Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 26 (2023), No. 6, pp. 1773–1799
DOI : 10.47974/JDMSC-1726

Generalized k–step Jacobsthal numbers coding and decoding method

Arun Kumar Yadav *
Manju Pruthi †
Department of Mathematics
Indira Gandhi University
Meerpur - Rewari
Haryana
India

Abstract

In this article, we create a coding/decoding method utilizing the generalized k-step Jacobsthal numbers and the relation between the components of the code matrix and error detection and correction have been established for all value of k. The method's accuracy is 93.33 percent for $k = 2$, the method's accuracy is 99.80 percent for $k = 3$ and the method accuracy is 99.998 percent for $n = 4$. In general, the accuracy of the approach improves with each successively higher value of n. We give two examples of generalized k – step jacobsthal sequences via blocking algorithm and solve these illustrations using python.

Subject Classification: *11B37, 11B39, 11B83, 11C20, 94B.*

Keywords: *Jacobsthal numbers, Coding-decoding, Recurrence relation, Error correction.*

1. Introduction

In 1999, Stakhov initiated the theory of Fibonacci coding and cryptography. Afterward, different authors formulated various interesting results using Fibonacci, Lucas and Pell coding and decoding methods. On the other hand, Basu and Prasad [2] introduced the coding theory using Fibonacci $p-$ numbers. Later, Basu et al. [3] developed the Fibonacci $n-$ step number coding/decoding method follow from Fibonacci matrices. In the same work, Basu and Das [4] extended this theory to generalized Fibonacci $n-$step polynomial. Sundarayya and Prasad [7] defined the coding theory on Pell –

* *E-mail:* **arun.math.rs@igu.ac.in (Corresponding Author)**
† *E-mail:* **manju.pruthi@yahoo.com**

Lucas p numbers using Pell – Lucas p −matrix. Recently, Tas, et al. [9] defined the Pell coding-decoding methods with applications and formulated the connections between the components of the code matrix. It has been given Pell blocking algorithm and generalized Pell blocking algorithm related to Pell G_n − matrices. Recently, calculate higher-order derivatives of the ratio of two differentiable functions, Feng Qi [6] used the Hessensberg determinant of generalised Fubini polynomials to arrive at an explicit formula. Several recurrence relations for Fubini and Eulerian polynomials were found, and a determinantal equation for Eulerian polynomials was provided.

In this paper, In section 1 give the basic results related to Jacobsthal sequences. In section 2, we give the generalized coding and decoding transformations and determinant relation between the components of code and message matrix for 2-step Jacobsthal numbers. In section 3 describe the coding and decoding 3-step Jacobsthal numbers allows for error detection and correction. In section 4 presents the generalized coding and decoding transformations and determinant connection between the code matrix and message matrix components for 4-step Jacobsthal numbers. In section 5 give coding and decoding for k-step Jabosthal numbers correct and detect errors. Use k-step Jacobsthal Numbers to determine message ability and finally conclude this result in section 6.

1.1 *Preliminaries*

The Jacobsthal sequence is described by the following recurrence relations $J_n = J_{n-1} + 2J_{n-2}$ with $n \geq 2$ with initial conditions $J_0 = 0, J_1 = 1$.

The generalized order - k Jacobsthal numbers have recently been described via recurrence relations in [12]:

$$J_n^i = J_{n-1}^i + 2J_{n-2}^i + J_{n-3}^i + \dots + J_{n-k}^i \tag{1}$$

For $n > 0$ and $1 \leq i \leq k$ with

$$J_n^i = \begin{cases} 1 \; if \; i + n = 1 \\ 0 \; otherwise \end{cases} for \; 1 - k \leq n \leq 0$$

where J_n^i is the n^{th} term of the i^{th} sequence.

Some number of terms of generalized order $-k$ Jacobsthal sequences are given in the following table:

	i/n	-2	-1	0	1	2	3	4	5	⋯
$k = 2$	1	0	0	1	1	3	5	11	21	⋯
	2	0	1	0	2	2	6	10	22	⋯
	1	0	0	1	1	3	6	13	28	⋯
$k = 3$	2	0	1	0	2	3	7	15	32	⋯
	3	1	0	0	1	1	3	6	13	⋯
	1	0	0	1	1	3	6	14	30	⋯
	2	0	1	0	2	3	8	16	37	⋯

Contd...

$k = 4$	3	1	0	0	1	2	4	9	20	⋯
	4	0	0	0	1	1	3	6	14	⋯
⋮				⋮					⋮	⋮

For $n \geq 1$, $\alpha_n = \lim_{n\to\infty} \frac{J_n^{(i)}}{J_{n-1}^{(i)}}$ exists, called nth −Jacobsthal constant and is the real root ≥ 1 of the equation $x^k - x^{k-1} - 2x^{k-2} - x^{k-3} - \cdots - x - 1 = 0$.
For odd k, there is one real root, which is always more than 1, while for even k, there are two real roots, one greater than 1 and other less than 1.Here, $\alpha_1 = 1, \alpha_2 = 1.61803,\ \alpha_3 = 2.14789,\ \alpha_4 = 2.20556, etc. \lim_{n\to\infty} \alpha_n = 3$.
In [12], Matrix C of order k was presented in the form

$$C = \begin{pmatrix} 1 & 2 & 1 & \cdots & 1 & 1 \\ 1 & 0 & 0 & \cdots & 0 & 0 \\ 0 & 1 & 0 & \cdots & 0 & 0 \\ \vdots & \vdots & \vdots & \cdots & \vdots & \vdots \\ 0 & 0 & 0 & \cdots & 1 & 0 \end{pmatrix} \tag{2}$$

Define a $k -$ square matrix B_k to deal with the $k -$ sequences of the generalized k −step Jacobsthal number as follows

$$C^k = B_k = \begin{pmatrix} J_n^1 & J_n^2 & \cdots & J_n^k \\ J_{n-1}^1 & J_{n-1}^2 & \cdots & J_{n-1}^k \\ \vdots & \vdots & \cdots & \vdots \\ J_{n-(k+1)}^1 & J_{n-(k+1)}^2 & \cdots & J_{n-(k+1)}^k \end{pmatrix} \tag{3}$$

using equations (2) and (3), we have

$$\mathrm{Det}(C^k) = \mathrm{Det}(B_k) = \begin{cases} (-2)^n, k = 2 \\ 1, k\ is\ odd \\ -1, k\ is\ even (k \neq 2) \end{cases} \quad \text{for } n > 0 \tag{4}$$

2. Generalized $k -$ Step Jacobsthal Coding & Decoding Method

This section introduces a novel coding and decoding theory using extended k-step Jacobsthal numbers. and express the message in this manner as a non-singular matrix. Consider the $k -$ square matrix B_k as coding matrix and decoding matrix $(B_k)^{-1}$. Now, we represent transformations:

Jacobsthal Coding Transformations	**Jacobsthal Decoding Transformations**
$M \times B_k = E$	$E \times (B_k)^{-1} = M$

2.1 *Illustration of Generalized* $2 -$ *Step Jacobsthal Coding & Decoding Method:*

Let us suppose that $M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$, Where $m_i \in \boldsymbol{Z}^+$, Where M is message matrix.

Consider $k = 2, n = 1, 1 \leq i \leq k$, Construct the coding matrix B_k:

$$B_2 = \begin{pmatrix} J_n^1 & J_n^2 \\ J_{n-1}^1 & J_{n-1}^2 \end{pmatrix}$$

Put $n = 1$,

$$B_2 = \begin{pmatrix} J_1^1 & J_1^2 \\ J_0^1 & J_0^2 \end{pmatrix} = \begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix}$$

Then, the inverse of B_2 is given by

$$(B_2)^{-1} = \frac{1}{1.0 - 1.2}\begin{pmatrix} 0 & -2 \\ -1 & 1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix}$$

Now,

$$E = M \times B_2 = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} m_1 + m_2 & 2m_1 \\ m_3 + m_4 & 2m_3 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$$

where $e_1 = m_1 + m_2, e_3 = m_3 + m_4, e_2 = 2m_1$ & $e_4 = 2m_3$
Then, using decoding transformation

$$M = E \times (B_2)^{-1} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \cdot \frac{1}{2}\begin{pmatrix} 0 & 2 \\ 1 & -1 \end{pmatrix} = \frac{1}{2}\begin{pmatrix} e_2 & 2e_1 - e_2 \\ e_4 & 2e_3 - e_4 \end{pmatrix}$$

$$= \begin{pmatrix} \frac{e_2}{2} & e_1 - \frac{e_2}{2} \\ e_4 & e_3 - \frac{e_4}{2} \end{pmatrix} = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}.$$

2.2 *Determinant connections between the components of the code and Message Matrix M for* $k = 2, n > 0$ *is*

$$Det(E) = Det(M \times B_k) = Det(M) \times Det(B_k) = Det(M) \times (-2)^n \tag{5}$$

2.3 *The Relationship between the components of code matrix for* $k = 2,$ $1 \leq i \leq 2$

The code & initial message matrix as follows:

$$E = M \times B_k = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} \times \begin{pmatrix} J_n^1 & J_n^2 \\ J_{n-1}^1 & J_{n-1}^2 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix}$$

and
$$M = E \times (B_k)^{-1} = \begin{pmatrix} e_1 & e_2 \\ e_3 & e_4 \end{pmatrix} \times \frac{1}{(-2)^n}\begin{pmatrix} J_{n-1}^2 & -J_n^2 \\ -J_{n-1}^1 & J_n^1 \end{pmatrix}$$

$$\begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix} = \frac{1}{(-2)^n}\begin{pmatrix} e_1 J_{n-1}^2 - e_2 J_{n-1}^1 & -e_1 J_n^2 + e_2 J_n^1 \\ e_3 J_{n-1}^2 - e_4 J_{n-1}^1 & -e_3 J_n^2 + e_4 J_n^1 \end{pmatrix}$$

We know that m_1, m_2, m_3 *and* $m_4 \in Z^+$. So, we get

$$m_1 = \frac{e_1 J_{n-1}^2 - e_2 J_{n-1}^1}{(-2)^n} \geq 0, m_2 = \frac{-e_1 J_n^2 + e_2 J_n^1}{(-2)^n} \geq 0$$

$$m_3 = \frac{e_3 J_{n-1}^2 - e_4 J_{n-1}^1}{(-2)^n} \geq 0, m_4 = \frac{-e_3 J_n^2 + e_4 J_n^1}{(-2)^n} \geq 0$$

Case (i) When n is odd ($\boldsymbol{n = 2t + 1}$)

$$-e_1\, J_{n-1}^2 + e_2\, J_{n-1}^1 \geq 0 \tag{6}$$

$$e_1 J_n^2 - e_2 J_n^1 \geq 0 \tag{7}$$

$$-e_3\, J_{n-1}^2 + e_4\, J_{n-1}^1 \geq 0 \tag{8}$$

$$e_3\, J_n^2 - e_4\, J_n^1 \geq 0 \tag{9}$$

From equations (6) and (7), we get

$$\frac{J_n^1}{J_n^2} \leq \frac{e_1}{e_2} \leq \frac{J_{n-1}^1}{J_{n-1}^2} \tag{10}$$

From equations (8) and (9), we get

$$\frac{J_n^1}{J_n^2} \leq \frac{e_3}{e_4} \leq \frac{J_{n-1}^1}{J_{n-1}^2} \tag{11}$$

From equations (10) and (11), we get

$$\frac{e_1}{e_2} \approx \alpha \text{ and } \frac{e_3}{e_4} \approx \alpha \tag{12}$$

where $\alpha = \frac{1+\sqrt{5}}{2}$ is the “golden ratio”.

