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Normalization, commutativity and centralization of *TFSM*(*G*)

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Abstract

This paper aims to introduce the normalization, commutativity and centralization of *TFSM*(*G*) and investigate some of their basic properties. Also we investigate image and pre image of them under group homomorphisms.

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1. Introduction

A multiset in mathematics is a generalization of the concept of a set. Its a collection of unordered numbers (or other elements), where every element x occurs a finite number of times. The difference between sets and multisets is in how they address multiples: a set includes any number at most once, while a multiset allows for multiple instances of the same number. Multisets are important in both math and computer science. They help us keep track of elements in databases, and are the backbone of modern combinatorics. A fuzzy concept is a concept of which the boundaries of application can vary considerably according to context or conditions, instead of being fixed once and for all.This means the concept is vague in some way, lacking a fixed, precise meaning, without however being unclear or meaningless altogether. The notion of a fuzzy

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subset of a set is due to Lotfi Zadeh ([32]). The notion of fuzzy subgroup was introduced by Rosenfeld [28]. In 1979 Anthony and Sherwood [1] redefined a fuzzy subgroup of a group using the concept of triangular norm (t-norm, for short). Recently, by Shinoj et al. [29], the concept of fuzzy multigroups was introduced as an application of fuzzy multisets to group theory, and some properties of fuzzy multigroups were presented. In fact, fuzzy multigroup is a generalization of fuzzy groups. The author by using norms, investigated some properties of fuzzy algebraic structures [3-27]. The author in [26], introduced fuzzy multigroups under t-norms written $TFSM(G)$ and investigated some properties and results about them. In this paper, we define normal of $A \in TFSM(G)$ written $NTFSM(G)$ and we prove that if $A, B \in NTFSM(G)$, then $A \cap B \in NTFSM(G)$. Also we prove that image and pre image of them is also normal and we define normality between two $A, B \in TFSM(G)$ and investigate them under group homomorphisms. Next, we define commutativity of $A \in TFSM(G)$ and obtain some results about them and commutator of $x, y \in G$. Later, we introduce centeralization of $A \in TFSM(G)$ written $Z(A)$ and characterize them and prove that $Z(A)$ is normal subgroup of G and finally, we investigate $Z(A)$ under group homomorphisms.

2. Preliminaries

In this section, we present some basic definitions and results that are used throughout the paper.

Definition 2.1 : ([30]) Let $X = \{x_1, x_2, ..., x_n, ...\}$ be a set. A multiset A over X is a cardinal-valued function, that is, $C_A : X \to \mathbb{N}$ such that $x \in Dom(A)$ implies $A(x)$ is a cardinal and $A(x) = C_A(x) > 0$, where $C_A(x)$, denotes the number of times an object x occur in A. Whenever $C_A(x) = 0$, implies $x \notin Dom(A)$. The set X is called the ground or generic set of the class of all multisets (for short, msets) containing objects from X.

A multiset $A = [a, a, b, b, c, c, c]$ can be represented as $A = [a, b, c]_{2,2,3}$ or $A = [a^2, b^2, c^3]$ or $\{\frac{a}{2}, \frac{b}{2}, \frac{c}{3}\}$ Difierent forms of representing multiset exist other than this. See [7, 17, 28] for details. We denote the set of all multisets by $MS(X)$.

Definition 2.2 : ([31]) Let A and B be two multisets over X, then A is called a submultiset of B written as $A \subseteq B$ if $C_A(x) \leq C_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper submultiset of B and denoted as $A \subset B$. Note that a multiset is called the parent in relation to its submultiset. Also two multisets A and B over X are comparable to each other if $A \subseteq B$ or $B \subseteq A$.

Definition 2.3 : ([27]) A group is a non-empty set G on which there is a binary operation $(a,b) \to ab$ such that

(1) if a and b belong to G then ab is also in G (closure),

(2) $a(bc) = (ab)c$ for all $a,b,c \in G$ (associativity),

(3) there is an element $e \in G$ such that $ae = ea = a$ for all $a \in G$ (identity),

(4) if $a \in G$, then there is an element $a^{-1} \in G$ such that $aa^{-1} = a^{-1}a = e$ (inverse).

One can easily check that this implies the unicity of the identity and of the inverse. A group G is called abelian if the binary operation is commutative, i.e., $ab = ba$ for all $a,b \in G$.

Remark 2.4 : ([27]) There are two standard notations for the binary group operation: either the additive notation, that is $(a,b) \to a+b$ in which case the identity is denoted by 0, or the multiplicative notation, that is $(a,b) \to ab$ for which the identity is denoted by e.

Proposition 2.5 : ([27]) Let G be a group. Let H be a non-empty subset of G. The following are equivalent:

(1) H is a subgroup of G.

(2) $x, y \in H$ implies $xy^{-1} \in H$ for all x, y.

Definition 2.6 : ([27]) Let G be a group and H be a subgroup of G. We say that H is normal subgroup of G written $H \trianglerighteq G$ if $ghg^{-1} \in H$ for all $h \in H$ and $g \in G$.

Definition 2.7 : ([27]) Let G be an arbitrary group with a multiplicative binary operation and identity e. A fuzzy subset of G, we mean a function from G into [0, 1]. The set of all fuzzy subsets of G is called the [0, 1] -power set of G and is denoted $[0,1]^G$.

Definition 2.8 : ([2]) Let X be a set. A fuzzy multiset A of X is characterized by a count membership function

$$CM_A : X \to [0,1]$$

of which the value is a multiset of the unit interval $I = [0,1]$. That is,

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n, ...\} \forall x \in X,$$

where $\mu^1, \mu^2, ..., \mu^n, ... \in [0,1]$ such that

$$(\mu^1 \geq \mu^2 \geq ... \geq \mu^n \geq ...).$$

Whenever the fuzzy multiset is finite, we write

$$CM_A(x) = \{\mu^1, \mu^2, ..., \mu^n\},$$

where $\mu^1, \mu^2, ..., \mu^n \in [0,1]$ such that

$$(\mu^1 \geq \mu^2 \geq ... \geq \mu^n),$$

or simply

$$CM_A(x) = \{\mu^i\},$$

for $\mu^i \in [0,1]$ and $i = 1, 2, ..., n$.

Now, a fuzzy multiset A is given as

$$A = \left\{ \frac{CM_A(x)}{x} : x \in X \right\} \text{ or } A = \{(CM_A(x), x) : x \in X\}.$$

The set of all fuzzy multisets is depicted by $FMS(X)$.

