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Fuzzy visible submodules in the class of fuzzy faithful multiplication modules

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Abstract

In this paper, the behavior of fuzzy visible submodules and its effect of it in the class of fuzzy faithful multiplication modules have been studied. Some relationship between fuzzy visible modules and fuzzy residual quotient ideals has been presented. Also, other properties with important results have been studied.

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Keywords: *Fuzzy visible modules, Fuzzy residual ideals, Fuzzy annihilator ideals, Fuzzy multiplication submodules, Fuzzy pure ideals.*

1. Introduction

¥ is a ring with an identity that is commutative in this paper, and $\mathcal{M}$ is a unitary module over ¥ and $\forall I \subseteq$ ¥, where I is a fuzzy ideal, $I(0)=1$. In 1975, the fuzzy module has been presented by C.V. Negoita, and D.A. Ralescu, [1]. Many other researchers studied this concept as in [2, 3, 4]. In 2021, N.S.Buthyna and K.M.Sajda [5] have introduced the notions of the fuzzy visible submodule, where a proper fuzzy submodule θ is visible if $\theta = I\,\theta$ for each nonempty fuzzy ideal I of ¥. This paper proposes the study of the effect and behavior of fuzzy

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visible submodules within the class of fuzzy multiplication modules. Many characteristics and problems have been proved.

2. Preliminaries

This section contains the basic concepts that will be needed in the next results.

Definition 2.1: Let $¥ \neq \emptyset$ be a set and $= [0,1]$. A function $\mathcal{B}: ¥ \longrightarrow [0,1]$ is called a fuzzy set in ¥, [6].
Let FF(¥) represents the collection of all fuzzy subsets of ¥.

Definition 2.2: Let $\mathcal{L} \neq \emptyset$ be a set, and $\mathbb{N} \in \mathrm{FF}(\mathcal{L})$, for all,$h \in [0,1]$, the set $\mathbb{N}_h = \{a \in \mathcal{L} : \mathbb{N}(x) \geq h\}$ is said to be a level set of $\mathcal{L}$ for $\mathbb{N}$, [7].

Definition 2.3: Assume $\mathcal{M}$ is an ¥ -module and $\mathbb{F} \in \mathrm{FF}(\mathcal{M})$. $\mathbb{F}$ is known as a fuzzy module of an ¥ -module $\mathcal{M}$ if,
1. $\mathbb{F}(i - e) \geq \min\{\mathbb{F}(i), \mathbb{F}(e)\}, \forall\, i, e \in \mathcal{M}$,
2. $\mathbb{F}(ri) \geq \mathbb{F}(i), \forall i \in M, r \in ¥$,
3. $\mathbb{F}(0) = 1$, [2].

Let $\mathrm{FM}(\mathcal{M}) = \{\mathbb{F} \in \mathrm{FF}(\mathcal{M}): \mathbb{F}\ is \text{ a fuzzy } ¥ - \text{ module}\}$,[5].

Definition 2.4: Let $\mathcal{E}$, $\mathbb{F} \in FM(\mathcal{M})$ and $\mathcal{E} \subseteq \mathbb{F}$. Then $\mathcal{E}$ is referred to as a fuzzy submodule of , [4].
Let $S(\mathbb{F}) = \{\mathcal{E} \in FM(\mathcal{M}): \mathcal{E} \subseteq \mathbb{F}\}$, *[5].*

Definition 2.5: Let $\mathcal{T} \in \mathrm{FF}(¥)$. $\mathcal{T}$ is known as a fuzzy ideal of ¥, if $\forall\, \omega_1, \omega_2 \in \mathbb{R}$:
1. $\mathcal{T}(\omega_1 - \omega_2) \geq min\{\mathcal{T}(\omega_1), \mathcal{T}(\omega_2)\}$
2. $\mathcal{T}(\omega_1 . \omega_2) \geq max\{\mathcal{T}(\omega_1), \mathcal{T}(\omega_2)\}$, [10].

Let $FI(¥) = \{\mathcal{T} \in FF(¥): \mathcal{T}$ is a fuzzy ideal of ¥}, [5].