Case (ii) When n is even $(n = 2t)$, We find the similar relationships described in (12).

2.4 *Error Detection / Correction for Generalized $k -$ Step Jacobsthal Coding and Decoding Process*

In this section, the code matrix E may contain mistakes as a result of a few channel-related factors. With the help of the determinant's properties, we attempt to identify and fix these errors in this process.

For $k = 2$, $1 \leq i \leq 2$, and the message matrix M is given by

$$M = \begin{pmatrix} m_1 & m_2 \\ m_3 & m_4 \end{pmatrix}$$

Then, Det $(M) = m_1 m_4 - m_3 m_2$

The code matrix $E = M \times B_k$ & Det $(E) = \text{Det}(M) \times (-2)^n$ for $k = 2$. From the connection between the determinant of E and M

$$\text{Det}(E) = \begin{cases} (-2)^n \text{Det}(M)\,, n\ is\ odd \\ (2)^n \text{Det}(M)\,, n\ is\ even \end{cases}$$

The process of coding and decoding, it is possible to find mistakes in a coded message and fix them. Based on the property of a matrix determinant, the errors are found and fixed. We are unable to identify the specific code message component that is broken. We make assumptions about diverse scenarios, such as one error, two errors, etc., to identify damaged elements. At this point, the first occurrence of the code matrix E that contains a single mistake is being taken into consideration. It is simple to determine that E contains a single error in four different cases:

$$i)\ \begin{pmatrix} q_1 & e_2 \\ e_3 & e_4 \end{pmatrix}\ ii)\ \begin{pmatrix} e_1 & q_2 \\ e_3 & e_4 \end{pmatrix}$$

$$iii)\ \begin{pmatrix} e_1 & e_2 \\ q_3 & e_4 \end{pmatrix}\ iv)\ \begin{pmatrix} e_1 & e_2 \\ e_3 & q_4 \end{pmatrix}$$

Where q_i are the possible damaged elements

The four separate situations mentioned before are now checked using the relations listed below:

$$q_1 e_4 - e_2 e_3 = (-2)^n Det(M) \tag{13}$$

$$e_1 e_4 - q_2 e_3 = (-2)^n Det(M) \tag{14}$$

$$e_1 e_4 - q_3 e_2 = (-2)^n Det(M) \tag{15}$$

$$e_1 q_4 - e_2 e_3 = (-2)^n Det(M) \tag{16}$$

using the above equations, we get

$$q_1 = \frac{(-2)^n Det(M) + e_2 e_3}{e_4},\ q_2 = \frac{(-2)^{n-1} Det(M) + e_1 e_4}{e_3} \qquad (17), (18)$$

$$q_3 = \frac{(-2)^{n-1} Det(M) + e_1 e_4}{e_4},\ q_4 = \frac{(-2)^n Det(M) + e_2 e_3}{e_1} \qquad (19), (20)$$

Since the components of message matrix M are +ve integer. The equations from (17) to (18) should have integer solutions. If these equations don't have integer solutions, then we know that the scenarios we came up with containing a single mistake are accurate, or else we know that there was a mistake somewhere in the checking element Det (M). In the event that Det (M) returns a wrong value, we check the validity of the code matrix E by applying the relations found in (12). In the code matrix E, check each instance of a double error in the same manner.

For example, In this instance, there are double errors in the code matrix.

$$\begin{pmatrix} q_1 & q_2 \\ e_3 & e_4 \end{pmatrix} \tag{21}$$

Where q_1, q_2 be possible "damaged" element. Using the determinant relation $\mathrm{Det}(E) = \mathrm{Det}(M) \times (-2)^n$, For the matrix, we can formulate the following equation:

$$q_1 e_4 - q_2 e_3 = (-2)^n Det(M) \tag{22}$$

Also, we know that the relation between q_1 *and* q_2 is

$$q_1 \approx \alpha q_2 \tag{23}$$

Clearly, Relation (22) be Diophantine equation. There are several solutions to the Diophantine equation (22). We must select q_1 and q_2 solutions that satisfy the equation (23).

In the same way, the code matrix E, we fix the triple error to make sure that $\begin{pmatrix} q_1 & q_2 \\ q_3 & e_4 \end{pmatrix}$, Where q_1, q_2 and q_3 be the possible "damaged" element.

Due to our correctness technique, we are only able to fix 14 out of the 15 scenarios. Corrective potential of our technique is defined as 14/15 =0.9333 ≅ 93.33%

3. Illustration of Generalized 3 − Step Jacobsthal Coding & Decoding Method:

Suppose that M is message matrix:

$$M = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}, \text{ Where } m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9 \in \boldsymbol{Z}^+$$

Consider $k = 3$, and $1 \le i \le k$, Construct the coding matrix B_k:

$$B_3 = \begin{pmatrix} J_n^1 & J_n^2 & J_n^3 \\ J_{n-1}^1 & J_{n-1}^2 & J_{n-1}^3 \\ J_{n-2}^1 & J_{n-2}^2 & J_{n-2}^3 \end{pmatrix}$$

Put $n = 1$,

$$B_3 = \begin{pmatrix} J_1^1 & J_1^2 & J_1^3 \\ J_0^1 & J_0^2 & J_0^3 \\ J_{-1}^1 & J_{-1}^2 & J_{-1}^3 \end{pmatrix} = \begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

Then, the inverse of B_3 is given by

$$(B_3)^{-1} = \frac{1}{1}\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix}$$

Now, we find the code matrix E.

$$E = M \times B_3 = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}$$

$$= \begin{pmatrix} m_1 + m_2 & 2m_1 + m_3 & m_1 \\ m_4 + m_5 & 2m_4 + m_6 & m_4 \\ m_7 + m_8 & 2m_7 + m_9 & m_7 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}$$

where $e_1 = m_1 + m_2$, $e_2 = 2m_1 + m_3$, $e_3 = m_1$, $e_4 = m_4 + m_5$, $e_5 = 2m_4 + m_6$, $e_6 = m_4$, $e_7 = m_7 + m_8$, $e_8 = 2m_7 + m_9$, $e_9 = m_7$.

Then, by decoding transformation

$$M = E \times (B_3)^{-1} = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}.$$

$$\begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -1 & -2 \end{pmatrix} = \begin{pmatrix} e_3 & e_1 - e_3 & e_2 - 2e_3 \\ e_6 & e_4 - e_6 & e_5 - 2e_6 \\ e_9 & e_7 - e_9 & e_8 - 2e_9 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}.$$

3.2 *Determinant connections between the components of code and Message Matrix M for* $k = 3, n > 0$ *is*

$$Det(E) = Det(M \times B_k) = Det(M) \times (DetB_k) = Det(M).1 \quad (3.1)$$

3.3 *The Relationship between the components of the code matrix for* $k = 3$, $1 \leq i \leq 3$ *define as*

$$E = M \times B_k = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix} \times \begin{pmatrix} J_n^1 & J_n^2 & J_n^3 \\ J_{n-1}^1 & J_{n-1}^2 & J_{n-1}^3 \\ J_{n-2}^1 & J_{n-2}^2 & J_{n-2}^3 \end{pmatrix} = \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}$$

And $M = E \times (B_k)^{-1}$

$$= \begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} \times$$

$$\begin{pmatrix} J_{n-1}^2 J_{n-2}^3 - J_{n-1}^3 J_{n-2}^2 & -J_n^2 J_{n-2}^3 + J_n^3 J_{n-2}^2 & J_n^2 J_{n-1}^3 - J_n^3 J_{n-1}^2 \\ -J_{n-1}^1 J_{n-2}^3 - J_{n-1}^2 J_{n-2}^1 & J_n^1 J_{n-2}^3 - J_n^3 J_{n-2}^1 & J_n^1 J_{n-1}^3 - J_n^3 J_{n-1}^1 \\ J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1 & -J_n^1 J_{n-2}^2 - J_n^2 J_{n-2}^1 & J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1 \end{pmatrix} = \begin{pmatrix} m_1 & m_2 & m_3 \\ m_4 & m_5 & m_6 \\ m_7 & m_8 & m_9 \end{pmatrix}$$

Where $m_1, m_2, m_3, m_4, m_5, m_6, m_7, m_8, m_9 \in \boldsymbol{Z}^+$ So, we get

$$m_1 = e_1(J_{n-1}^2 J_{n-2}^3 - J_{n-1}^3 J_{n-2}^2) + e_2(-J_{n-1}^1 J_{n-2}^3 - J_{n-1}^2 J_{n-2}^1) \quad (3.2)$$
$$+e_3 (J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1) \geq 0$$

$$m_2 = e_1(-J_n^2 J_{n-2}^3 + J_n^3 J_{n-2}^2) + e_2(J_n^1 J_{n-2}^3 - J_n^3 J_{n-2}^1) \quad (3.3)$$
$$+e_3(-J_n^1 J_{n-2}^2 - J_n^2 J_{n-2}^1) \geq 0$$

$$m_3 = e_1(J_n^2 J_{n-1}^3 - J_n^3 J_{n-1}^2) + e_2(J_n^1 J_{n-1}^3 - J_n^3 J_{n-1}^1) \quad (3.4)$$
$$+ e_3(J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1 \geq 0$$

$$m_4 = e_4(J_{n-1}^2 J_{n-2}^3 - J_{n-1}^3 J_{n-2}^2) + e_5(-J_{n-1}^1 J_{n-2}^3 - J_{n-1}^2 J_{n-2}^1) \quad (3.5)$$
$$+e_6 (J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1) \geq 0$$

$$m_5 = e_4(-J_n^2 J_{n-2}^3 + J_n^3 J_{n-2}^2) + e_5(J_n^1 J_{n-2}^3 - J_n^3 J_{n-2}^1) \quad (3.6)$$
$$+e_6(-J_n^1 J_{n-2}^2 - J_n^2 J_{n-2}^1) \geq 0$$

$$m_6 = e_4(J_n^2 J_{n-1}^3 - J_n^3 J_{n-1}^2) + e_5(J_n^1 J_{n-1}^3 - J_n^3 J_{n-1}^1) \quad (3.7)$$
$$+ e_6(J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1 \geq 0$$

$$m_7 = e_7(J_{n-1}^2 J_{n-2}^3 - J_{n-1}^3 J_{n-2}^2) + e_8(-J_{n-1}^1 J_{n-2}^3 - J_{n-1}^2 J_{n-2}^1) \quad (3.8)$$
$$+e_9 (J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1) \geq 0$$

$$m_8 = e_7(-J_n^2 J_{n-2}^3 + J_n^3 J_{n-2}^2) + e_8(J_n^1 J_{n-2}^3 - J_n^3 J_{n-2}^1) \quad (3.9)$$
$$+e_9(-J_n^1 J_{n-2}^2 - J_n^2 J_{n-2}^1) \geq 0$$

$$m_9 = e_7(J_n^2 J_{n-1}^3 - J_n^3 J_{n-1}^2) + e_8(J_n^1 J_{n-1}^3 - J_n^3 J_{n-1}^1) \quad (3.10)$$
$$+ e_9(J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1 \geq 0$$