Example 2.9 : ([2]) Assume that $X = \{a, b, c\}$ is a set. Then for $CM_A(a) = \{1, 0.5, 0.4\}$ and $CM_A(b) = \{0.9, 0.6\}$ and $CM_A(c) = \{0\}$ we get that A is a fuzzy multiset of X written as

$$A = \left\{ \frac{1, 0.5, 0.4}{a}, \frac{0.9, 0.6}{b} \right\}.$$

Definition 2.10 : ([2]) Let $A, B \in FMS(X)$. Then A is called a fuzzy submultiset of B written as $A \subseteq B$ if $CM_A(x) \leq CM_B(x)$ for all $x \in X$. Also, if $A \subseteq B$ and $A \neq B$, then A is called a proper fuzzy submultiset of B and denoted as $A \subset B$.

Definition 2.11 : ([27]) A t-norm T is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(T1) $T(x, 1) = x$ (neutral element),

(T2) $T(x, y) \leq T(x, z)$ if $y \leq z$ (monotonicity),

(T3) $T(x, y) = T(y, x)$ (commutativity),

(T4) $T(x, T(y, z)) = T(T(x, y), z)$ (associativity), for all $x, y, z \in [0,1]$.

We say that T be idempotent if $T(x,x) = x$ for all $x \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

Example 2.12 : ([27])

(1) Standard intersection t-norm $T_m(x,y) = \min\{x,y\}$.

(2) Bounded sum t-norm $T_b(x,y) = \max\{0, x+y-1\}$.

(3) algebraic product t-norm $T_p(x,y) = xy$.

(4) Drastic T-norm

$$T_D(x,y) = \begin{cases} y & \text{if } x = 1 \\ x & \text{if } y = 1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum t-norm

$$T_{nM}(x,y) = \begin{cases} \min\{x,y\} & \text{if } x+y > 1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product t-norm

$$T_{H_0}(x,y) = \begin{cases} 0 & \text{if } x = y = 0 \\ \dfrac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic t-norm is the pointwise smallest t-norm and the minimum is the pointwise largest t-norm: $T_D(x,y) \leq T(x,y) \leq T_m(x,y)$ for all $x, y \in [0,1]$.

Lemma 2.13 : *([27]) Let T be a t-norm. Then*

$$T(T(x,y), T(w,z)) = T(T(x,w), T(y,z)),$$

for all $x, y, w, z \in [0,1]$.

Definition 2.14 : ([26]) Let $A \in FMS(G)$. Then A is said to be a fuzzy multigroup of G under t-norm T if it satisfies the following two conditions:

(1) $CM_A(xy) \geq T(CM_A(x), CM_A(y))$,

(2) $CM_A(x^{-1}) \geq CM_A(x)$,

for all $x, y \in G$.

The set of all fuzzy multisets of G under t-norm T is depicted by $TFMS(G)$.

Theorem 2.15 : *([26]) Let* $A \in TFMS(G)$. *If T be idempotent, then for all* $x \in G$, *and* $n \geq 1$,

(1) $CM_A(e) \geq CM_A(x)$;

(2) $CM_A(x^n) \geq CM_A(x)$;

(3) $CM_A(x) = CM_A(x^{-1})$.

Proposition 2.16 : ([26]) Let G and H be groups and $f : G \to H$ be an epimorphism. If $A \in TFMS(G)$, then $f(A) \in TFMS(H)$.

Proposition 2.17 : ([26]) Let G and H be groups and $f : G \to H$ be a homomorphism. If $B \in TFMS(H)$, then $f^{-1}(B) \in TFMS(G)$.

Definition 2.18 : ([26]) Let $A, B \in TFMS(G)$. Then the intersection of A and B denoted as $A \cap B$ is governed by

$$CM_{A \cap B}(x) = T(CM_A(x), CM_B(x))$$

for all $x \in G$.

Proposition 2.19 : ([26]) Let $A, B \in TFMS(G)$. Then $A \cap B \in TFMS(G)$.

Definition 2.20 : ([26]) Let $A, B \in TFMS(G)$. Then the product of A and B denoted as $A \circ B$ is governed by

$$CM_{A \circ B}(x) = \begin{cases} \sup\limits_{x=yz} T(CM_A(y), CM_B(z)) & \text{if } x = yz \\ 0 & \text{otherwise.} \end{cases}$$

3. Normalization in *TFMS*(*G*)

Definition 3.1 : We say that $A \in TFMS(G)$ is a normal of G if for all $x, y \in G$,

$$CM_A(xyx^{-1}) = CM_A(y).$$

Also we denote by $NTFMS(G)$ set of all normal fuzzy multisets of G under t-norm T.

Proposition 3.2 : Let G be a group and H be commutative group and $f : G \to H$ be an epimorphism. If $A \in NTFMS(G)$, then $f(A) \in NTFMS(H)$.

Proof : As Proposition 2.16 we get that $f(A) \in TFMS(H)$. Let $x, y \in H$ and since *f* is a surjection so $f(u) = x$ for some $u \in G$. Then

$$f(CM_A)(xyx^{-1})$$

$$= \sup\{CM_A(w) \mid w \in G, f(w) = xyx^{-1}\}$$

$$= \sup\{CM_A(u^{-1}wu) \mid w \in G, f(u^{-1}wu) = y\}$$

$$= \sup\{CM_A(u^{-1}wu) \mid w \in G, f(u^{-1})f(w)f(u) = y\}$$

$$= \sup\{CM_A(u^{-1}wu) \mid w \in G, f^{-1}(u)f(w)f(u) = y\}$$

$$= \sup\{CM_A(u^{-1}wu) \mid w \in G, f(w)f^{-1}(u)f(u) = y\}$$

$$= \sup\{CM_A(w) \mid w \in G, f(w) = y\}$$

$$= f(CM_A)(y).$$

Therefore $f(A) \in NTFMS(H)$. □

Proposition 3.3 : Let G be a group and H be commutative group and $f : G \to H$ be a homomorphism. If $B \in NTFMS(H)$, then $f^{-1}(B) \in NTFMS(G)$.

