.*
To date, There are defined a J *−Prime submodule as follows:*

Proposition 2.10: The following are comparable only in has been case V has been interested in a proper submodule of an R-module W:

1. *V* has been an interesting J-Prime submodule of *W*.
2. All possible ideals of the I of R and its sub modules T of W, if $IT \subseteq V + J(W)$, afterward perhaps $T \subseteq V$ or $I \subseteq [V{:}W]$.
3. The submodule $[V + J(W){:}_W < r >] = V$ to stay everyone $r \in R, r \notin [V{:}W]$.
4. $[V + J(W){:}< y >] = [V{:}W]$ $\forall$ $y \in W$ and $y \notin V$.
5. $[V + J(W){:}T] = [V{:}W]$ to stay everyone submodule T of W having an appropriate V-containment.

Proof: **(1)** ⇒ (2) Let's pretend T was a submodule of W while I was a cool ideal of R that should be said $IT \subseteq V + J(W)$. Suppose that $T \nsubseteq V$, afterward there exhaust bents $y \in T$ and $y \notin V$. As a corollary, it stands to reason that to stay everyone $\in I$, Apparently, Irrefutably, There are $ay \in V + J(W)$. Nevertheless, V has been interested a J-Prime submodule and $y \notin V$, Consequently $y \in [V: W]$ and accordingly $I \subseteq [V: W]$

(2) ⇒ (3) Suppose $r \in R$ and $r \notin [V: W]$. Suppose that $x \in [V + J(W):_W < r >]$, afterward $rx \in V + J(W)$ and accordingly $< r >< x > \subseteq V + J(W)$. By (2) we obtain that $< x > \subseteq V$. Consequently $x \in V$. What has been points to has been $[V + J(W):_W < r >] = V$

(3) ⇒ (4) Suppose $y \in W$ and $y \notin V$. As a corollary, it stands to reason that $[V: W] \subseteq [V + J(W): < y >]$. Presently Suppose $r \in [V + J(W): < y >]$. If $\notin [V: W]$, by (3) $[V + J(W):_W < r >] = V$. Nevertheless $y \in [V + J(W):_W < r >]$, afterward, $y \in V$, which themselves has been interesting an inconshas beentency, and which, as a result, $r \in [V: W]$ to stay everyone $y \in W$ and $y \notin V$.

(4) ⇒ (5) clear.

(5) ⇒ (1)Suppose $rx \in V + J(W)$, to stay some of that $r \in R$ and $x \in W$. Suppose that $x \notin V$, subsequently, $T = V + < x >$ has been interesting a submodule of W includes V properly, consequently by (5) we obtain that $[V + J(W): T] = [V: W]$. Nevertheless $r \in [V + J(W): T]$, afterward $r \in [V: W]$. And has been suggests that V has been an interesting J-Prime submodule of W.

Definition 2.11: Suppose W possesses an R-module that has been greater than zero. If $\{0_W\}$ has been interesting a J-Prime submodule of W, afterward W has been called the J-Prime module.

The accompa particular ing specification of a J-Prime module follows as a corollary from the premhas beene (2.10).

Corollary 2.12: *Suppose W has been greater than zero R-module. It's possible to equate:*

1. *W has been an interesting J-Prime R-module.*
2. *To stay everyone sub module T of W, in addition to a particular ideal, I of R, if* $IT \subseteq J(W)$*, afterward perhaps* $K = 0_W$ *or* $I \subseteq Ann(W)$*.*
3. *The submodule* $[J(W):_W < r >] = 0_W$ *to stay everyone* $\in R$ *,* $r \notin Ann(W)$*.*
4. $[J(W): < y >] = Ann(W)\ \forall\ y \in W\ ;\ y \neq 0$.
5. $[J(W): T] = Ann(W)$ *to stay everyone greater than zero submodule T of W.*

Please evaluate the following assertion against the assertion (2.13) in [2].