From equations (3.2), (3.3) and (3.4), we get

$$e_3\,(J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1) \geq e_2(-J_{n-1}^1 J_{n-2}^3 - J_{n-1}^2 J_{n-2}^1)$$
$$+e_1(J_{n-1}^2 J_{n-2}^3 - J_{n-1}^3 J_{n-2}^2) \quad (3.11)$$

$$e_3(-J_n^1 J_{n-2}^2 - J_n^2 J_{n-2}^1) \geq e_2(J_n^1 J_{n-2}^3 - J_n^3 J_{n-2}^1) + e_1(-J_n^2 J_{n-2}^3 + J_n^3 J_{n-2}^2) \quad (3.12)$$

$$e_3(J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1 \geq e_2(J_n^1 J_{n-1}^3 - J_n^3 J_{n-1}^1) + e_1(J_n^2 J_{n-1}^3 - J_n^3 J_{n-1}^2) \quad (3.13)$$

Dividing both side by e_1 of equation (3.11), (3.12), (3.13) , we have

$$\frac{e_3}{e_1}\,(J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1)$$
$$\geq \frac{e_2}{e_1}(-J_{n-1}^1 J_{n-2}^3 - J_{n-1}^2 J_{n-2}^1) + (J_{n-1}^2 J_{n-2}^3 - J_{n-1}^3 J_{n-2}^2) \quad (3.14)$$

$$\frac{e_3}{e_1}(J_n^1 J_{n-2}^2 + J_n^2 J_{n-2}^1) \leq \frac{e_2}{e_1}(-J_n^1 J_{n-2}^3 + J_n^3 J_{n-2}^1) + (J_n^2 J_{n-2}^3 - J_n^3 J_{n-2}^2) \quad (3.15)$$

$$\frac{e_3}{e_1}(J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1) \geq \frac{e_2}{e_1}(J_n^1 J_{n-1}^3 - J_n^3 J_{n-1}^1) + (J_n^2 J_{n-1}^3 - J_n^3 J_{n-1}^2) \quad (3.16)$$

Let $a = J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1$, $b = J_n^1 J_{n-2}^2 + J_n^2 J_{n-2}^1$, $c = J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1$

Now $3^3 = 27$ cases arise for $a >=<, b >=<, c >=<$.

Case: a) If a, b, c all are positive, then

From equation (3.14) , we get

$$\frac{e_3}{e_1} \geq \alpha, \text{ where } \alpha = \frac{e_2}{e_1}\left(\frac{-J_{n-1}^1 J_{n-2}^3 - J_{n-1}^2 J_{n-2}^1}{J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1}\right) + \left(\frac{J_{n-1}^2 J_{n-2}^3 - J_{n-1}^3 J_{n-2}^2}{J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1}\right) \quad (3.17)$$

From equation (3.15) , we get

$$\frac{e_3}{e_1} \leq \beta, \text{ where } \beta = \frac{e_2}{e_1}\left(\frac{-J_n^1 J_{n-2}^3 + J_n^3 J_{n-2}^1}{J_n^1 J_{n-2}^2 + J_n^2 J_{n-2}^1}\right) + \left(\frac{J_n^2 J_{n-2}^3 - J_n^3 J_{n-2}^2}{J_n^1 J_{n-2}^2 + J_n^2 J_{n-2}^1}\right) \quad (3.18)$$

From equation (3.16) , we get

$$\frac{e_3}{e_1} \geq \gamma, \text{ where } \gamma = \frac{e_2}{e_1}\left(\frac{J_n^1 J_{n-1}^3 - J_n^3 J_{n-1}^1}{J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1}\right) + \left(\frac{J_n^2 J_{n-1}^3 - J_n^3 J_{n-1}^2}{J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1}\right) \quad (3.19)$$

From equations (3.17) and (3.18) , we get

$$\frac{e_1}{e_2} \geq min\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}\right\}, \text{ using equation (3.1)} \quad (3.20)$$

From equations (3.17) and (3.18) , we get

$$\frac{e_1}{e_2} \leq max\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}\right\}, \text{using equation (3.1)} \tag{3.21}$$

From equations (3.20) and (3.21) , we get

$$min\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}\right\} \leq \frac{e_1}{e_2} \leq max\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}\right\} \tag{3.22}$$

Similarly, for rest cases

$$min\left\{\frac{J_n^2}{J_n^3}, \frac{J_{n-1}^2}{J_{n-1}^3}, \frac{J_{n-2}^2}{J_{n-2}^3}\right\} \leq \frac{e_2}{e_3} \leq max\left\{\frac{J_n^2}{J_n^3}, \frac{J_{n-1}^2}{J_{n-1}^3}, \frac{J_{n-2}^2}{J_{n-2}^3}\right\} \tag{3.23}$$

and

$$min\left\{\frac{J_n^1}{J_n^3}, \frac{J_{n-1}^1}{J_{n-1}^3}, \frac{J_{n-2}^1}{J_{n-2}^3}\right\} \leq \frac{e_1}{e_3} \leq max\left\{\frac{J_n^1}{J_n^3}, \frac{J_{n-1}^1}{J_{n-1}^3}, \frac{J_{n-2}^1}{J_{n-2}^3}\right\} \tag{3.24}$$

Case: b) If $a = 0, b > 0, c > 0$, then

From equation (3.17) and (3.18) we get

$$\frac{e_1}{e_2} \geq min\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}\right\}, since\ a = 0 \tag{3.24}$$

From equation (3.18) and (3.19), we get

$$\frac{e_1}{e_2} \leq max\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}\right\}, \text{using equation (3.1), and } a = 0 \tag{3.25}$$

From equation (3.24) and (3.25), we get

$$min\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}\right\} \leq \frac{e_1}{e_2} \leq max\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}\right\} \tag{3.26}$$

Case: c) If $a < 0, b < 0, c < 0$, then

From equation (3.14) , we get

$$\frac{e_3}{e_1} \leq \alpha, \text{ where } \alpha = \frac{e_2}{e_1}\left(\frac{-J_{n-1}^1 J_{n-2}^3 - J_{n-1}^2 J_{n-2}^1}{J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1}\right) + \left(\frac{J_{n-1}^2 J_{n-2}^3 - J_{n-1}^3 J_{n-2}^2}{J_{n-1}^1 J_{n-2}^2 - J_{n-1}^2 J_{n-2}^1}\right) \tag{3.27}$$

From equation (3.15) , we get

$$\frac{e_3}{e_1} \geq \beta, \text{ where } \beta = \frac{e_2}{e_1}\left(\frac{-J_n^1 J_{n-2}^3 + J_n^3 J_{n-2}^1}{J_n^1 J_{n-2}^2 + J_n^2 J_{n-2}^1}\right) + \left(\frac{J_n^2 J_{n-2}^3 - J_n^3 J_{n-2}^2}{J_n^1 J_{n-2}^2 + J_n^2 J_{n-2}^1}\right) \tag{3.28}$$

From equation (3.16) , we get

$$\frac{e_3}{e_1} \leq \gamma, \text{ where } \gamma = \frac{e_2}{e_1}\left(\frac{J_n^1 J_{n-1}^3 - J_n^3 J_{n-1}^1}{J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1}\right) + \left(\frac{J_n^2 J_{n-1}^3 - J_n^3 J_{n-1}^2}{J_n^1 J_{n-1}^2 - J_n^2 J_{n-1}^1}\right) \tag{3.29}$$

From equations (2.27) and (2.28) , we get

$$\frac{e_1}{e_2} \geq min\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}\right\}, \text{using equation (3.1)} \tag{3.30}$$

From equations (2.28) and (2.29) , we get

$$\frac{e_1}{e_2} \leq max\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}\right\}, \text{using equation (3.1)} \tag{3.31}$$

From equations (2.30) and (2.31) , we get

$$min\left\{\frac{J_n^1}{J_n^2},\frac{J_{n-1}^1}{J_{n-1}^2},\frac{J_{n-2}^1}{J_{n-2}^2}\right\}\le\frac{e_1}{e_2}\le max\left\{\frac{J_n^1}{J_n^2},\frac{J_{n-1}^1}{J_{n-1}^2},\frac{J_{n-2}^1}{J_{n-2}^2}\right\} \tag{3.32}$$

Similarly, $$min\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3}\right\}\le\frac{e_2}{e_3}\le max\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3}\right\} \tag{3.33}$$

and $$min\left\{\frac{J_n^1}{J_n^3},\frac{J_{n-1}^1}{J_{n-1}^3},\frac{J_{n-2}^1}{J_{n-2}^3}\right\}\le\frac{e_1}{e_3}\le max\left\{\frac{J_n^1}{J_n^3},\frac{J_{n-1}^1}{J_{n-1}^3},\frac{J_{n-2}^1}{J_{n-2}^3}\right\} \tag{3.34}$$

Similarly, in rest cases

$$min\left\{\frac{J_n^1}{J_n^2},\frac{J_{n-1}^1}{J_{n-1}^2},\frac{J_{n-2}^1}{J_{n-2}^2}\right\}\le\frac{e_1}{e_2}\le max\left\{\frac{J_n^1}{J_n^2},\frac{J_{n-1}^1}{J_{n-1}^2},\frac{J_{n-2}^1}{J_{n-2}^2}\right\} \tag{3.35}$$

$$min\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3}\right\}\le\frac{e_2}{e_3}\le max\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3}\right\} \tag{3.36}$$

$$min\left\{\frac{J_n^1}{J_n^3},\frac{J_{n-1}^1}{J_{n-1}^3},\frac{J_{n-2}^1}{J_{n-2}^3}\right\}\le\frac{e_1}{e_3}\le max\left\{\frac{J_n^1}{J_n^3},\frac{J_{n-1}^1}{J_{n-1}^3},\frac{J_{n-2}^1}{J_{n-2}^3}\right\} \tag{3.37}$$

Therefore, for large value of k,

$$\frac{e_1}{e_2}\approx r_3,\frac{e_2}{e_3}\approx r_3,\frac{e_1}{e_3}\approx r_3 \tag{3.38}$$

Similarly, $$\frac{e_4}{e_5}\approx r_3,\frac{e_5}{e_6}\approx r_3,\frac{e_4}{e_6}\approx r_3 \tag{3.39}$$

$$\frac{e_7}{e_8}\approx r_3,\frac{e_8}{e_9}\approx r_3,\frac{e_7}{e_9}\approx r_3 \tag{3.40}$$

3.4 *Error Detection / Correction for Generalized* $3-$ *Step Jacobsthal Coding and Decoding Process*

In this section, the code matrix E may contain mistakes as a result of a few channel-related factors. With the help of the determinant's properties, we attempt to identify and fix these errors in this process.