Proof : From Proposition 2.17 we have that $f^{-1}(B) \in TFMS(G)$. Now for any $x, y \in G$, we have

$$f^{-1}(CM_B)(xyx^{-1}) = CM_B(f(xyx^{-1}))$$

$$= CM_B(f(x)f(y)f(x^{-1})) = CM_B(f(x)f(y)f^{-1}(x))$$

$$= CM_B(f(y)f(x)f^{-1}(x)) = CM_B(f(y))$$

$$= f^{-1}(CM_B)(y).$$

Hence $f^{-1}(B) \in NTFMS(G)$. □

Proposition 3.4 : Let $A, B \in NTFMS(G)$, Then $A \cap B \in NTFMS(G)$.

Proof : Let $x, y, \in G$. Then

$$CM_{A\cap B}(xyx^{-1}) = T(CM_A(xyx^{-1}), CM_B(xyx^{-1}))$$

$$= T(CM_A(y), CM_B(y)) = CM_{A\cap B}(y).$$

Thus $A \cap B \in NTFMS(G)$. □

Remark 3.5 : Let $\{A_i\}_{i\in I} \in NTFMS(G)$. Then $\cap_{i\in I} A_i \in NTFMS(G)$.

Definition 3.6 : Let $A, B \in TFMS(G)$ and $A \subseteq B$. Then A is called a normal of B, written $A \rhd\lhd B$, if for all $x, y \in G$,

$$CM_A(xyx^{-1}) \geq T(CM_A(y), CM_B(x)).$$

Proposition 3.7 :

(1) If G_1 and G_2 are subgroups of G, then $G_1 \trianglerighteq G_2$ if and only if $1_{G_1} \rhd\lhd 1_{G_2}$.

(2) If T be idempotent t-norm, then every fuzzy multisets is a normal fuzzy multisets of itself under t-norm T.

(3) If $A \in NTFMS(G)$, then $A \rhd\lhd 1_G$.

Proof :

(1) Let $x \in G_2$ and $y \in G_1$ and then $1_{G_1}(y) = 1$ and $1_{G_2}(x) = 1$.
If $G_1 \trianglerighteq G_2$, then $xyx^{-1} \in G_1$ and we have

$$1_{G_1}(xyx^{-1}) = 1 = T(1,1) = T(1_{G_1}(y), 1_{G_2}(x))$$

and thus $1_{G_1} \rhd\lhd 1_{G_2}$.
Conversely, let $1_{G_1} \rhd\lhd 1_{G_2}$. Then

$$1_{G_1}(xyx^{-1}) \geq T(1_{G_1}(y), 1_{G_2}(x)) = T(1,1) = 1$$

and then $1_{G_1}(xyx^{-1}) = 1$ so $xyx^{-1} \in G_1$ and then $G_1 \trianglerighteq G_2$.

(2) Let $A \in TFMS(G)$ and $x, y \in G$. It follows that

$$CM_A(xyx^{-1}) \geq T(CM_A(xy), CM_A(x^{-1}))$$

$$\geq T(T(CM_A(x), CM_A(y)), CM_A(x))$$

$$= T(T(CM_A(y), CM_A(x)), CM_A(x))$$

$$= T(CM_A(y), T(CM_A(x), CM_A(x)))$$

$$= T(CM_A(y), CM_A(x)).$$

Thus $A \rhd\lhd A$.

(3) Suppose $A \in NTFMS(G)$ and $x, y \in G$. Then

$$CM_A(xyx^{-1}) \geq CM_A(y) = T(CM_A(y), 1) = T(CM_A(y), 1_G(x)).$$

Hence $A \rhd\lhd 1_G$. □

Proposition 3.8 : Let T be idempotent t-norm. If $A \in NTFMS(G)$ and $B \in TFMS(G)$, then $A \cap B \rhd\lhd B$.

Proof : From Proposition 2.19 we have $(A \cap B) \subseteq B$ and $(A \cap B) \in TFMS(G)$. Now for all $x, y \in G$ we have

$$CM_{A\cap B}(xyx^{-1})$$

$$= T(CM_A(xyx^{-1}), CM_B(xyx^{-1}))$$

$$= T(CM_A(y), CM_B(xyx^{-1}))$$

$$\geq T(CM_A(y), T(CM_B(xy), CM_B(x^{-1})))$$

$$\geq T(CM_A(y), T(CM_B(xy), CM_B(x)))$$

$$\geq T(CM_A(y), T(T(CM_B(x), CM_B(y)), CM_B(x)))$$

$$= T(CM_A(y), T(T(CM_B(x), CM_B(x)), CM_B(y)))$$

$$= T(CM_A(y), T(CM_B(x), CM_B(y)))$$

$$= T(CM_A(y), T(CM_B(y), CM_B(x)))$$

$$= T(T(CM_A(y), CM_B(y)), CM_B(x))$$

$$= T(CM_{A\cap B}(y), CM_B(x)).$$

Hence $A \cap B \rhd\lhd B$. □

Proposition 3.9 : Let T be idempotent t-norm and $A,B,C \in TFMS(G)$. If $A,B \rhd\lhd C$, then $A \cap B \rhd\lhd C$.

Proof : As Proposition 2.19 we have $A \cap B \in TFMS(G)$ and $A \cap B \subseteq C$. If $x,y \in G$, then

$$CM_{A\cap B}(xyx^{-1})$$

$$= T(CM_A(xyx^{-1}), CM_B(xyx^{-1}))$$

$$\geq T(T(CM_A(y), CM_C(x)), T(CM_B(y), CM_C(x)))$$

$$= T(T(CM_A(y), CM_B(y)), T(CM_C(x), CM_C(x)))(\text{Lemma 2.13})$$

$$= T(CM_{A\cap B}(y), CM_C(x))$$

Therefore $A \cap B \rhd\lhd C$. $\square$

Remark 3.10 : Let $\{A_i\}_{i\in I} \in TFMS(G)$ such that $\{A_i\}_{i\in I} \rhd\lhd C$. Then $\cap_{i\in I} A_i \rhd\lhd C$.

Proposition 3.11 : Let G and H be groups and $f: G \to H$ be an epimorphism. If $A,B \in TFMS(G)$ and $A \rhd\lhd B$, then $f(A) \rhd\lhd f(B)$.