Definition 2.6: Let $\mathbb{F} \in FM(\mathcal{M})$. $\mathbb{F}$ is called a multiplication fuzzy module if and only if , $\forall \mathcal{E} \in FS(\mathbb{F})$, there exists $\mathcal{T} \in FI(\mathbb{R})$ such that $\mathcal{E} = \mathcal{T}\mathbb{F}$, [9].

Definition 2.7: Let $\mathbb{F} \in FM(\mathcal{M})$ and $\mathcal{E}, \mathfrak{D} \in FS(\mathbb{F})$. Then $(\mathcal{E}: \mathfrak{D}) \in \mathrm{FF}(¥)$ is a residual quotient ideal of $\mathcal{E}$ and $\mathfrak{D}$ defined by:
$(\mathcal{E}: \mathfrak{D})(s) = sup\{h \in [0,1]: s_h \mathfrak{D} \subseteq \mathcal{E}\}, for\ all\ s \in ¥\}$, [2].

Proposition 2.8: If $\mathcal{E}, \mathfrak{D} \in FS(\mathbb{P})$, then $(\mathcal{E}: \mathfrak{D}) \in FI(¥)$, [2].

Definition 2.9: Let $\mathfrak{D} \in FS(\mathbb{P}) \neq \emptyset$. $\mathfrak{D}$'s fuzzy annihilator, denoted by $F - ann\mathfrak{D}$ where it is defined as:
$F - ann\mathfrak{D}(s) = sup\{t : t \in [0,1], s_t \mathfrak{D} \subseteq 0_1\}$, for all $s \in ¥$, [10].

Note that $F - ann\mathfrak{D}$=$(0_1 : \mathfrak{D})$, [10]. Hence $((F - ann\ \mathfrak{D}))_h \subseteq F\text{-}ann\ \mathfrak{D}_h$, [11].

Definition 2.10: Let $\mathbb{P} \in FM(\mathcal{M})$ *and* $\mathbb{P} \neq \mathfrak{Q} \in FS(\mathbb{P})$. Then $\mathfrak{Q}$ is denoted by fuzzy visible submodule such that $\mathfrak{Q} = \mathcal{T}\mathfrak{Q}$ for every non-empty $\mathcal{T} \in FI(¥)$. If $\mathcal{T} \in FI(¥)$ is said visible if it is visible of $M(¥)$, [5].
Let $V(\mathbb{P}) = \{\mathbb{P} \neq \mathfrak{Q} \in FS(\mathbb{P}) : \mathfrak{Q}$ is fuzzy visible of $\mathbb{P}\}$.

3. Fuzzy Visible Submodules of the Fuzzy Faithful Multiplication Modules

In this part our interest is to study the behavior and effects of fuzzy visible submodules in the set of fuzzy faithful multiplication modules. Several more properties and results that interpret this relationship have been established.

Proposition 3.1: Assume that $\mathbb{P} \in FM(\mathcal{M})$ be a multiplication cancellation ¥ – module. Then the below are equal:
1. $\theta \in FV(\mathbb{P})$.
2. $(\theta : \mathbb{P}) \in FV(¥)$

Proof: (1) $\Longrightarrow$ (2) Let $\theta \in FS(\mathbb{P})$, then θ_t is visible by[5, prop. 3.2] and θ_t is multiplication by [8, prop. 3.3.12], therefore $(\theta_t : \mathbb{P}_t)$ is a visible ideal of ¥ by [12, prop. 1.2.4]. But $(\theta : \mathbb{P})_t \subseteq (\theta_t : \mathbb{P}_t)$, so by [12, prop. 1.1.7], we have $(\theta : \mathbb{P})_t$ is visible. Therefore $(\theta : \mathbb{P}) \in FV(\mathbb{P})$.