For $k=3$, $1\le i\le 3$, and the message matrix M is given by $\begin{pmatrix} m_1 & m_2 & m_3\\ m_4 & m_5 & m_6\\ m_7 & m_8 & m_9\end{pmatrix}$

Det $(M)=m_1(m_5m_9-m_6m_8)+m_2(m_7m_6-m_9m_4)+m_3(m_4m_8-m_5m_7)$

The code matrix $E=M\times B_k$ & Det $(E)=$ Det$(M)\times(-1)^k$

From the connection between the determinant of E and M

$$\text{Det}(E)=\begin{cases}\ \ \text{Det}(M)\ ,n\ is\ odd\\ -\text{Det}(M)\ ,n\ is\ even\end{cases}$$

The process of coding and decoding makes it possible to detect and correct mistakes in the coded message. The error detection and correction are performed based on a matrix determinant's property. The receiver can determine the truth of code message E by comparing the determinants retrieved from the channel.

We are unable to identify the specific code message component that is broken. We make assumptions about diverse scenarios, such as one error, two errors, etc., to identify damaged elements. It is simple to determine that E contains a single error in nine different cases:

$$i)\begin{pmatrix} q_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} ii)\begin{pmatrix} e_1 & q_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} iii)\begin{pmatrix} e_1 & e_2 & q_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix}$$

$$iv)\begin{pmatrix} e_1 & e_2 & e_3 \\ q_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} v)\begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & q_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} vi)\begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & q_6 \\ e_7 & e_8 & e_9 \end{pmatrix}$$

$$vii)\begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ q_7 & e_8 & e_9 \end{pmatrix} viii)\begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & q_8 & e_9 \end{pmatrix} ix)\begin{pmatrix} e_1 & e_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & q_9 \end{pmatrix}$$

Where q_i are the possible damaged elements.

The nine separate situations mentioned before are now checked using the relations listed below:

$$q_1(e_5e_9 - e_6e_8) + e_2(e_7e_6 - e_9e_4) + e_3(e_4e_8 - e_5e_7) = \det(M) \tag{3.41}$$

$$e_1(e_5e_9 - e_6e_8) + q_2(e_7e_6 - e_9e_4) + e_3(e_4e_8 - e_5e_7) = \det(M) \tag{3.42}$$

$$e_1(e_5e_9 - e_6e_8) + e_2(e_7e_6 - e_9e_4) + q_3(e_4e_8 - e_5e_7) = \det(M) \tag{3.43}$$

$$q_4(e_3e_8 - e_2e_9) + e_5(e_1e_9 - e_3e_7) + e_3(e_2e_7 - e_1e_8) = \det(M) \tag{3.44}$$

$$e_4(e_3e_8 - e_2e_9) + q_5(e_1e_9 - e_3e_7) + e_3(e_2e_7 - e_1e_8) = \det(M) \tag{3.45}$$

$$e_4(e_3e_8 - e_2e_9) + e_5(e_1e_9 - e_3e_7) + q_6(e_2e_7 - e_1e_8) = \det(M) \tag{3.46}$$

$$q_7(e_2e_6 - e_3e_5) + e_8(e_3e_4 - e_1e_6) + e_9(e_1e_5 - e_2e_4) = \det(M) \tag{3.47}$$

$$e_7(e_2e_6 - e_3e_5) + q_8(e_3e_4 - e_1e_6) + e_9(e_1e_5 - e_2e_4) = \det(M) \tag{3.48}$$

$$e_7(e_2e_6 - e_3e_5) + e_8(e_3e_4 - e_1e_6) + q_9(e_1e_5 - e_2e_4) = \det(M) \tag{3.49}$$

using the above equations, we get

$$q_1 = \frac{\det(M) - e_2(e_7e_6 - e_9e_4) - e_3(e_4e_8 - e_5e_7)}{(e_5e_9 - e_6e_8)} \tag{3.50}$$

$$q_2 = \frac{\det(M) - e_1(e_5e_9 - e_6e_8) - e_3(e_4e_8 - e_5e_7)}{(e_7e_6 - e_9e_4)} \tag{3.51}$$

$$q_3 = \frac{\det(M) - e_1(e_5e_9 - e_6e_8) - e_2(e_7e_6 - e_9e_4)}{(e_4e_8 - e_5e_7)} \tag{3.52}$$

$$q_4 = \frac{\det(M) - e_5(e_1e_9 - e_3e_7) - e_3(e_2e_7 - e_1e_8)}{(e_3e_8 - e_2e_9)} \tag{3.53}$$

$$q_5 = \frac{\det(M) - e_4(e_3e_8 - e_2e_9) - e_3(e_2e_7 - e_1e_8)}{(e_1e_9 - e_3e_7)} \tag{3.54}$$

$$q_6 = \frac{\det(M) - e_4(e_3e_8 - e_2e_9) + e_5(e_1e_9 - e_3e_7)}{(e_2e_7 - e_1e_8)} \tag{3.55}$$

$$q_7 = \frac{\det(M) - e_8(e_3e_4 - e_1e_6) - e_9(e_1e_5 - e_2e_4)}{(e_2e_6 - e_3e_5)} \tag{3.56}$$

$$q_8 = \frac{\det(M) - e_7(e_2e_6 - e_3e_5) - e_9(e_1e_5 - e_2e_4)}{(e_3e_4 - e_1e_6)} \tag{3.57}$$

$$q_9 = \frac{\det(M) - e_7(e_2e_6 - e_3e_5) - e_8(e_3e_4 - e_1e_6)}{(e_1e_5 - e_2e_4)} \tag{3.58}$$

Since the components of message matrix M are +ve integer. The equations from (3.50) to (3.58) should have integer solutions. If these equations do not have integer solutions, we discover that our scenarios involving a single error are correct or that a mistake could have occurred in the checking element Det (M). we utilize equation from (3.38), (3.39), (3.40) to determine whether the code matrix E is valid.

Now check each instance of a double error in the same manner. For example, In this instance, there are double errors in the code matrix.

$$\begin{pmatrix} q_1 & q_2 & e_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} \tag{3.59}$$

Where q_1, q_2 be the possible "damaged" element. Using the determinant relation $\mathrm{Det}(E) = \mathrm{Det}(M) \times (-1)^k$, For the matrix, we can formulate the following equation:

$$q_1(e_5e_9 - e_6e_8) + q_2(e_7e_6 - e_9e_4) + e_3(e_4e_8 - e_5e_7) = (-1)^k.Det(M) \tag{3.60}$$

Also, we know that the relation between q_1 and q_2 is

$$q_1 \approx r_3 q_2 \tag{3.61}$$

Clearly, Relation (3.60) be Diophantine equation. There are several solutions to the Diophantine equation (3.60). We must select $q_1$and q_2 solutions that satisfy the equation (3.61).

In the same way, for triple error in code matrix

$$\begin{pmatrix} q_1 & q_2 & q_3 \\ e_4 & e_5 & e_6 \\ e_7 & e_8 & e_9 \end{pmatrix} \tag{3.62}$$

Where q_1, q_2 and q_3 be the possible "damaged" element.

It's easy to see that there are 9_{C_2} =36 different types of double error in the code matrix E, and we fix them all using the same method. Hence, $9_{C_1} + 9_{C_2} + 9_{C_3} + 9_{C_4} + 9_{C_5} + 9_{C_6} + 9_{C_7} + 9_{C_8} + 9_{C_9} = 511$cases of errors in E. We use the same method to demonstrate that it is feasible to fix any triple-fold, four-fold,..., eight-fold mistakes in the code matrix E.

As a result, our approach has the potential to rectify eight situations of this method is $\frac{510}{511} = 0.9980 = 99.80\%$ because the code matrix's nine-fold error cannot be fixed.

4. Illustration of Generalized $k-$ Step Jacobsthal Coding & Decoding Method:

Suppose that M is message matrix

$$M=\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}$$, Where $m_{ij} \geq 0$ for $i,j=1,2,3,4$.

Consider $k=4$, and $1 \leq i \leq k$, Construct the coding matrix B_k:

$$B_4=\begin{pmatrix} J_n^1 & J_n^2 & J_n^3 & J_n^4 \\ J_{n-1}^1 & J_{n-1}^2 & J_{n-1}^3 & J_{n-1}^4 \\ J_{n-2}^1 & J_{n-2}^2 & J_{n-2}^3 & J_{n-2}^4 \\ J_{n-3}^1 & J_{n-3}^2 & J_{n-3}^3 & J_{n-4}^4 \end{pmatrix}$$

Put $n=1$,

$$B_4=\begin{pmatrix} J_1^1 & J_1^2 & J_1^3 & J_1^4 \\ J_0^1 & J_0^2 & J_0^3 & J_0^4 \\ J_{-1}^1 & J_{-1}^2 & J_{-1}^3 & J_{-1}^4 \\ J_{-2}^1 & J_{-2}^2 & J_{-2}^3 & J_{-2}^4 \end{pmatrix}=\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

Then, the inverse of B_4 is given by

$$(B_4)^{-1}=\frac{1}{-1}\begin{pmatrix} 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \\ -1 & 1 & 2 & 1 \end{pmatrix}=\begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & -1 \end{pmatrix}$$

Now, we find the code matrix E.

$$E=M\times B_4=\begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}\begin{pmatrix} 1 & 2 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$$

$$=\begin{pmatrix} m_{11}+m_{12} & 2m_{11}+m_{13} & m_{11}+m_{14} & m_{11} \\ m_{21}+m_{22} & 2m_{21}+m_{23} & m_{21}+m_{24} & m_{21} \\ m_{31}+m_{32} & 2m_{31}+m_{33} & m_{31}+m_{34} & m_{31} \\ m_{21}+m_{22} & 2m_{41}+m_{43} & m_{41}+m_{42} & m_{41} \end{pmatrix}=\begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix}$$

where $e_{11}=m_{11}+m_{12}$, $e_{12}=2m_{11}+m_{13}$, $e_{13}=m_{11}+m_{14}$, $e_{14}=m_{11}$, $e_{21}=m_{21}+m_{22}$, $e_{22}=2m_{21}+m_{23}$, $e_{23}=m_{21}+m_{24}$, $e_{24}=m_{21}$, $e_{31}=m_{31}+m_{32}$, $e_{32}=2m_{31}+m_{33}$, $e_{33}=m_{31}+m_{34}$, $e_{34}=m_{31}$, $e_{41}=m_{41}+m_{42}$, $e_{42}=2m_{41}+m_{43}$, $e_{43}=m_{41}+m_{44}$, $e_{44}=m_{41}$.