Proof: By Proposition 2.16 we have $f(\mu), f(\nu) \in TFMS(G)$ and $f(A) \subseteq f(B)$. Let $x,y \in H$ and $u,v \in G$. Then

$$f(CM_A)(xyx^{-1}) = \sup\{CM_A(z) \mid z \in G, f(z) = xyx^{-1}\}$$

$$= \sup\{CM_A(uvu^{-1}) \mid u,v \in G, f(u) = x, f(v) = y\}$$

$$\geq \sup\{T(CM_A(v), CM_B(u)) \mid f(u) = x, f(v) = y\}$$

$$= T(\sup\{CM_A(v) \mid y = f(v)\}, \sup\{CM_B(u) \mid x = f(u)\})$$

$$= T(f(CM_A)(y), f(CM_B)(x)).$$

Hence $f(A) \rhd\lhd f(B)$. $\square$

Proposition 3.12 : Let G and H be groups and $f: G \to H$ be a homomorphism. If $A,B \in TFMS(H)$ and $A \rhd\lhd B$, then $f^{-1}(A) \rhd\lhd f^{-1}(B)$.

Proof : From Proposition 2.17 we have $f^{-1}(A), f^{-1}(B) \in TFMS(H)$ and also $f^{-1}(A) \subseteq f^{-1}(B)$. Let $x, y \in G$. Now

$$f^{-1}(CM_A)(xyx^{-1}) = CM_A(f(xyx^{-1}))$$

$$= CM_A(f(x)f(y)f^{-1}(x)) \geq T(CM_A(f(y)), CM_B(f(x)))$$

$$= T(f^{-1}(CM_A)(y), f^{-1}(CM_B)(x)).$$

Hence $f^{-1}(A) \triangleright\triangleleft f^{-1}(B)$. □

4. Commutativity and centeralization in *TFMS*(*G*)

Definition 4.1 : Let $A \in TFMS(G)$. Then A is abelian if

$$CM_A(xy) = CM_A(yx)$$

for all $x, y \in G$.

Corollary 4.2 : *Let $A \in TFMS(G)$. If G is an abelian group, then A is commutative or abelian.*

Remark 4.3 : Let $A, B \in TFMS(G)$ such that $A \subseteq B$ and $CM_A(e) = CM_B(e)$, where *e* is the identity element of *G*. If *B* is abelian, then *A* is abelian. Let *G* be a group and $x, y \in G$. Recall that a commutator of *x* and *y* in *G* is defined by

$$[x, y] = x^{-1}y^{-1}xy.$$

Proposition 4.4 : Let $A \in TFMS(G)$ and A is abelian. If T be idempotent t-norm, then

(1) $CM_A([x, y]) = CM_A(e)$,

(2) $CM_A([x, y]) \geq CM_A(x)$,

(3) $CM_A([x, y]) \geq CM_A(y)$,

for all $x, y \in G$.

Proof : Let $x, y \in G$ such that *x* and *y* commute with each other. Now

(1)

$$CM_A([x, y]) = CM_A(x^{-1}y^{-1}xy) = CM_A(x^{-1}xy^{-1}y)$$

$$\geq T(CM_A(x^{-1}x), CM_A(y^{-1}y)) = T(CM_A(e), CM_A(e))$$

$$= CM_A(e) \geq CM_A(x^{-1}y^{-1}xy) = CM_A([x,y]). \text{ (Theorem 2.15)}$$

Thus $CM_A([x,y]) = CM_A(e)$.

(2)

$$CM_A([x,y]) = CM_A(x^{-1}y^{-1}xy) \geq T(CM_A(x^{-1}), CM_A(y^{-1}xy))$$

$$\geq T(CM_A(x), CM_A(y^{-1}xy)) = T(CM_A(x), CM_A(xy^{-1}y))$$

$$= T(CM_A(x), CM_A(xe)) = T(CM_A(x), CM_A(x)) = CM_A(x).$$

Then $CM_A([x,y]) \geq CM_A(x)$.

(3)

$$CM_A([x,y]) = CM_A(x^{-1}y^{-1}xy) = CM_A(x^{-1}xy^{-1}y)$$

$$= CM_A(ey^{-1}y) \geq T(CM_A(e), CM_A(y^{-1}y)) \geq T(CM_A(y^{-1}y), CM_A(y^{-1}y))$$

$$= CM_A(y^{-1}y) \geq T(CM_A(y^{-1}), CM_A(y)) \geq T(CM_A(y), CM_A(y)) = CM_A(y).$$

Then $CM_A([x,y]) \geq CM_A(y)$. □

Proposition 4.5 : Let $A \in TFMS(G)$ and A is abelian. If T be idempotent t-norm and $n \in \mathbb{N}$, then

$$CM_A((xy)^n) = CM_A(x^n y^n)$$

for all $x, y \in G$.

Proof : Let $x, y \in G$ such that x and y commute with each other. Then

$$CM_A((xy)^n) = CM_A(xyxyxy...xyxyxy) = CM_A(xyxyxy...xyxy^2x[x,y])$$

$$\geq T(CM_A(xyxyxy...xyxy^2x), CM_A([x,y])) = T(CM_A(xyxyxy...xyxy^2x), CM_A(e))$$

$$\geq T(CM_A(xyxyxy...xyxy^2x), CM_A(xyxyxy...xyxy^2x)) = CM_A(xyxyxy...xyxy^2x)$$

$$= CM_A(x^2yxyxy...xyxy^2) = CM_A(x^2y...xy^3x) = CM_A(x^2y...xy^3x[x,y])$$

$$\geq T(CM_A(x^2y...xy^3x), CM_A([x,y])) \geq T(CM_A(x^2y...xy^3x), CM_A(e))$$

$$\geq T(CM_A(x^2y...xy^3x), CM_A(x^2y...xy^3x)) = CM_A(x^2y...xy^3x)$$

$$= CM_A(x^3y...xy^3) \geq ... \geq CM_A(x^{n-1}yxy^{n-1})$$

$$= CM_A(x^{n-1}xy^n[x, y^{n-1}]) \geq T(CM_A(x^{n-1}xy^n), CM_A([x, y^{n-1}]))$$

$$\geq T(CM_A(x^{n-1}xy^n), CM_A(e)) \geq T(CM_A(x^{n-1}xy^n), CM_A(x^{n-1}xy^n))$$

$$= CM_A(x^{n-1}xy^n) = CM_A(x^ny^n) = CM_A(x^{n-1}y^ny)$$

$$= CM_A(x^{n-1}yxy^{n-1}[y^{n-1}, x]) \geq T(CM_A(x^{n-1}yxy^{n-1}), CM_A([y^{n-1}, x]))$$