(2) $\Longrightarrow$ (1) Let $(\theta : \mathbb{P}) \in FI(¥)$. Let $a_t \subseteq \theta$. Then $< a_t > \subseteq \theta$ and hence $(< a_t > : \mathbb{P}) \subseteq (\theta : \mathbb{P}) = I(\theta : \mathbb{P})$ for every $I \in FI(¥)$. Therefore $(< a_t > : \mathbb{P})\mathbb{P} \subseteq I(\theta : \mathbb{P})\ \mathbb{P} = I\theta$ (since $\mathbb{P}$ is a multiplication), then $< a_t > \subseteq I\theta$ and hence $a_t \subseteq I\theta$. Therefore $\theta \subseteq I\theta$ but $I\theta \subseteq \theta$. Thus $\theta = I\theta$, therefore $\theta \in FV(\mathbb{P})$.

Corollary 3.2: *If* $\mathbb{P} \neq \mathfrak{W} \in FS(\mathbb{P})$*. If* $\mathbb{P}$ *a finitely generated faithful multiplication (f.g.f.m)* $fuzzy$ ¥ *– module, then* $\in FV(\mathbb{P}\ )$ *iff a fuzzy ideal* $(\mathfrak{W} : \mathbb{P}) \in FV(¥\ )$.

Proof: By [11, prop. 3.3 and 3.13], we have $\mathbb{P}$ is a cancellation fuzzy module. Now, by prop. 3.1 ,the result is true.

Corollary 3.3: *Let* $\mathbb{P} \in FM(\mathcal{M})$. *If* $\mathbb{P}$ *is a fuzzy faithful cyclic over integral domain* ¥ *. Then* $\mathbb{P} \neq \mathfrak{W} \in FV(\mathbb{P}\ )$ *is iff a fuzzy ideal* $(\mathfrak{W} : \mathbb{P}) \in FV(¥\ )$.

Proof: Since $\mathbb{P}$ cyclic fuzzy ¥ – module ,then $\mathbb{P}$ is multiplication by [8] and hence $\mathbb{P}_t$ is multiplication by[8,prop. 3.3.12] but $\mathbb{P}_t$ is faithful module by [11, prop. 2.12], where ¥ is integral domain then $\mathbb{P}_t$ is a cancellation by [12, proof of corollary1.2.6). Therefore $\mathbb{P}$ is cancellation by [11, prop. 2.3], and hence by prop. 3.1 the result is achieved.

Proposition 3.4: Let $\mathcal{F} \in FM(\mathcal{M})$ .If $\mathcal{F}$ is a fuzzy cancellation multiplication over ¥, and $\mathfrak{T} \in FS(\mathcal{F})$,where $\mathfrak{T}$ is visible, then for each non – empty $\mathfrak{K}$ ∈*FI(*ℝ*)*, result from this $\mathfrak{K}(\mathfrak{T}:\mathcal{F}) = (\mathfrak{K}\mathfrak{T}:\mathcal{F})$.

Proof: Let $\in FS(\mathcal{F})$. Since $\mathcal{F}$ is a fuzzy multiplication, so $\mathfrak{T} = \mathfrak{K}\mathcal{F}$ for some $\mathfrak{K} \in FI(¥)$. Therefore $(\mathfrak{T}:\mathcal{F}) = (\mathfrak{K}\mathcal{F}:\mathcal{F}) = \mathfrak{K}$ [11, prop. 2.6(4)] because of $\mathcal{F}$ is cancellation, so $(\mathfrak{T}:\mathcal{F}) \cdot (\mathfrak{T}:\mathcal{F}) = \mathfrak{K}(\mathfrak{T}:\mathcal{F})$, from visiblity of $\mathfrak{T}$ we get the fuzzy ideal $(\mathfrak{T}:\mathcal{F}) \in FV(¥)$ by prop. 3.1 so from [5, prop. 3.17] $(\mathfrak{T}:\mathcal{F})$ is pure, then $(\mathfrak{T}:\mathcal{F})(\mathfrak{T}:\mathcal{F}) = (\mathfrak{T}:\mathcal{F}) \cap (\mathfrak{T}:\mathcal{F})$, then $(\mathfrak{T}:\mathcal{F}) \cap (\mathfrak{T}:\mathcal{F}) = \mathfrak{K}(\mathfrak{T}:\mathcal{F})$ and hence $(\mathfrak{T}:\mathcal{F}) = \mathfrak{K}(\mathfrak{T}:\mathcal{F})$. Thus $(\mathfrak{K}\mathfrak{T}:\mathcal{F}) = \mathfrak{K}(\mathfrak{T}:\mathcal{F})$ by visibility of $\mathfrak{T}$.