Then, by decoding transformation

$$M = E \times (B_3)^{-1} = \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -1 & -2 & -1 \end{pmatrix}$$

$$= \begin{pmatrix} e_{14} & e_{11} - e_{14} & e_{12} - 2e_{14} & e_{13} - e_{14} \\ e_{24} & e_{21} - e_{24} & e_{22} - 2e_{24} & e_{23} - e_{24} \\ e_{34} & e_{31} - e_{34} & e_{32} - 2e_{34} & e_{33} - e_{34} \\ e_{44} & e_{41} - e_{44} & e_{42} - 2e_{44} & e_{43} - e_{44} \end{pmatrix} = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}$$

4.2 *Determinant connections between the components of code and Message Matrix M for* $k = 4$,

$$Det(E) = (DetM \times B_k) = Det(M) \times Det(B_k) = Det(M) \times -1 = -Det(M) \tag{4.0}$$

4.3 *The Relationship between the components of code matrix for* $k = 4$, $1 \leq i \leq 4$.

$$E = M \times B_k = \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix} \times \begin{pmatrix} J_n^1 & J_n^2 & J_n^3 & J_n^4 \\ J_{n-1}^1 & J_{n-1}^2 & J_{n-1}^3 & J_{n-1}^4 \\ J_{n-2}^1 & J_{n-2}^2 & J_{n-2}^3 & J_{n-2}^4 \\ J_{n-3}^1 & J_{n-3}^2 & J_{n-3}^3 & J_{n-3}^4 \end{pmatrix}$$

$$= \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix}$$

and

$$M = E \times (B_k)^{-1} =$$

$$\begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} \begin{pmatrix} J_n^1 & J_n^2 & J_n^3 & J_n^4 \\ J_{n-1}^1 & J_{n-1}^2 & J_{n-1}^3 & J_{n-1}^4 \\ J_{n-2}^1 & J_{n-2}^2 & J_{n-2}^3 & J_{n-2}^4 \\ J_{n-3}^1 & J_{n-3}^2 & J_{n-3}^3 & J_{n-3}^4 \end{pmatrix}^{-1}$$

$$= (-1)^k \begin{pmatrix} e_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} \begin{pmatrix} h_{11} & h_{12} & h_{13} & h_{14} \\ h_{21} & h_{22} & h_{23} & h_{24} \\ h_{31} & h_{32} & h_{33} & h_{34} \\ h_{41} & h_{42} & h_{43} & h_{44} \end{pmatrix}$$

$$= \begin{pmatrix} m_{11} & m_{12} & m_{13} & m_{14} \\ m_{21} & m_{22} & m_{23} & m_{24} \\ m_{31} & m_{32} & m_{33} & m_{34} \\ m_{41} & m_{42} & m_{43} & m_{44} \end{pmatrix}$$

where h_{ij} are cofactors of B_k for $i, j = 1,2,3,4$ and $m_{ij} \geq 0$, for $i, j =$ 1,2,3,4 So, we get

$$m_{11} = e_{11}(h_{11}) + e_{12}(h_{21}) + e_{13}(h_{31}) + e_{14}(h_{41}) \geq 0 \tag{4.1}$$

$$m_{12} = e_{11}(h_{12}) + e_{12}(h_{22}) + e_{13}(h_{32}) + e_{14}(h_{42}) \geq 0 \tag{4.2}$$
$$m_{13} = e_{11}(h_{13}) + e_{12}(h_{23}) + e_{13}(h_{33}) + e_{14}(h_{43}) \geq 0 \tag{4.3}$$
$$m_{14} = e_{11}(h_{14}) + e_{12}(h_{24}) + e_{13}(h_{34}) + e_{14}(h_{44}) \geq 0 \tag{4.4}$$
$$m_{21} = e_{21}(h_{11}) + e_{22}(h_{21}) + e_{23}(h_{31}) + e_{24}(h_{41}) \geq 0 \tag{4.5}$$
$$m_{22} = e_{21}(h_{12}) + e_{22}(h_{22}) + e_{23}(h_{32}) + e_{24}(h_{42}) \geq 0 \tag{4.6}$$
$$m_{23} = e_{21}(h_{13}) + e_{22}(h_{23}) + e_{23}(h_{33}) + e_{24}(h_{43}) \geq 0 \tag{4.7}$$
$$m_{24} = e_{21}(h_{14}) + e_{22}(h_{24}) + e_{23}(h_{34}) + e_{24}(h_{44}) \geq 0 \tag{4.8}$$
$$m_{31} = e_{31}(h_{11}) + e_{32}(h_{21}) + e_{33}(h_{31}) + e_{34}(h_{41}) \geq 0 \tag{4.9}$$
$$m_{32} = e_{31}(h_{12}) + e_{32}(h_{22}) + e_{33}(h_{32}) + e_{34}(h_{42}) \geq 0 \tag{4.10}$$
$$m_{33} = e_{31}(h_{13}) + e_{32}(h_{23}) + e_{33}(h_{33}) + e_{34}(h_{43}) \geq 0 \tag{4.11}$$
$$m_{34} = e_{31}(h_{14}) + e_{32}(h_{24}) + e_{33}(h_{34}) + e_{34}(h_{44}) \geq 0 \tag{4.12}$$
$$m_{41} = e_{41}(h_{11}) + e_{42}(h_{21}) + e_{43}(h_{31}) + e_{24}(h_{41}) \geq 0 \tag{4.13}$$
$$m_{42} = e_{41}(h_{12}) + e_{42}(h_{22}) + e_{43}(h_{32}) + e_{24}(h_{42}) \geq 0 \tag{4.14}$$
$$m_{43} = e_{41}(h_{13}) + e_{42}(h_{23}) + e_{43}(h_{33}) + e_{24}(h_{43}) \geq 0 \tag{4.15}$$
$$m_{44} = e_{41}(h_{14}) + e_{42}(h_{33}) + e_{43}(h_{34}) + e_{24}(h_{44}) \geq 0 \tag{4.16}$$

From (4.1), we have
$$e_{11}(h_{11}) + e_{12}(h_{21}) + e_{13}(h_{31}) + e_{14}(h_{41}) \geq 0 \tag{4.17}$$
From (4.2), we have
$$e_{11}(h_{12}) + e_{12}(h_{22}) + e_{13}(h_{32}) + e_{14}(h_{42}) \geq 0 \tag{4.18}$$
From (4.3), we have
$$e_{11}(h_{13}) + e_{12}(h_{23}) + e_{13}(h_{33}) + e_{14}(h_{43}) \geq 0 \tag{4.19}$$
From (4.4), we have
$$e_{11}(h_{14}) + e_{12}(h_{24}) + e_{13}(h_{34}) + e_{14}(h_{44}) \geq 0 \tag{4.20}$$
Dividing both sides by $e_{11}(> 0)$ of (4.17) to (4.20), we have
$$(h_{11}) + \frac{e_{12}}{e_{11}}(h_{21}) + \frac{e_{13}}{e_{11}}(h_{31}) + \frac{e_{14}}{e_{11}}(h_{41}) \geq 0 \tag{4.21}$$
$$(h_{12}) + \frac{e_{12}}{e_{11}}(h_{22}) + \frac{e_{13}}{e_{11}}(h_{32}) + \frac{e_{14}}{e_{11}}(h_{42}) \geq 0 \tag{4.22}$$
$$(h_{13}) + \frac{e_{13}}{e_{11}}(h_{23}) + \frac{e_{13}}{e_{11}}(h_{33}) + \frac{e_{14}}{e_{11}}(h_{43}) \geq 0 \tag{4.23}$$
$$(h_{14}) + \frac{e_{13}}{e_{11}}(h_{24}) + \frac{e_{13}}{e_{11}}(h_{34}) + \frac{e_{14}}{e_{11}}(h_{44}) \geq 0 \tag{4.24}$$
Now $3^8 = 6561$ cases arise for h_{ij} >=< for $i = 3,4$ and $j = 1,2,3,4$.

Case c) : $h_{ij} > 0$ for $i = 3, 4$ and $j = 1, 2, 3, 4$.

From (4.21), we have

$$\frac{e_{13}}{e_{11}}(h_{31}) + \frac{e_{14}}{e_{11}}(h_{41}) \geq -(h_{11} + \frac{e_{12}}{e_{11}} h_{21}) \tag{4.25}$$

From (4.22), we have

$$\frac{e_{13}}{e_{11}}(h_{32}) + \frac{e_{14}}{e_{11}}(h_{42}) \geq -(h_{12} + \frac{e_{12}}{e_{11}} h_{22}) \tag{4.26}$$

From (4.23), we have

$$\frac{e_{13}}{e_{11}}(h_{33}) + \frac{e_{14}}{e_{11}}(h_{43}) \geq -(h_{13} + \frac{e_{13}}{e_{11}} h_{23}) \tag{4.27}$$

From (4.24), we have

$$\frac{e_{13}}{e_{11}}(h_{34}) + \frac{e_{14}}{e_{11}}(h_{44}) \geq -(h_{14} + \frac{e_{13}}{e_{11}} h_{24}) \tag{4.28}$$

From equations (4.25) to (4.28) and using (2.1), we have

$$min\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}, \frac{J_{n-3}^1}{J_{n-3}^2}\right\} \leq \frac{e_{11}}{e_{12}} \leq max\left\{\frac{J_n^1}{J_n^2}, \frac{J_{n-1}^1}{J_{n-1}^2}, \frac{J_{n-2}^1}{J_{n-2}^2}, \frac{J_{n-3}^2}{J_{n-3}^3}\right\} \tag{4.29}$$

Similarly, for rest cases

$$min\left\{\frac{J_n^2}{J_n^3}, \frac{J_{n-1}^2}{J_{n-1}^3}, \frac{J_{n-2}^2}{J_{n-2}^3}, \frac{J_{n-3}^2}{J_{n-3}^3}\right\} \leq \frac{e_{11}}{e_{13}} \leq max\left\{\frac{J_n^2}{J_n^3}, \frac{J_{n-1}^2}{J_{n-1}^3}, \frac{J_{n-2}^2}{J_{n-2}^3}, \frac{J_{n-3}^2}{J_{n-3}^3}\right\} \tag{4.30}$$

and $$min\left\{\frac{J_n^1}{J_n^3}, \frac{J_{n-1}^1}{J_{n-1}^3}, \frac{J_{n-2}^1}{J_{n-2}^3}, \frac{J_{n-3}^1}{J_{n-3}^3}\right\} \leq \frac{e_{11}}{e_{13}} \leq max\left\{\frac{J_n^1}{J_n^3}, \frac{J_{n-1}^1}{J_{n-1}^3}, \frac{J_{n-2}^1}{J_{n-2}^3}, \frac{J_{n-3}^1}{J_{n-3}^3}\right\} \tag{4.31}$$

$$min\left\{\frac{J_n^1}{J_n^4}, \frac{J_{n-1}^1}{J_{n-1}^4}, \frac{J_{n-2}^1}{J_{n-2}^4}, \frac{J_{n-3}^1}{J_{n-3}^4}\right\} \leq \frac{e_{11}}{e_{14}} \leq max\left\{\frac{J_n^1}{J_n^4}, \frac{J_{n-1}^1}{J_{n-1}^4}, \frac{J_{n-2}^1}{J_{n-2}^4}, \frac{J_{n-3}^1}{J_{n-3}^4}\right\} \tag{4.32}$$

$$min\left\{\frac{J_n^2}{J_n^3}, \frac{J_{n-1}^2}{J_{n-1}^3}, \frac{J_{n-2}^2}{J_{n-2}^3}, \frac{J_{n-3}^2}{J_{n-3}^3}\right\} \leq \frac{e_{12}}{e_{13}} \leq max\left\{\frac{J_n^2}{J_n^3}, \frac{J_{n-1}^2}{J_{n-1}^3}, \frac{J_{n-2}^2}{J_{n-2}^3}, \frac{J_{n-3}^2}{J_{n-3}^3}\right\} \tag{4.33}$$

$$min\left\{\frac{J_n^2}{J_n^4}, \frac{J_{n-1}^2}{J_{n-1}^4}, \frac{J_{n-2}^2}{J_{n-2}^4}, \frac{J_{n-3}^2}{J_{n-3}^4}\right\} \leq \frac{e_{12}}{e_{14}} \leq max\left\{\frac{J_n^2}{J_n^4}, \frac{J_{n-1}^2}{J_{n-1}^4}, \frac{J_{n-2}^2}{J_{n-2}^4}, \frac{J_{n-3}^2}{J_{n-3}^4}\right\} \tag{4.34}$$

$$min\left\{\frac{J_n^3}{J_n^4}, \frac{J_{n-1}^3}{J_{n-1}^4}, \frac{J_{n-2}^3}{J_{n-2}^4}, \frac{J_{n-3}^3}{J_{n-3}^4}\right\} \leq \frac{e_{13}}{e_{14}} \leq max\left\{\frac{J_n^3}{J_n^4}, \frac{J_{n-1}^3}{J_{n-1}^4}, \frac{J_{n-2}^3}{J_{n-2}^4}, \frac{J_{n-3}^3}{J_{n-3}^4}\right\} \tag{4.35}$$

Case b): $h_{31} = 0, h_{32} > 0, h_{33} > 0, h_{34} < 0, h_{41} > 0, h_{42} = 0, h_{43} > 0,$ $h_{44} < 0$.