$$= T(CM_A(x^{n-1}yxy^{n-1}), CM_A(e))$$

$$\geq T(CM_A(x^{n-1}yxy^{n-1}), CM_A(x^{n-1}yxy^{n-1})) = CM_A(x^{n-1}yxy^{n-1})$$

$$\geq ... \geq CM_A(xyxy...xyxy^2x) = CM_A(xyxy...xyxy[x,y])$$

$$\geq T(CM_A(xyxy...xyxy), CM_A([x,y])) = T(CM_A(xyxy...xyxy), CM_A(e))$$

$$\geq T(CM_A(xyxy...xyxy), T(CM_A(xyxy...xyxy)) = CM_A(xyxy...xyxy)$$

$$= CM_A((xy)^n).$$

Therefore

$$CM_A((xy)^n) = CM_A(x^ny^n).$$

□

Definition 4.6 : Let $A \in TFMS(G)$. Define centeralizer of A as $Z(A) = \{x \in G \mid CM_A(xy) = CM_A(yx) \text{ and } CM_A(xyz) = CM_A(yxz) \text{ for all } y, z \in G\}$.

Proposition 4.7 : Let $A \in TFMS(G)$. Then $x \in Z(A)$ if and only if $CM_A(xyx^{-1}y^{-1}) = CM_A(e)$ for all $y \in G$.

Proof : Let $x \in Z(A)$. Then for all $y \in G$ we get that

$$CM_A(xyx^{-1}y^{-1}) = CM_A(yxx^{-1}y^{-1})$$

$$= CM_A(yey^{-1}) = CM_A(yy^{-1}) = CM_A(e).$$

Conversely, let $CM_A(xyx^{-1}y^{-1}) = CM_A(e)$ for all $y \in G$. We prove that $x \in Z(A)$. As $CM_A(xyx^{-1}y^{-1}) = CM_A(e)$ so $CM_A(xyx^{-1}y^{-1}y) = CM_A(ey)$ then $CM_A(xyx^{-1}e) = CM_A(y)$ and then $CM_A(xyx^{-1}) = CM_A(y)$ and $CM_A(xyx^{-1}x) = CM_A(yx)$ and so $CM_A(xye) = CM_A(yx)$ then

$$CM_A(xy) = CM_A(yx). \qquad (a)$$

As

$$CM_A(xyz(yxz)^{-1}) = CM_A(xyzz^{-1}x^{-1}y^{-1})$$

$$= CM_A(xyex^{-1}y^{-1}) = CM_A(xyx^{-1}y^{-1}) = CM_A(e)$$

so $CM_A(xyz(yxz)^{-1}) = CM_A(e)$ and then $CM_A(xyz(yxz)^{-1}yxz) = CM_A(eyxz)$ and so $CM_A(xyze) = CM_A(yxz)$ and

$$CM_A(xyz) = CM_A(yxz). \qquad (b)$$

Now from (a) and (b) we get that $x \in Z(A)$. □

Corollary 4.8 : *Let $A \in TFMS(G)$. Then $CM_A(xyx^{-1}y^{-1}) = CM_A(e)$ for all $x, y \in G$ if and only if A is abelian.*

Proof : Let $CM_A(xyx^{-1}y^{-1}) = CM_A(e)$ for all $x, y \in G$ then $CM_A(xyx^{-1}y^{-1}y) = CM_A(ey)$ so $CM_A(xyx^{-1}e) = CM_A(y)$ and $CM_A(xyx^{-1}) = CM_A(y)$ and $CM_A(xyx^{-1}x) = CM_A(yx)$ and $CM_A(xye) = CM_A(yx)$ and so $CM_A(xy) = CM_A(yx)$ therefore A is abelian.

Conversely, if A is abelian, then $CM_A(xy) = CM_A(yx)$ for all $x, y \in G$. Then

$$CM_A(xyx^{-1}) = CM_A(yxx^{-1}) = CM_A(ye) = CM_A(y).$$

Also $CM_A(xyx^{-1}y^{-1}) = CM_A(yy^{-1}) = CM_A(e)$. □

Proposition 4.9 : Let $A \in TFMS(G)$. Then

(1) $Z(A)$ is a subgroup of G.

(2) $Z(A) \trianglerighteq G$.

Proof : Let $x, y, z, w \in G$.

(1) If $x, y \in Z(A)$, then we prove that $xy^{-1} \in Z(A)$. Now

$$CM_A((xy^{-1})z) = CM_A(xy^{-1}z) = CM_A(y^{-1}xz)$$

$$= CM_A(y^{-1}xzyy^{-1}) = CM_A(y^{-1}xyzy^{-1}) = CM_A(y^{-1}yxzy^{-1})$$

$$= CM_A(exzy^{-1}) = CM_A(xzy^{-1}) = CM_A(zxy^{-1}) = CM_A(z(xy^{-1}))$$

and then

$$CM_A((xy^{-1})z) = CM_A(z(xy^{-1})). \qquad (a)$$

Also

$$CM_A((xy^{-1})zw) = CM_A(xy^{-1}zw) = CM_A(y^{-1}xzw)$$

$$= CM_A(y^{-1}zxw) = CM_A(y^{-1}zxyy^{-1}w) = CM_A(y^{-1}zyxy^{-1}w)$$

$$= CM_A(y^{-1}yzxy^{-1}w) = CM_A(ezxy^{-1}w) = CM_A(zxy^{-1}w)$$

$$= CM_A(z(xy^{-1})w)$$

and then

$$CM_A((xy^{-1})zw) = CM_A(z(xy^{-1})w). \qquad (b)$$

Thus (a) and (b) give us $xy^{-1} \in Z(A)$ and from Proposition 2.5 we get that $Z(A)$ is a subgroup of G.