Proposition 2.13: Suppose W and $W`$ be R-modules and $\emptyset: W \to W`$ become an epimorphosis been m of Ker $\emptyset \ll W$.

1. If V has been interesting a J-Prime submodule of W as to stay Ker $\emptyset \subseteq V$, afterward $\emptyset(V)$ has been interesting a J-Prime submodule of $W`$.

2. If $V`$ has been interesting a J-Prime submodule of $W`$ afterward $\emptyset^{-1}(V`)$ has been interesting a J-Prime submodule of W.

Proof:

(1) Counterclaim $\emptyset(V)$ has been a proper submodule of $W`$, even be to stay e if $\emptyset(V) = W`$, afterward $\emptyset(m) \in \emptyset(V)$ to stay everyone $m \in W$, Consequently, there has been t $n \in V$ interesting that $\emptyset(m) = \emptyset(n)$, What has been points to has been $m - n \in Ker\ \emptyset \subseteq V$, Consequently $W = V$, in which case there has been a major in cons has been teeny, consequently $\emptyset(V) \neq W`$. Presently suppose that $rm \in \emptyset(V) + J(W`)$, to stay with some of that $r \in R$ and $m \in W`$, without even $\emptyset$ has been interesting an epimorph has been m, Consequently there has been t $x \in W$ having stated that $\emptyset(x) = m.$ Afterward $\emptyset(rx) = r\emptyset(x) = rm \in \emptyset(V) + \emptyset\big(J(W)\big)$, consequently, there has been $y \in V$ and $w \in J(W)$ having stated that $\emptyset(rx) = \emptyset(y) + \emptyset(w).$ Afterward $rx - (y + w) \in\ Ker\ \emptyset\ \subseteq V.$ Consequently $rx\ \in\ V\ + J(W)$, Nevertheless V has been interesting a J-Prime submodule of W, meaning that has been implies s$x \in V$ or $r \in [V\colon W]$. Afterward perhaps $\emptyset(x) \in \emptyset(V)$, and consequently $m \in \emptyset(V)$ or $rW \subseteq V$, afterward $r\emptyset(W) \subseteq \emptyset(V)$. Consequently $rW` \subseteq \emptyset(V)$, has been gives us $r \in [\emptyset(V)\colon W`]$. Subsequently $r \in [\emptyset(V)\colon W`]$. That's to say that $\emptyset(V)$ has been interesting a J-Prime submodule of $W`$.

(2) $\emptyset^{-1}(V`)$ has been interesting a proper submodule of $W$. Even be to stay e if $\emptyset^{-1}(V`) = W$, afterward $V` = \emptyset(W) = W`$, which has been a conflict worth noting. Presently suppose $r \in R$ and $y \in W$ have stated that $ry \in \emptyset^{-1}(V`) + J(W)$, Consequently $ry \in \emptyset^{-1}(V`) + \emptyset^{-1}\big(J(W`)\big)$.Subsequently $r\emptyset(y) \in V` + J(W`)$. Nevertheless $V`$ has been interesting a J-Prime submodule of $W`$, afterward perhaps $\emptyset(y) \in V`$ or $r \in [V`\colon W`]$, consequently perhaps $y \in \emptyset^{-1}(V`)$ or $rW` \subseteq V`$, subsequently $r\emptyset(W) \subseteq V`$, afterward $rW\ \subseteq \emptyset^{-1}(V`)$. Consequently $r \in [\emptyset^{-1}(V`)\colon W]$, and has been suggests that $r \in [\emptyset^{-1}(V`)\colon W]$. Afterward $\emptyset^{-1}(V`)$ has been interesting a J-Prime submodule of W.

Corollary 2.14: *Suppose V has been a J-Prime submodule of W and T has been interesting a submodule of W as to stay $T \subseteq V$ and T<< W, afterward $\frac{V}{T}$ has been a J-Prime submodule of $\frac{W}{T}$.*

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