From (4.21), we have

$$\frac{e_{14}}{e_{11}}(h_{41}) \geq -(h_{11} + \frac{e_{12}}{e_{11}} h_{21}) \tag{4.36}$$

From (4.22), we have

$$\frac{e_{13}}{e_{11}}(h_{32}) \geq -(h_{12} + \frac{e_{12}}{e_{11}} h_{22}) \tag{4.37}$$

From (4.23), we have

$$\frac{e_{13}}{e_{11}}(h_{33}) + \frac{e_{14}}{e_{11}}(h_{43}) \geq -(h_{13} + \frac{e_{13}}{e_{11}} h_{23}) \tag{4.38}$$

From (4.24), we have

$$\frac{e_{13}}{e_{11}}(h_{34})+\frac{e_{14}}{e_{11}}(h_{44})\leq -(h_{14}+\frac{e_{13}}{e_{11}}h_{24}) \tag{4.39}$$

From equations (4.36) to (4.39) and using (2.1), we have

$$min\left\{\frac{J_n^1}{J_n^2},\frac{J_{n-1}^1}{J_{n-1}^2},\frac{J_{n-2}^1}{J_{n-2}^2},\frac{J_{n-3}^1}{J_{n-3}^2}\right\}\leq\frac{e_{11}}{e_{12}}\leq max\left\{\frac{J_n^1}{J_n^2},\frac{J_{n-1}^1}{J_{n-1}^2},\frac{J_{n-2}^1}{J_{n-2}^2},\frac{J_{n-3}^2}{J_{n-3}^3}\right\} \tag{4.40}$$

Similarly, for rest cases

$$min\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3},\frac{J_{n-3}^2}{J_{n-3}^3}\right\}\leq\frac{e_{11}}{e_{13}}\leq max\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3},\frac{J_{n-3}^2}{J_{n-3}^3}\right\} \tag{4.41}$$

and

$$min\left\{\frac{J_n^1}{J_n^3},\frac{J_{n-1}^1}{J_{n-1}^3},\frac{J_{n-2}^1}{J_{n-2}^3},\frac{J_{n-3}^1}{J_{n-3}^3}\right\}\leq\frac{e_{11}}{e_{13}}\leq max\left\{\frac{J_n^1}{J_n^3},\frac{J_{n-1}^1}{J_{n-1}^3},\frac{J_{n-2}^1}{J_{n-2}^3},\frac{J_{n-3}^1}{J_{n-3}^3}\right\} \tag{4.42}$$

$$min\left\{\frac{J_n^1}{J_n^4},\frac{J_{n-1}^1}{J_{n-1}^4},\frac{J_{n-2}^1}{J_{n-2}^4},\frac{J_{n-3}^1}{J_{n-3}^4}\right\}\leq\frac{e_{11}}{e_{14}}\leq max\left\{\frac{J_n^1}{J_n^4},\frac{J_{n-1}^1}{J_{n-1}^4},\frac{J_{n-2}^1}{J_{n-2}^4},\frac{J_{n-3}^1}{J_{n-3}^4}\right\} \tag{4.33}$$

$$min\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3},\frac{J_{n-3}^2}{J_{n-3}^3}\right\}\leq\frac{e_{12}}{e_{13}}\leq max\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3},\frac{J_{n-3}^2}{J_{n-3}^3}\right\} \tag{4.34}$$

$$min\left\{\frac{J_n^2}{J_n^4},\frac{J_{n-1}^2}{J_{n-1}^4},\frac{J_{n-2}^2}{J_{n-2}^4},\frac{J_{n-3}^2}{J_{n-3}^4}\right\}\leq\frac{e_{12}}{e_{14}}\leq max\left\{\frac{J_n^2}{J_n^4},\frac{J_{n-1}^2}{J_{n-1}^4},\frac{J_{n-2}^2}{J_{n-2}^4},\frac{J_{n-3}^2}{J_{n-3}^4}\right\} \tag{4.35}$$

$$min\left\{\frac{J_n^3}{J_n^4},\frac{J_{n-1}^3}{J_{n-1}^4},\frac{J_{n-2}^3}{J_{n-2}^4},\frac{J_{n-3}^3}{J_{n-3}^4}\right\}\leq\frac{e_{13}}{e_{14}}\leq max\left\{\frac{J_n^3}{J_n^4},\frac{J_{n-1}^3}{J_{n-1}^4},\frac{J_{n-2}^3}{J_{n-2}^4},\frac{J_{n-3}^3}{J_{n-3}^4}\right\} \tag{4.36}$$

Case c) : $h_{ij} < 0$ for $i = 3,4$ and $j = 1,2,3,4$.

From (4.21), we have

$$\frac{e_{13}}{e_{11}}(h_{31})+\frac{e_{14}}{e_{11}}(h_{41})\geq -(h_{11}+\frac{e_{12}}{e_{11}}\ h_{21}) \tag{4.37}$$

From (4.22), we have

$$\frac{e_{13}}{e_{11}}(h_{32})+\frac{e_{14}}{e_{11}}(h_{42})\geq -(h_{12}+\frac{e_{12}}{e_{11}}h_{22}) \tag{4.38}$$

From (4.23), we have

$$\frac{e_{13}}{e_{11}}(h_{33})+\frac{e_{14}}{e_{11}}(h_{43})\geq -(h_{13}+\frac{e_{13}}{e_{11}}h_{23}) \tag{4.39}$$

From (4.24), we have

$$\frac{e_{13}}{e_{11}}(h_{34})+\frac{e_{14}}{e_{11}}(h_{44})\geq -(h_{14}+\frac{e_{13}}{e_{11}}h_{24}) \tag{4.40}$$

From equations (2.40) to (2.43) and using (2.1), we have

$$min\left\{\frac{J_n^1}{J_n^2},\frac{J_{n-1}^1}{J_{n-1}^2},\frac{J_{n-2}^1}{J_{n-2}^2},\frac{J_{n-3}^1}{J_{n-3}^2}\right\}\leq\frac{e_{11}}{e_{12}}\leq max\left\{\frac{J_n^1}{J_n^2},\frac{J_{n-1}^1}{J_{n-1}^2},\frac{J_{n-2}^1}{J_{n-2}^2},\frac{J_{n-3}^2}{J_{n-3}^3}\right\} \tag{4.41}$$

Similarly, for rest cases

$$min\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3},\frac{J_{n-3}^2}{J_{n-3}^3}\right\}\leq\frac{e_{11}}{e_{13}}\leq max\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3},\frac{J_{n-3}^2}{J_{n-3}^3}\right\} \tag{4.42}$$

and

$$min\left\{\frac{J_n^1}{J_n^3},\frac{J_{n-1}^1}{J_{n-1}^3},\frac{J_{n-2}^1}{J_{n-2}^3},\frac{J_{n-3}^1}{J_{n-3}^3}\right\} \le \frac{e_{11}}{e_{13}} \le max\left\{\frac{J_n^1}{J_n^3},\frac{J_{n-1}^1}{J_{n-1}^3},\frac{J_{n-2}^1}{J_{n-2}^3},\frac{J_{n-3}^1}{J_{n-3}^3}\right\} \tag{4.43}$$

$$min\left\{\frac{J_n^1}{J_n^4},\frac{J_{n-1}^1}{J_{n-1}^4},\frac{J_{n-2}^1}{J_{n-2}^4},\frac{J_{n-3}^1}{J_{n-3}^4}\right\} \le \frac{e_{11}}{e_{14}} \le max\left\{\frac{J_n^1}{J_n^4},\frac{J_{n-1}^1}{J_{n-1}^4},\frac{J_{n-2}^1}{J_{n-2}^4},\frac{J_{n-3}^1}{J_{n-3}^4}\right\} \tag{4.44}$$

$$min\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3},\frac{J_{n-3}^2}{J_{n-3}^3}\right\} \le \frac{e_{12}}{e_{13}} \le max\left\{\frac{J_n^2}{J_n^3},\frac{J_{n-1}^2}{J_{n-1}^3},\frac{J_{n-2}^2}{J_{n-2}^3},\frac{J_{n-3}^2}{J_{n-3}^3}\right\} \tag{4.45}$$

$$min\left\{\frac{J_n^2}{J_n^4},\frac{J_{n-1}^2}{J_{n-1}^4},\frac{J_{n-2}^2}{J_{n-2}^4},\frac{J_{n-3}^2}{J_{n-3}^4}\right\} \le \frac{e_{12}}{e_{14}} \le max\left\{\frac{J_n^2}{J_n^4},\frac{J_{n-1}^2}{J_{n-1}^4},\frac{J_{n-2}^2}{J_{n-2}^4},\frac{J_{n-3}^2}{J_{n-3}^4}\right\} \tag{4.46}$$

$$min\left\{\frac{J_n^3}{J_n^4},\frac{J_{n-1}^3}{J_{n-1}^4},\frac{J_{n-2}^3}{J_{n-2}^4},\frac{J_{n-3}^3}{J_{n-3}^4}\right\} \le \frac{e_{13}}{e_{14}} \le max\left\{\frac{J_n^3}{J_n^4},\frac{J_{n-1}^3}{J_{n-1}^4},\frac{J_{n-2}^3}{J_{n-2}^4},\frac{J_{n-3}^3}{J_{n-3}^4}\right\} \tag{4.47}$$

Therefore, for large value of k,

$$\frac{e_{11}}{e_{12}} \approx r_4\,, \frac{e_{11}}{e_{13}} \approx r_4, \frac{e_{11}}{e_{14}} \approx r_4 \tag{4.48}$$

$$\text{Similarly, } \frac{e_{12}}{e_{13}} \approx r_4, \frac{e_{12}}{e_{13}} \approx r_4, \tag{4.49}$$

$$\frac{e_{13}}{e_{14}} \approx r_4 \text{ , where } r_4 \approx 2.20556 \tag{4.50}$$