Proposition 3.5: Let $\mathbb{P} \in FM(\mathcal{M})$. If $\mathbb{P}$ is a *(f.g.f.m)* fuzzy ¥ – module and $I \in FI(¥)$ is a proper , then the following hold:
1. $I \in. FV(¥)$ if and only if $I\,\mathbb{P} \in FV(\mathbb{P})$.
2. Suppose that $\mathfrak{H} \in FV(\mathbb{P})$, then $F - ann(\mathfrak{H}) = F - ann(\mathfrak{H}:\mathbb{P})$.

Proof:
1. From visibility of I we have I_t is a visible [5, prop. 3.2] and $\mathbb{P}_t$ is a finitely generated by [11, prop. 2.12] and faithful multiplication module as in corollary 3.3, therefore by [12, prop. 1.2.9]. We now have $I_t\ \mathbb{P}_t$ is a visible of $\mathbb{P}_t$. But $(I\mathbb{P})_t = I_t\ \mathbb{P}_t$ by [3, prop. 4.3] and hence $(I\mathbb{P})_t$ is visible, and hence $I\mathbb{P} \in FV(\mathbb{P})$ by [5, prop. 3.2] .Conversely, if $I\mathbb{P} \in FV(\mathbb{P})$, then $KI\mathbb{P} = I\mathbb{P}$ for a non-empty K ∈*FI(¥)*. But $\mathbb{P}$ is fuzzy *(f.g.f.m)* module and hence $\mathbb{P}$ is cancellation, therefore *KI=I* by[11]. Then $\in FV(¥)$ is visible.
2. Let $x_t \subseteq F - ann(\mathfrak{H}:\mathbb{P})$. Then $x_t(\mathfrak{H}:\mathbb{P}) \subseteq 0_1$, then $x_t\mathfrak{H} = x_t(\mathfrak{H}:\mathbb{P})\mathfrak{H}$ (since $\mathfrak{H} \in FV(\mathbb{P})$), then $x_t\mathfrak{H} \subseteq 0_1$.Therefore $x_t \in F - ann\mathfrak{H}$, so $F - ann(\mathfrak{H}:\mathbb{P}) \subseteq F - ann\mathfrak{H}$. Now, from visibility of $\mathfrak{H}$, then $\mathfrak{H} = I\mathfrak{H}\ \forall$ non − empty $I \in FI(¥)$ and by [5, prop. 3.17], we have $\mathfrak{H}$ is a pure. Because of this truth, we write $I\mathfrak{H} = \mathfrak{H} \cap I\mathbb{P}\ by$ [8, Definition 2.1.1], ∀I ∈*FI(¥)*. Put $I = F - ann\mathfrak{H}$ and hence $(F - ann\mathfrak{H})\mathfrak{H} = \mathfrak{H} \cap (F - ann\mathfrak{H})\mathbb{P}$, but $(F - ann\mathfrak{H})\mathfrak{H} = 0_1$, then $\mathfrak{H} \cap (F - ann\mathfrak{H})\mathbb{P} \subseteq 0_1$and hence $(\mathfrak{H} \cap (F - ann\mathfrak{H})\mathbb{P} : \mathbb{P}) \subseteq (0_1 : \mathbb{P})$, then $(\mathfrak{H}:\mathbb{P}) \cap \big((F - ann\mathfrak{H})\mathbb{P} : \mathbb{P}\big) \subseteq (0_1:\mathbb{P})$. So $(\mathfrak{H} : \mathbb{P}) \cap (I\mathbb{P} : \mathbb{P}) \subseteq (0_1 : \mathbb{P})$. Therefore $(\mathfrak{H}:\mathbb{P}) \cap I \subseteq (0_1:\mathbb{P})$ (since $\mathbb{P}$ is a fuzzy cancellation) and hence $(\mathfrak{H} : \mathbb{P})I \subseteq (0_1 : \mathbb{P})$ (since $(\mathfrak{H}:\mathbb{P}) \in FV(¥)$ and hence pure). Then $(\mathfrak{H} : \mathbb{P})F - ann\mathfrak{H} \subseteq (0_1 : \mathbb{P}) = F - ann\mathbb{P} = 0_1$ because of $\mathbb{P}$ is faithful, then $F - ann\mathfrak{H} \subseteq F - ann(\mathfrak{H} : \mathbb{P})$, therefore $F - ann\mathfrak{H} = F - ann(\mathfrak{H} : \mathbb{P})$.