(2) Let $g \in G$ and $x \in Z(A)$ and we must prove that $gxg^{-1} \in Z(A)$. Now

$$CM_A((gxg^{-1})y) = CM_A(gxg^{-1}y) = CM_A(xgg^{-1}y) = CM_A(xey)$$

$$= CM_A(xy) = CM_A(yx) = CM_A(yxgg^{-1}) = CM_A(ygxg^{-1})$$

$$= CM_A(y(gxg^{-1}))$$

and so

$$CM_A((gxg^{-1})y) = CM_A(y(gxg^{-1})). \qquad (a)$$

Also

$$CM_A((gxg^{-1})zw) = CM_A(gxg^{-1}zw) = CM_A(xgg^{-1}zw)$$

$$= CM_A(xezw) = CM_A(xzw) = CM_A(zxw) = CM_A(zxgg^{-1}w)$$

$$= CM_A(zgxg^{-1}w) = CM_A(z(gxg^{-1})w)$$

then

$$CM_A((gxg^{-1})zw) = CM_A(z(gxg^{-1})w). \qquad (b)$$

Therefore from (a) and (b) we get that $gxg^{-1} \in Z(A)$ and then $Z(A) \trianglerighteq G.$ □

Proposition 4.10 : Let $A, B \in TFMS(G)$. Then $Z(A) \cap Z(B) \subseteq Z(A \cap B)$.

Proof:Let $x \in Z(A) \cap Z(B)$ and $y, z \in G$. As $x \in Z(A)$ so $CM_A(xy) = CM_A(yx)$ and $CM_A(xyz) = CM_A(yxz)$ and as $x \in Z(B)$ so $CM_B(xy) = CM_B(yx)$ and $CM_B(xyz) = CM_B(yxz)$. Now

$$CM_{A\cap B}(xy) = T(CM_A(xy), CM_B(xy)) = T(CM_A(yx), CM_B(yx)) = CM_{A\cap B}(yx).$$

Thus

$$CM_{A\cap B}(xy) = CM_{A\cap B}(yx). \qquad (a)$$

Also

$$CM_{A\cap B}(xyz) = T(CM_A(xyz), CM_B(xyz)) = T(CM_A(yxz), CM_B(yxz)) = CM_{A\cap B}(yxz).$$

Then

$$CM_{A\cap B}(xyz) = CM_{A\cap B}(yxz). \qquad (b)$$

Therefore from (a) and (b) we have that $x \in Z(A \cap B)$ and then $Z(A) \cap Z(B) \subseteq Z(A \cap B)$. □

Proposition 4.11 : Let $A, B \in TFMS(G)$ and $CM_A(e) = CM_B(e)$. Then

$$Z(A) \cap Z(B) = Z(A \cap B).$$

Proof : As Proposition 4.10 we have that $Z(A) \cap Z(B) \subseteq Z(A \cap B)$. We must prove that $Z(A \cap B) \subseteq Z(A) \cap Z(B)$. If $x \in Z(A \cap B)$, then from Proposition 4.7 for all $y \in G$ we get that

$$CM_{A\cap B}(xyx^{-1}y^{-1}) = CM_{A\cap B}(e)$$

$$\Leftrightarrow T(CM_A(xyx^{-1}y^{-1}), CM_B(xyx^{-1}y^{-1})) = T(CM_A(e), CM_B(e))$$

$$\Leftrightarrow CM_A(xyx^{-1}y^{-1}) = CM_A(e) \text{ and } CM_B(xyx^{-1}y^{-1}) = CM_B(e)$$

$$\Leftrightarrow x \in Z(A) \text{ and } x \in Z(B)$$

$$\Leftrightarrow x \in Z(A) \cap Z(B).$$

□

Corollary 4.12 : *Let* $A, B \in TFMS(G)$ *such that* $A \subseteq B$ *and* $CM_A(e) = CM_B(e)$. *Then* $Z(A) \subseteq Z(B)$.

Proof : Let $x \in Z(A)$ and then from Proposition 4.7 we have that $CM_A(xyx^{-1}y^{-1}) = CM_A(e)$ for all $y \in G$. As $A \subseteq B$ so $CM_A(xyx^{-1}y^{-1}) \leq CM_B(xyx^{-1}y^{-1})$ and from Proposition 4.7 we get that $CM_B(xyx^{-1}y^{-1}) \leq CM_B(e)$. Now

$$CM_B(e) = CM_A(e) = CM_A(xyx^{-1}y^{-1}) \leq CM_B(xyx^{-1}y^{-1}) \leq CM_B(e)$$

and then $CM_B(xyx^{-1}y^{-1}) = CM_B(e)$ and from Proposition 4.7 we get $x \in Z(B)$. □

Proposition 4.13 : Let $A, B \in TFMS(G)$. Then $Z(A)Z(B) \subseteq Z(A \circ B)$.

Proof : Let $x \in Z(A)$ and $y \in Z(B)$ and we prove that $xy \in Z(A \circ B)$. Let $z, w \in G$ and we have

$$CM_{(A \circ B)}((xy)z) = CM_{(A \circ B)}(xyz)$$

$$= \sup\{T(CM_A(a), CM_B(b)) \mid a, b \in G, xyz = ab\}$$

$$= \sup\{T(CM_A(xyzb^{-1}), CM_B(b)) \mid a, b \in G, xyzb^{-1}b = ab\}$$

$$= \sup\{T(CM_A(xzyb^{-1}), CM_B(b)) \mid a, b \in G, xzyb^{-1}b = ab\}$$

$$= \sup\{T(CM_A(a), CM_B(b)) \mid a, b \in G, xzy = ab\}$$

$$= \sup\{T(CM_A(a), CM_B(a^{-1}xzy)) \mid a, b \in G, aa^{-1}xzy = ab\}$$

$$= \sup\{T(CM_A(a), CM_B(a^{-1}zxy)) \mid a, b \in G, aa^{-1}zxy = ab\}$$

$$= \sup\{T(CM_A(a), CM_B(b)) \mid a, b \in G, zxy = ab\}$$

$$= \sup\{T(CM_A(a), CM_B(b)) \mid a, b \in G, z(xy) = ab\}$$

$$= CM_{(A \circ B)}(z(xy)).$$

Thus

$$CM_{(A \circ B)}((xy)z) = CM_{(A \circ B)}(z(xy)). \quad (a)$$

Also

$$CM_{(A \circ B)}((xy)zw) = CM_{(A \circ B)}(xyzw)$$

$$= \sup\{T(CM_A(a), CM_B(b)) \mid a, b \in G, xyzw = ab\}$$

$$= \sup\{T(CM_A(xyzwb^{-1}), CM_B(b)) \mid a, b \in G, xyzwb^{-1}b = ab\}$$

$$= \sup\{T(CM_A(xzywb^{-1}), CM_B(b)) \mid a, b \in G, xzywb^{-1}b = ab\}$$

$$= \sup\{T(CM_A(a), CM_B(b)) \mid a, b \in G, xzyw = ab\}$$

$$= \sup\{T(CM_A(a), CM_B(a^{-1}xzyw)) \mid a, b \in G, aa^{-1}xzyw = ab\}$$

$$= \sup\{T(CM_A(a), CM_B(a^{-1}zxyw)) \mid a, b \in G, aa^{-1}zxyw = ab\}$$

$$= \sup\{T(CM_A(a), CM_B(b)) \mid a, b \in G, zxyw = ab\}$$

$$= \sup\{T(CM_A(a), CM_B(b)) \mid a, b \in G, z(xy)w = ab\}$$

$$= CM_{(A \circ B)}(z(xy)w).$$

Then

$$CM_{(A \circ B)}((xy)zw) = CM_{(A \circ B)}(z(xy)w). \quad (b)$$

Therefore (a) and (b) give us $xy \in Z(A \circ B)$ and so $Z(A)Z(B) \subseteq Z(A \circ B)$. □