4.4 *Generalized Relations*

Similar to this, we may define the generalized relation between the elements of the code matrix for every k as follows:

$$min\left\{\frac{J_{n-r}^1}{J_{n-r}^2}:\ r = 0,1,2,3,\dots,k+1\right\} \le \frac{e_{i1}}{e_{i2}} \le max\left\{\frac{J_{n-r}^1}{J_{n-r}^2}:\ r = 0,1,2,3,\dots,k+1\right\} \tag{4.51}$$

$$min\left\{\frac{J_{n-r}^1}{J_{n-r}^3}:\ r = 0,1,2,3,\dots,k+1\right\} \le \frac{e_{i1}}{e_{i3}} \le max\left\{\frac{J_{n-r}^1}{J_{n-r}^3}:\ r = 0,1,2,3,\dots,k+1\right\} \tag{4.52}$$

$$min\left\{\frac{J_{n-r}^1}{J_{n-r}^4}:\ r = 0,1,2,3,\dots,k+1\right\} \le \frac{e_{i1}}{e_{i4}} \le max\left\{\frac{J_{n-r}^1}{J_{n-r}^4}:\ r = 0,1,2,3,\dots,k+1\right\} \tag{4.53}$$

$$\vdots\,\vdots$$

$$min\left\{\frac{J_{n-r}^1}{J_{n-r}^k}:\ r = 0,1,2,3,\dots,k+1\right\} \le \frac{e_{i1}}{e_{ik}} \le max\left\{\frac{J_{n-r}^1}{J_{n-r}^k}:\ r = 0,1,2,3,\dots,k+1\right\} \tag{4.54}$$

$$min\left\{\frac{J_{n-r}^2}{J_{n-r}^3}:\ r = 0,1,2,3,\dots,k+1\right\} \le \frac{e_{i2}}{e_{i3}} \le max\left\{\frac{J_{n-r}^2}{J_{n-r}^3}:\ r = 0,1,2,3,\dots,k+1\right\} \tag{4.55}$$

$$min\left\{\frac{J_{n-r}^2}{J_{n-r}^4}:\ r = 0,1,2,3,\dots,k+1\right\} \le \frac{e_{i2}}{e_{i4}} \le max\left\{\frac{J_{n-r}^2}{J_{n-r}^4}:\ r = 0,1,2,3,\dots,k+1\right\} \tag{4.56}$$

$$\vdots\,\vdots$$

$$min\left\{\frac{J_{n-r}^2}{J_{n-r}^k}:\ r = 0,1,2,3,\dots,k+1\right\} \le \frac{e_{i2}}{e_{ik}} \le max\left\{\frac{J_{n-r}^2}{J_{n-r}^k}:\ r = 0,1,2,3,\dots,k+1\right\} \tag{4.57}$$

$$min\left\{\frac{J_{n-r}^{3}}{J_{n-r}^{4}}:\ r=0,1,2,3,\dots,k+1\right\}\leq\frac{e_{i3}}{e_{i4}}\leq max\left\{\frac{J_{n-r}^{3}}{J_{n-r}^{4}}:\ r=0,1,2,3,\dots,k+1\right\} \tag{4.58}$$

$$min\left\{\frac{J_{n-r}^{3}}{J_{n-r}^{5}}:\ r=0,1,2,3,\dots,k+1\right\}\leq\frac{e_{i3}}{e_{i5}}\leq max\left\{\frac{J_{n-r}^{3}}{J_{n-r}^{5}}:\ r=0,1,2,3,\dots,k+1\right\} \tag{4.59}$$

$$\vdots\ \vdots$$

$$min\left\{\frac{J_{n-r}^{3}}{J_{n-r}^{k}}:\ r=0,1,2,3,\dots,k+1\right\}\leq\frac{e_{i3}}{e_{ik}}\leq max\left\{\frac{J_{n-r}^{3}}{J_{n-r}^{k}}:\ r=0,1,2,3,\dots,k+1\right\} \tag{4.60}$$

and so on

$$min\left\{\frac{J_{n-r}^{3}}{J_{n-r}^{k}}:\ r=0,1,2,3,\dots,k+1\right\}\leq\frac{e_{i(k-1)}}{e_{ik}}\leq max\left\{\frac{J_{n-r}^{3}}{J_{n-r}^{k}}:\ r=0,1,2,3,\dots,k+1\right\} \tag{4.61}$$

for $i=1,2,3,\dots,k$. For large k, we arrive at

$$\frac{e_{i1}}{e_{i2}}\approx r_1,\frac{e_{i1}}{e_{i3}}\approx r_1,\dots,\frac{e_{i1}}{e_{ik}}\approx r_1 \tag{4.62}$$

Similarly, $$\frac{e_{i2}}{e_{i3}}\approx r_2,\frac{e_{i2}}{e_{i4}}\approx r_2,\dots,\frac{e_{i2}}{e_{ik}}\approx r_2 \tag{4.63}$$

$$\frac{e_{i3}}{e_{i4}}\approx r_3,\frac{e_{i3}}{e_{i5}}\approx r_3,\dots,\frac{e_{i3}}{e_{ik}}\approx r_3 \tag{4.64}$$

$$\vdots$$

$$\frac{e_{i(k-1)}}{e_{ik}}\approx r_k \tag{4.65}$$

4.5 *Error Detection / Correction Process*

In present study, we begin by developing for the simplex instance 93.33% accuracy for $k=2$, 99.80% accuracy for $k=3$, and above. Errors in the code massage E may be found and fixed using the encoding/decoding procedure. In order to spot and fix errors, scientists use a characteristic of matrices expressed as (4.0). After calculating the determinant of the massage matrix M, we send it to a communication channel. The code matrix E's checking elements are Det(M). After obtaining the code matrix and confirming elements Det(M), we construct the matrix e's determinant and compare it to Det(M) via equation (4.0). If the relation (4.0) is true, we infer that the elements of the code matrix E were broadcast across the communication channel without errors. otherwise there are errors and then we try to correct these errors using the relations (4.0) , (4.48), (4.49), (4.50).

In this section, we have single error in the code matrix E. It is clear that there are sixteen variants of single error in the code matrix E. As an example, one of them is

$$\begin{pmatrix} q_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix}, \text{ Where } q_{11} \text{ is error elements} \tag{4.66}$$

Using equation (4.0), then

$$\begin{vmatrix} q_{11} & e_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{vmatrix} = (-1)^k DetM. \quad (4.67)$$

There are sixteen equations, such as (4.66) for sixteen possible variants of a single error, if q_{11} are the potential destroyed elements for the elements e_{ij} for $i, j = 1, 2, 3, 4$. However, we must only select the correct variant from these cases of the solution q_{ij} for $i, j = 1, 2, 3, 4$ which satisfies the relations (4.48), (4.49) and (4.50). In sixteen equations don't resolve to integers, Consequently, we know that either our assumption that there is only one error in the checking element Det(M) is incorrect or there are several errors. We verify the integrity of the encoding matrix E with the help of the approximation equalities (4.48), (4.49), and (4.50).In the code matrix E, check each instance of a double error in the same manner. For example, In this instance, there are double errors in the code matrix.

$$\begin{pmatrix} q_{11} & q_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} \quad (4.68)$$

Where q_{11}, q_{12} be the possible "damaged" element. Using the determinant relation $\mathrm{Det}(E) = \mathrm{Det}(M) \times (-1)$, For the matrix, we can formulate the following equation:

$$\begin{vmatrix} q_{11} & q_{12} & e_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{vmatrix} = (-1) DetM \quad (4.69)$$

Also, we know that the relation between q_{11} *and* q_{12} is

$$q_{11} \approx r_4 q_{12} \quad (4.70)$$

Clearly, Relation (4.69) be Diophantine equation. There are several solutions to the Diophantine equation (4.69). We must select $q_1$and q_2 solutions that satisfy the equation (4.70).

In the same way, we fix triple error to make sure that

$$\begin{pmatrix} q_{11} & q_{12} & q_{13} & e_{14} \\ e_{21} & e_{22} & e_{23} & e_{24} \\ e_{31} & e_{32} & e_{33} & e_{34} \\ e_{41} & e_{42} & e_{43} & e_{44} \end{pmatrix} \quad (4.71)$$

Where q_{11}, q_{12} and q_{13} be the possible "damaged" element.

We use a similar strategy to repair all double faults in the code matrix E, which has $16_{C_2} = 120$ variations. Hence $16_{C_1} + 16_{C_2} + 16_{C_3} + 16_{C_4} + 16_{C_5} + 16_{C_6} + \cdots + 16_{C_{16}} = 65535$ cases of errors.

We use the same method to demonstrate that it is feasible to fix any triple-fold, four-fold, … , fifteen –fold errors in E.

As a result, our approach has the potential to rectify eight situations of this method is $\frac{65534}{65535} = 0.999980 = 99.998\%$

The likelihood of fixing method faults for k=m is typically is $\frac{2^{m^2}-2}{2^{m^2}-1}$

Therefore, for large value of k the possibility to correct errors of the methods is $\frac{2^{m^2}-2}{2^{m^2}-1} \approx 1 = 100\%$

5. Blocking Method for Generalized $k-$ step Jacobsthal Numbers.

We develop encoding and decoding techniques here that make use of generalized k-step Jacobsthal numbers. Our message's size is calculated by placing it in a matrix of even dimensions and inserting zeroes between each pair of words until we reach the end of the message. We may design a coding scheme by splitting the message square matrix M of size 2m into block matrices, say T_i $(1 \leq i \leq m^2)$ of size 2×2, from left to right.

We now describe the notation that was utilized during the coding process. Assume that matrices T_i & E_i are

$$T_i = \begin{pmatrix} t_1^i & t_2^i \\ t_3^i & t_4^i \end{pmatrix} \text{ and } E_i = \begin{pmatrix} e_1^i & e_2^i \\ e_3^i & e_4^i \end{pmatrix}$$

We use the matrix B_k and rewrite the components of this matrix to

$$B_k = \begin{pmatrix} J_n^1 & J_n^2 \\ J_{n-1}^1 & J_{n-1}^2 \end{pmatrix}$$

The no. of the block matrices T_i by b. Using b, we select a number n as follows:

$$n = \begin{cases} 3, b \leq 3 \\ \left|\frac{b}{2}\right|, b > 3 \end{cases}$$

We write the following character table mod (22) after selecting n. Each character in the table is represented by a number between n and $n + 21$. We begin n for the first character.

A	n	**G**	$n+6$	**M**	$n+12$	**S**	$n+18$
B	$n+1$	**H**	$n+7$	**N**	$n+13$	**T**	$n+19$
C	$n+2$	**I**	$n+8$	**O**	$n+14$	**U**	$n+20$
D	$n+3$	**J**	$n+9$	**P**	$n+15$	**0**	$n+21$
E	$n+4$	**K**	$n+10$	**Q**	$n+16$		
F	$n+5$	**L**	$n+11$	**R**	$n+17$		

Now, we provide the following k-step Jacobsthal blocking technique using the encoding and decoding Algorithm.

	Coding Algorithm Process
I.	Massage matrix M divided into 2×2 blocks T_i $(1 \leq i \leq m^2)$.
II.	Select the value of n
III.	Resolve t_j^i $(1 \leq j \leq 4)$.
IV.	Find determinant $(T_i) \rightarrow d_i$.
V.	Construct $= [d_i, t_r^i]_{r \in \{1,3,4\}}$.
VI.	Algorithm completed.