Definition 3.6: Let $\mathfrak{H} \in FS(\mathbb{P})$. $\mathfrak{H}$ is said to be fuzzy multiplication if and only if $\mathcal{K} \cap \mathfrak{H} = (\mathcal{K}:\mathfrak{H})\mathfrak{H}\ \forall\mathcal{K} \in FS(\mathbb{P})$.

Proposition 3.7: Let $\mathfrak{H} \in FS(\mathcal{F})$.If $\mathfrak{H}$ is a visible and $\mathcal{F}$ is a *(f.g.f.m)* fuzzy ¥ – module, then $\mathfrak{H}$ is a fuzzy multiplication.

Proof: Let $\in FS(\mathcal{F})$. Since $\mathcal{F}$ is a fuzzy *(f.g.f.m)*, then $\mathcal{F}$ is cancelation by [11, Prop. 3.3 and prop. 3.13] also $(\mathfrak{Y}:\mathcal{F})\mathcal{F} = \mathfrak{Y}$. By visibility of $\mathfrak{H}$ we now have $\mathfrak{H} = \mathcal{E}\mathfrak{H}\ \forall \mathcal{E} \in FI(¥)$. Hence $\mathfrak{Y} \cap \mathfrak{H} \subseteq \mathfrak{H} = \mathcal{E}\mathfrak{H}$. Put $\mathcal{E} = (\mathfrak{Y}:\mathfrak{H})$, then $\mathfrak{Y} \cap \mathfrak{H} \subseteq (\mathfrak{Y}:\mathfrak{H})\mathfrak{H}$.

Now, it is clear that $(\mathfrak{Y}:\mathfrak{H})\mathfrak{H} \subseteq \mathcal{F}$, then $(\mathfrak{Y}:\mathcal{F})(\mathfrak{Y}:\mathfrak{H})\mathfrak{H} \subseteq (\mathfrak{Y}:\mathcal{F})\mathcal{F} = \mathfrak{Y}$, which implies that $(\mathfrak{Y}:\mathcal{F})(\mathfrak{Y}:\mathfrak{H})\mathfrak{H} \cap \mathfrak{H} \subseteq \mathfrak{Y} \cap \mathfrak{H}$ and hence $(\mathfrak{Y}:\mathcal{F})\mathfrak{H} = \mathfrak{H}$, then $(\mathfrak{Y}:\mathfrak{H})\mathfrak{H} \subseteq \mathfrak{Y} \cap \mathfrak{H}$. Therefore $\cap\, \mathfrak{H} = (\mathfrak{Y}:\mathfrak{H})\mathfrak{H}$.

Proposition 3.8: Let $\mathbb{P}$ be a *(f.g.f.m)* fuzzy ¥ – module and $\mathfrak{H} \in FS(\mathbb{P})$ be visible. Then $\bigcap_{\mathscr{b}\in\mathfrak{k}} j_{\mathscr{b}}\mathfrak{H} = (\bigcap_{\mathscr{b}\in\mathfrak{k}} j_{\mathscr{b}})\mathfrak{H}$, $\forall$ nonempty collection $j_{\mathscr{b}}(\mathscr{b} \in \mathfrak{k})$ of fuzzy visible ideals of ¥.