Proposition 4.14 : Let G and H be groups and $f : G \to H$ be a homomorphism. If $A \in TFMS(G)$, then $f(Z(A)) \subseteq Z(f(A))$.

Proof : Let $x \in f(Z(A))$ then $\exists u \in Z(A)$ such that $x = f(u)$ and $u \in G.$ we prove that $x \in Z(f(A)).$ Now

$$CM_{f(A)}(xy) = \sup\{CM_A(uv) \mid u, v \in G, f(uv) = xy, f(u) = x\}$$

$$= \sup\{CM_A(uv) \mid u, v \in G, f(u)f(v) = xy, f(u) = x, f(v) = y\}$$

$$= \sup\{CM_A(vu) \mid u, v \in G, f(v)f(u) = yx, f(u) = x, f(v) = y\}$$

$$= \sup\{CM_A(vu) \mid u, v \in G, f(vu) = yx, f(u) = x, f(v) = y\}$$

$$= CM_{f(A)}(yx).$$

Thus

$$CM_{f(A)}(xy) = CM_{f(A)}(yx). (a)$$

Also

$$CM_{f(A)}(xyz) = \sup\{CM_A(uvw) \mid u, v, w \in G, f(uvw) = xyz, f(u) = x\}$$

$$= \sup\{CM_A(uvw) \mid u, v, w \in G, f(u)f(v)f(w) = xyz, f(u) = x\}$$

$$= \sup\{CM_A(vuw) \mid u, v, w \in G, f(v)f(u)f(w) = yxz, f(u) = x\}$$

$$= CM_{f(A)}(yxz).$$

Then

$$CM_{f(A)}(xyz) = CM_{f(A)}(yxz). \qquad (b)$$

Therefore (a) and (b) give us that $x \in Z(f(A))$ and then $f(Z(A)) \subseteq Z(f(A)).$ □

Proposition 4.15 : Let G and H be groups and $f : G \to H$ be a homomorphism. If $B \in TFMS(H),$ then $f^{-1}(Z(B)) = Z(f^{-1}(B)).$

Proof : At first we prove that $f^{-1}(Z(B)) \subseteq Z(f^{-1}(B)).$ Let $x \in f^{-1}(Z(B))$ then $f(x) \in Z(B).$ Now for all $y, z \in G$ we get that

$$CM_{f^{-1}(B)}(xy) = CM_B(f(xy)) = CM_B(f(x)f(y))$$

$$= CM_B(f(y)f(x)) = CM_B(f(yx)) = CM_{f^{-1}(B)}(yx).$$

Then

$$CM_{f^{-1}(B)}(xy) = CM_{f^{-1}(B)}(yx). \qquad (a)$$

Also

$$CM_{f^{-1}(B)}(xyz) = CM_B(f(xyz)) = CM_B(f(x)f(y)f(z))$$

$$= CM_B(f(y)f(x)f(z)) = CM_B(f(yxz)) = CM_{f^{-1}(B)}(yxz).$$

Then

$$CM_{f^{-1}(B)}(xyz) = CM_{f^{-1}(B)}(yxz). \qquad (b)$$

Thus from (a) and (b) we get that $x \in Z(f^{-1}(B))$ and then

$$f^{-1}(Z(B)) \subseteq Z(f^{-1}(B)). \qquad (c)$$

Now we must prove that $Z(f^{-1}(B)) \subseteq f^{-1}(Z(B))$. Let $x, y, z \in G$ and $u, v, w \in H$ such that $f(x) = u$, $f(y) = v$ and $f(z) = w$. If $x \in Z(f^{-1}(B))$, then

$$CM_B(uv) = CM_B(f(x)f(y)) = CM_B(f(xy))$$

$$= CM_{f^{-1}(B)}(xy) = CM_{f^{-1}(B)}(yx) = CM_B(f(yx))$$

$$= CM_B(f(y)f(x)) = CM_B(vu)$$

and then

$$CM_B(uv) = CM_B(vu). \qquad (d)$$

Also

$$CM_B(uvw) = CM_B(f(x)f(y)f(z)) = CM_B(f(xyz))$$

$$= CM_{f^{-1}(B)}(xyz) = CM_{f^{-1}(B)}(yxz) = CM_B(f(yxz))$$

$$= CM_B(f(y)f(x)f(z)) = CM_B(vuw)$$

and then

$$CM_B(uvw) = CM_B(vuw). \quad (e)$$

Now from (d) and (e) we get that $u \in Z(B)$ and then $f(x) \in Z(B)$ and thus $x \in f^{-1}(Z(B))$. Then

$$Z(f^{-1}(B)) \subseteq f^{-1}(Z(B)). \qquad (m)$$

Therefore from (c) and (m) we get that

$$f^{-1}(Z(B)) = Z(f^{-1}(B)).$$

□

Proposition 4.16 : Let G and H be abelian groups and $f : G \to H$ be an epimorphism. Let $A \in TFMS(G)$ and $B \in TFMS(H)$ such that A and B are abelian. Then

(1) $f(A) \in TFMS(H)$ is abelian.

(2) $f^{-1}(B) \in TFMS(G)$ is abelian.