	Decoding Algorithm Process
I.	Take the Matrix $(B_k)^n$.
II.	Resolve J_i $(j \in 1,2,3,4)$.
III.	Solve the relation $J_1 t_3^i + J_3 t_4^i \rightarrow e_3^i$ $(1 \leq i \leq m^2)$.
IV.	Solve the relation $J_2 t_3^i + J_4 t_4^i \rightarrow e_4^i$ $(1 \leq i \leq m^2)$.
V.	Solve $(-2)^n \times d_i = e_4^i(J_1 t_1^i + J_3 x_i) - e_3^i(J_2 t_1^i + J_4 x_i)$.
VI.	Putting for $x_i = t_2^i$.
VII.	Make T_i.
VIII.	Make M.
IX.	Algorithm completed.

Illustration 4.1: Examine the message matrix for the subsequent message text "ARUN"

We write the message matrix using the above message text $M = \begin{pmatrix} A & R \\ U & N \end{pmatrix}$

Solution:

Coding Algorithm

I. Massage matrix M divided into 2×2 blocks $T_1 = \begin{pmatrix} A & R \\ U & N \end{pmatrix}$

II. Here, $b = 1 < 3$ then $n = 3$. For the message matrix M, we apply the following character table:

A	R	U	N
3	20	23	16

III. The block $T_1'^s$ components are as follows:

$$t_1^1 = 3, t_2^1 = 20, t_3^1 = 23, t_4^1 = 16$$

IV. Find determinant d_1 of the block T_1:

$$d_1 = \det\ (T_1) = -412$$

V. Obtain the following matrix P utilizing (3) & (4)

$$P = [-412\ 3\ 23\ 16]$$

VI. Algorithm completed

Decoding Algorithm

I. Take the Matrix $(B_2)^3 = \begin{pmatrix} J_n^1 & J_n^2 \\ J_{n-1}^1 & J_{n-1}^2 \end{pmatrix}^3 = \begin{pmatrix} 5 & 6 \\ 3 & 2 \end{pmatrix}$

II. Here $J_1 = 5, J_2 = 6, J_3 = 3,\ J_4 = 2$

III. To create the matrix E_i using the elements e_3^i: $e_3^1 = 163$

IV. To create the matrix E_i using the elements e_4^i: $e_4^1 = 170$

V. Find the element x_i :

$$\det(B_k).d_1 = e_4^1(J_1\, t_1^1 + J_3\, x_1) - e_3^1(J_2\, t_1^1 + J_4\, x_1)$$

$$(-2)^3.(-412) = 170(15 + 3x_1) - 163(18 + 2x_1)$$

$$\Rightarrow\ x_1 = 20$$

VI. Putting for $x_1 = t_2^1$

$$x_1 = t_2^1 = 20$$

VII. Create block matrix T_1:

$$T_1 = \begin{pmatrix} 3 & 20 \\ 23 & 16 \end{pmatrix}$$

VIII. Create message matrix M:

$$\mathrm{M} = \begin{pmatrix} 3 & 20 \\ 23 & 16 \end{pmatrix} = \begin{pmatrix} A & R \\ U & N \end{pmatrix}$$

IX. Algorithm completed.

Illustration 4.2: Examine the message matrix for the subsequent message text:

"MAM IS BEAUTIFUL"

We write the message matrix using the above message text

$$\mathrm{M} = \begin{pmatrix} M & A & M & 0 \\ I & S & 0 & B \\ E & A & U & T \\ I & F & U & L \end{pmatrix}$$

Solution:

I. Massage matrix M divided into 2×2 blocks:

$$B_1 = \begin{pmatrix} M & A \\ I & S \end{pmatrix}, B_2 = \begin{pmatrix} M & 0 \\ 0 & B \end{pmatrix}, B_3 = \begin{pmatrix} E & A \\ I & F \end{pmatrix}, B_4 = \begin{pmatrix} U & T \\ U & L \end{pmatrix}$$

II. Here, $b = 4 > 3$ then $n = \left|\frac{b}{2}\right| = \left|\frac{4}{2}\right| = 2$. For the message matrix M, we apply the following character table:

M	**A**	**M**	**0**	**I**	**S**	**0**	**B**
14	2	14	23	10	20	23	3
E	**A**	**U**	**T**	**I**	**F**	**U**	**L**
6	2	22	21	10	7	22	13

III. The component of the block T_i:

$t_1^1 = 14$	$t_2^1 = 2$	$t_3^1 = 10$	$t_4^1 = 20$
$t_1^2 = 14$	$t_2^2 = 23$	$t_3^2 = 23$	$t_4^2 = 3$
$t_1^3 = 6$	$t_2^3 = 2$	$t_3^3 = 10$	$t_4^3 = 7$
$t_1^4 = 22$	$t_2^4 = 21$	$t_3^4 = 22$	$t_4^4 = 13$

IV. Find determinant d_1 of the block T_1:
$d_1 = \det(T_1) = 260,\quad d_2 = \det(T_2) = -487,\quad d_3 = \det(T_3) = 22,$
$d_4 = \det(T_4) = -176$

V. Obtain the following matrix P utilizing (3) & (4):

$$P = \begin{pmatrix} 260 & 14 & 10 & 20 \\ -487 & 14 & 23 & 3 \\ 22 & 6 & 10 & 7 \\ -176 & 22 & 22 & 13 \end{pmatrix}$$

VI. Algorithm completed.

Decoding Algorithm

I. Take the Matrix $(B_2)^2 = \begin{pmatrix} J_n^1 & J_n^2 \\ J_{n-1}^1 & J_{n-1}^2 \end{pmatrix}^2 = \begin{pmatrix} 3 & 2 \\ 1 & 2 \end{pmatrix}$

II. Here $J_1 = 3, J_2 = 2, J_3 = 1, J_4 = 2$

III. To create the matrix E_i using the elements e_3^i:

$$e_3^1 = 50, e_3^2 = 72, e_3^3 = 37 e_4^4 = 70,$$

IV. To create the matrix E_i using the elements e_4^i:

$$e_4^1 = 60, e_4^2 = 52, e_4^3 = 34, e_4^4 = 70,$$

V. Find the element x_i :

$$\det(B_k)\,.\,d_1 = e_4^1(J_1\, t_1^1 + J_3\, x_1) - e_3^1(J_2\, t_1^1 + J_4\, x_1)$$

$$(-2)^2.(260) = 60(28 + x_1) - 50(28 + 2x_1)$$

$$\Rightarrow x_1 = 2$$

$$\det(B_k)\,.\,d_2 = e_4^2(J_1\, t_1^2 + J_3\, x_2) - e_3^1(J_2\, t_1^2 + J_4\, x_2)$$

$$(-2)^2.(-487) = 52(42 + x_2) - 72(28 + 2x_2)$$

$$\Rightarrow x_2 = 23$$

$$\det(B_k)\,.\,d_3 = e_4^3(J_1\, t_1^3\; + J_3\, x_3) - e_3^3(J_2\, t_1^3\; + J_4\, x_3)$$
$$(-2)^2.(22) = 34(18 + x_3) - 37(12 + 2x_3)$$
$$\Rightarrow\; x_3 = 2$$
$$\det(B_k)\,.\,d_4 = e_4^4(J_1\, t_1^4\; + J_3\, x_4) - e_3^4(J_2\, t_1^4\; + J_4\, x_4)$$
$$(-2)^2.(-176) = 70(66 + x_4) - 79(44 + 2x_4)$$
$$\Rightarrow\; x_4 = 21$$

VI. We rename x_i as follows:

$$x_1 = t_2^1 = 2, x_2 = t_2^2 = 23, x_3 = t_2^3 = 2, x_4 = t_2^4 = 21$$

VII. Create block matrix T_1:

$$T_1 = \begin{pmatrix} 14 & 2 \\ 10 & 20 \end{pmatrix}, T_2 = \begin{pmatrix} 14 & 23 \\ 23 & 3 \end{pmatrix}, T_3 = \begin{pmatrix} 6 & 2 \\ 10 & 7 \end{pmatrix}, T_4 = \begin{pmatrix} 22 & 21 \\ 22 & 13 \end{pmatrix}$$

VIII. Create message matrix M:

$$\mathrm{M} = \begin{pmatrix} 14 & 2 & 14 & 23 \\ 10 & 20 & 23 & 3 \\ 6 & 2 & 22 & 21 \\ 10 & 7 & 22 & 13 \end{pmatrix} = \begin{pmatrix} M & A & M & 0 \\ I & S & 0 & B \\ E & A & U & T \\ I & F & U & L \end{pmatrix}.$$

IX. Algorithm completed.

6. Conclusion

The generalized k-step Jacobsthal numbers coding and decoding method are discussed in the article. It may be simplified to the familiar algebraic action of multiplying matrices. The method's primary practical is that it targets big information units, specifically matrix elements, for error identification and repair. The method's accuracy for the k = 2 value is 93.33 percent, the method's accuracy is 99.80 percent for $k = 3$ and 99.998 percent for $n = 4$. In general, as n rises, the method's potential to be right increases.

References

[1] Al-Kateeb A., Generalization of Jacobsathal and Jacobsathal Lucas Number (2020). https://www.researchgate.net/publication/336086490.

[2] Basu, M., Prasad B., Coding theory on the m-extension of the Fibonacci p-numbers, *Chaos, Solutions, and Fractals* 42, 2522-2530, (2009).

[3] Basu, M., Das, M., Coding theory on Fibonacci n-step numbers, *Discrete Mathematics, Algorithms and Application*, Vol. 6, No. 2, 1450017 (2014).

[4] Basu, M., Das, M., Coding theory on generalized Fibonacci n-step polynomials, *Journal of Information and Optimization Sciences*, Vol. 38, No.1, 83-13 (2017).

[5] Esmaeili M., Esmaeili M., A Fibonacci-polynomial based coding method with error detection and correction, *Computers & Mathematics with Applications,* Vol. 60, Issue 10, 2738-2752, (2010).

[6] F. Qi, "Determinantal expressions and recurrence relations for Fubini and Eulerian polynomials," *J. Interdiscip. Math.,* vol. 22, no. 3, pp. 317–335 (2019), doi: 10.1080/09720502.2019.1624063.

[7] Sundarayya P., Vara Prasad. M.G., Coding theory on Pell-Lucas *p* numbers, *IOP Conf. Series: Journal of Physics: conf. series* 1344 (2019).

[8] Stakhov A., Massingue V., Sluchenkova A., "Introduction into Fibonacci Coding and Cryptography, (1999).

[9] Tas N., Ucar S., Ozgur N.: Pell Coding and Pell Decoding Methods with Some Applications, Contributions of Discrete Mathematics, Vol. 15, No. 1, 52–66 (2020).

[10] Tas N., Ucar S., Ozgur Y.,Kaymak O., A new coding/decoding algorithm using Fibonacci numbers, *Discrete Mathematics Algorithms and Applications,* (2017).

[11] Ucar S., Tas N., Ozgur N. Y., A new cryptography model via Fibonacci and Lucas numbers, *Mathematical Sciences and Applications* E-Notes, (2019). Doi: 10.36753/mathenot.559251.

[12] Yilmaz F., Bozkurt D., The generalized order –k Jacobsthal numbers, *Int. J. Comtemp. Math and Sciences,* vol. 4, no. 34, 1685-1694 (2009).

Received August, 2022

Revised December, 2022