Proof: By visibility of $\mathfrak{H}$ we now have $\mathfrak{H}_t$ which is a visible and from visibility of $j_{\mathscr{b}}$ we already have $(j_{\mathscr{b}})_t \in FV(¥)$. And since $\mathbb{P}$ is a *(f.g.f.m)* fuzzy module then $\mathbb{P}_t$ is a *(f.g.f.m)* module. Now, by [12, prop. 1.2.12], we have $\bigcap_{\mathscr{b}\in\mathfrak{k}}(j_{\mathscr{b}})_t\mathfrak{H}_t = (\bigcap_{\mathscr{b}\in\mathfrak{k}}(j_{\mathscr{b}})_t)\mathfrak{H}_t$, then $(\bigcap_{\mathscr{b}\in\mathfrak{k}} j_{\mathscr{b}}\mathfrak{H})_t = \left((\bigcap_{\mathscr{b}\in\tau} j_{\mathscr{b}})\mathfrak{H}\right)_t$, and hence $\bigcap_{\mathscr{b}\in\mathfrak{k}} j_{\mathscr{b}}\mathfrak{H} = (\bigcap_{\mathscr{b}\in\mathfrak{k}} j_{\mathscr{b}})\mathfrak{H}$.

Proposition 3.9: Let $\mathbb{P} \neq \mathfrak{W} \in FS(\mathbb{P})$ .If $\mathbb{P}$ is a fully cancellation fuzzy module over ¥ , where ¥ regular. Then $\mathfrak{W} \in FV(\mathbb{P})$.

Proof: Let $\mathbb{P} \neq \mathfrak{W} \in FS(\mathbb{P})$. Since $\mathbb{P}$ is a fuzzy module on regular ring ¥ then $\mathbb{P}$ is a fuzzy regular module [8, p59]. Then a fuzzy $\mathfrak{W}$ is pure by [8, prop. 1.2.2]. Let G $\in FI(¥)$; this leads to $G^2 = G$ since G is a fuzzy pure , then $G^2\mathfrak{W} = G\mathfrak{W}$ but $\mathbb{P}$ is a fuzzy fully cancellation and hence $G\mathfrak{W} = \mathfrak{W}$ by [13, Definition 2.1].

Proposition 3.10: Suppose that $\mathcal{F} \in FM(\mathcal{M})$. If $\mathcal{F}$ is an *(f.g.f.m)* fuzzy ¥ $-$module and $\mathcal{F} \neq \mathfrak{W} \in FS(\mathcal{F})$ so the following will be equivalent, where (5) → (1) holds if ¥ is P.I.D. and $\mathcal{F}_t$ is divisible $\forall\ t \in (0,1]$,

1. $\mathfrak{L} \in FV(\mathcal{F})$,
2. $\mathfrak{L}$ is a fuzzy idempotent and multiplication in $\mathcal{F}$,
3. $(\mathfrak{K} : \mathfrak{L})\mathfrak{L} = (\mathfrak{K} : \mathcal{F})\mathfrak{L}\ \forall\ \mathfrak{K} \in FS(\mathcal{F})$ and $\mathfrak{L}$ is a fuzzy multiplication.
4. $\mu d_t = (\mathfrak{L} : \mathcal{F})dt$,where $\mu \in FI(¥)$ defined as $\mu(x) = \begin{cases} 1 & \text{if } x \in ¥ \\ 0 & o.w. \end{cases}$.
5. If μ as in (4), then $\mu = (\mathfrak{L} : \mathcal{F}) + F - annd_t$, provided that $\mathfrak{L}$ is a cancellation.

Proof: **(1) →(2)** from (1) and by [5, proposition 3.27], we have $\mathfrak{L}$ is a fuzzy idempotent. And by proposition 3.7 , we have $\mathfrak{L}$ is a fuzzy multiplication.