Proof :

(1) As Proposition 2.16 we have that $f(A) \in TFMS(H)$. Let $x, y \in H$ then $\exists a, b \in G$ such that $x = f(a)$ and $y = f(b)$. Now

$$CM_{f(A)}(xy) = \sup\{CM_A(ab) \mid a, b \in G, f(ab) = xy\}$$

$$= \sup\{CM_A(ab) \mid a, b \in G, f(a)f(b) = xy\}$$

$$= \sup\{CM_A(ba) \mid a, b \in G, f(b)f(a) = yx\}$$

$$= \sup\{CM_A(ba) \mid a, b \in G, f(ba) = yx\}$$

$$= CM_{f(A)}(yx).$$

Therefore $f(A) \in TFMS(H)$ is abelian.

(2) From Proposition 2.17 we get that $f^{-1}(B) \in TFMS(G)$. Let $a, b \in G$ then

$$CM_{f^{-1}(B)}(ab) = CM_B(f(ab)) = CM_B(f(a)f(b))$$

$$= CM_B(f(b)f(a)) = CM_B(f(ba)) = CM_{f^{-1}(B)}(ba).$$

Thus $f^{-1}(B) \in TFMS(G)$ and it is abelian. □

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References

[1] J. M. Anthony and H. Sherwood, Fuzzy groups redefined, *J. Math. Anal. Appl.,* 69(1979), 124-130.

[2] S. Miyamoto, Basic operations of fuzzy multisets, *J. Japan Soci., Fuzzy Theory Systems,* 8 (1996), 639-645.

[3] R. Rasuli, Fuzzy Ideals of Subtraction Semigroups with Respect to A t-norm and A t-conorm, *The Journal of Fuzzy Mathematics Los Angeles,* 24 (4) (2016), 881-892.

[4] R. Rasuli, Fuzzy modules over a t-norm, I*nt. J. Open Problems Compt. Math.*, 9 (3) (2016), 12-18.

[5] R. Rasuli, Fuzzy Subrings over a t-norm, *The Journal of Fuzzy Mathematics Los Angeles*, 24 (4) (2016), 995-1000.

[6] R. Rasuli, Norms over intuitionistic fuzzy subrings and ideals of a ring, Notes on Intuitionistic Fuzzy Sets, 22 (5) (2016), 72-83.

[7] R. Rasuli, Norms over fuzzy Lie algebra, *Journal of New Theory,* 15(2017), 32-38.

[8] R. Rasuli, Fuzzy subgroups on direct product of groups over a t-norm, *Journal of Fuzzy Set Valued Analysis,* 3(2017), 96-101.

[9] R. Rasuli, Characterizations of intuitionistic fuzzy subsemirings of semirings and their homomorphisms by norms, *Journal of New Theory,* 18(2017), 39-52.

[10] R. Rasuli, Intuitionistic fuzzy subrings and ideals of a ring under norms, LAP LAMBERT Academic publishing, 2017, ISBN: 978-620-2-06926-7.

[11] R. Rasuli, Characterization of Q-Fuzzy subrings (Anti Q-Fuzzy Subrings) with respect to a T-norm (T-Conorms), *Journal of Information and Optimization Science*, 31(2018), 827-837.

[12] R. Rasuli, T-Fuzzy Submodules of $R \times M$, *Journal of New Theory,* 22(2018), 92-102.

[13] R. Rasuli, Fuzzy subgroups over a T-norm, *Journal of Information and Optimization Science,* 39(2018), 1757-1765.

[14] R. Rasuli, Fuzzy Sub-vector Spaces and Sub-bivector Spaces under t-Norms, *General Letters in Mathematics*, 5(2018), 47-57.

[15] R. Rasuli, Anti Fuzzy Submodules over A t -conorm and Some of Their Properties, *The Journal of Fuzzy Mathematics Los Angles*, 27(2019), 229-236.

[16] R. Rasuli, Artinian and Noetherian Fuzzy Rings, *Int. J. Open Problems Compt. Math.*, 12(2019), 1-7.

[17] R. Rasuli and H. Narghi, T-Norms Over Q-Fuzzy Subgroups of Group, *Jordan Journal of Mathematics and Statistics* (JJMS), 12(2019), 1-13.

[18] R. Rasuli, Fuzzy equivalence relation, fuzzy congrunce relation and fuzzy normal subgroups on group G over t-norms, *Asian Journal of Fuzzy and Applied Mathematics,* 7(2019), 14-28.

[19] R. Rasuli, Norms over anti fuzzy G-submodules, *MathLAB Journal,* 2(2019), 56-64.

[20] R. Rasuli, Norms over bifuzzy bi-ideals with operators in semigroups, Notes on Intuitionistic Fuzzy Sets , 25(2019), 1-11.

[21] R. Rasuli, Norms Over Basic Operations on Intuitionistic Fuzzy Sets, *The Journal of Fuzzy Mathematics Los Angles*, 27(3)(2019), 561-582.

[22] R. Rasuli, T-fuzzy Bi-ideals in Semirings, *Earthline Journal of Mathematical Sciences,* 27(1)(2019), 241-263.

[23] R. Rasuli, Anti Fuzzy Equivalence Relation on Rings with respect to t-conorm C, *Earthline Journal of Mathematical Sciences*, 3(1)(2020), 1-19.

[24] R. Rasuli, Anti Fuzzy Subbigroups of Bigroups under t -conorms, *The Journal of Fuzzy Mathematics Los Angles*, 28(1)(2020), 181-200.

[25] R. Rasuli, Anti Q-soft Normal Subgroups, *The Journal of Fuzzy Mathematics Los Angles,* 28(1)(2020), 237-248.

[26] R. Rasuli, t-norms over Fuzzy Multigroups, *Earthline Journal of Mathematical Sciences*, 3(2)(2020), 207-228.

[27] R. Rasuli, Intuitionistic fuzzy subgroups with respect to norms (T, S), Eng. Appl. Sci. Lett. (EASL), 3(2)(2020), 40-53.

[28] A. Rosenfeld, Fuzzy Groups, *J. Math. Anal. Appl.,* 35(1971), 512-517.

[29] T. K. Shinoj, A. Baby and J. J. Sunil, On some algebraic structures of fuzzy multisets, *Annals of Fuzzy Mathematics and Informatics*, 9(2015), 77-90.

[30] D. Singh, A. M. Ibrahim, T. Yohanna and J. N. Singh, An overview of the applications of multisets, *Novi Sad J. Math.*, 37 (2007), 73-92.

[31] A. Syropoulos, Mathematics of multisets, Springer-Verlag Berlin Heidelberg, (2001), 347-358.

[32] L. A. Zadeh, Fuzzy sets, Inform. Control., 8(1965), 338-353.

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