(2) → (3) From $\mathcal{F}$ is fuzzy multiplication, then $= (\mathfrak{L} : \mathcal{F})\mathcal{F}$, $\mathfrak{K} = (\mathfrak{K} : \mathcal{F})\mathcal{F}$, also from (2), $\mathfrak{L}$ is multiplication and hence we have $\mathfrak{K} = (\mathfrak{K}:\mathfrak{L})\mathfrak{L}$, then $\mathfrak{K} =(\mathfrak{K} : \mathfrak{L})(\mathfrak{L} : \mathcal{F})\mathcal{F} = (\mathfrak{K} : \mathcal{F})\mathcal{F}$. But $\mathcal{F}$ is cancellation so $(\mathfrak{K} : \mathfrak{L})(\mathfrak{L} : \mathcal{F}) = (\mathfrak{K} : \mathcal{F})$. Now, $(\mathfrak{K} : \mathfrak{L})(\mathfrak{L} : \mathcal{F})\mathfrak{L} = (\mathfrak{K} : \mathcal{F})\mathfrak{L}$, since $\mathfrak{L}$ is idempotent we have $(\mathfrak{K} : \mathfrak{L})\mathfrak{L} = (\mathfrak{K} : \mathcal{F})\mathfrak{L}$.

(3) → (4) From (3), we have $(\mathfrak{K} : \mathfrak{L})\,\mathfrak{L} = (\mathfrak{K} : \mathcal{F})\mathfrak{L}$, therefore $(\mathfrak{L} : \mathfrak{L})\,\mathfrak{L} = (\mathfrak{L} : \mathcal{F})\mathfrak{L}$, then $\mu\mathfrak{L} = (\mathfrak{L} : \mathcal{F})\mathfrak{L}$ and hence $\mathfrak{L} = (\mathfrak{L} : \mathcal{F})\mathfrak{L}$. Therefore $< d_t >= (\mathfrak{L} : \mathcal{F})\, d_t$.

(4) → (5) From (4), $(\mathfrak{L}: \mathcal{F})d_t =< d_t > \ \forall\, d_t \subseteq \mathfrak{L}$ then $\forall d_t \subseteq \mathfrak{L}, \exists y_\ell \subseteq (\mathfrak{L} : \mathcal{F})$ s. t. $d_t = y_\ell d_t \subseteq < y_\ell > \mathfrak{L} \Rightarrow \mathfrak{L} \subseteq < y_\ell > \mathfrak{L}$, but $< y_\ell > \mathfrak{L} \subseteq \mathfrak{L}$ so $< y_\ell > \mathfrak{L} = \mathfrak{L}$, since $\mathfrak{L}$ is a cancellation we have $< y_\ell >= \mu$. Now, $\forall y_\ell \subseteq (\mathfrak{L} : \mathcal{F})$ we have $(\mathfrak{L}: \mathcal{F}) = \mu$, from this we obtain

$\mu + F - annd_t = (\mathfrak{L}: \mathcal{F}) + F - annd_t$. Since $\mu \subseteq \mu + F - annd_t$.

Also, we know that $(\mathfrak{L}: \mathcal{F}) + F - annd_t \subseteq \mu$. Therefore $(\mathfrak{L}: \mathcal{F}) + F - annd_t = \mu$.

(5)→ (1) By (5) $\mu = (\mathfrak{L} : \mathcal{F}) + F - annd_t\ \forall d_t \in \mathfrak{L}$, then $\mu\mathfrak{L} = (\mathfrak{L} : \mathcal{F})\mathfrak{L} + (F - annd_t\mathfrak{L}\,)\mathfrak{L}$, therefore $\mathfrak{L} = (\mathfrak{L}: \mathcal{F})\,\mathfrak{L}$. Since $\mathfrak{L}$ is multiplication then from (4), we get $(\mathfrak{K} : \mathfrak{L})\,\mathfrak{L} = (\mathfrak{K} : \mathcal{F})\,\mathfrak{L}$. Also because that $\mathfrak{L}$ is multiplication then $\forall$ $I \in$ $FI(¥), \mathcal{F}I \cap \mathfrak{L} = \mathfrak{L}(\mathcal{F}I : \mathfrak{L}) = \mathfrak{L}(\mathcal{F}I : \mathcal{F}) = I\mathfrak{L}$. Therefore $\mathfrak{L}$ is a fuzzy pure and hence $\mathfrak{L}_t$ is a pure but ¥ is P.I.D. and $\mathcal{F}_t$ is divisible then by [12, proposition 1.1.20], we get $\mathfrak{L}_t$ is visible and hence $\mathfrak{L}$ is visible.

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