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Local antimagic vertex coloring for generalized friendship graphs

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Abstract

Let $G=(V,E)$ be a graph of order p and size q without isolated vertices. A bijection $f:E\to\{1,2,\cdots,q\}$ is called a local antimagic labeling if $w(u)\neq w(v)$ for all $uv\in E$, where the vertex weight $w(u)=\sum_{e\in E(u)} f(e)$ and $E(u)$ is the set of edges incident to the vertex $u\in V$. The local antimagic chromatic number $\chi_{la}(G)$ is defined to be the minimum number of colors(vertex weights) taken over all colorings of G induced by local antimagic labelings of G. In this paper, we study the local chromatic number for generalized friendship graph of complete graphs and cycles.

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1. Introduction

By a graph $G=(V,E)$ we mean a finite undirected graph without loops or multiple edges. The order and size of G are denoted by $|V|=p$ and $|E|=q$ respectively. For graph theoretic terminology we refer to the book by Chartrand and Lesniak [3].

Hartsfield and Ringel [4] introduced the concept of antimagic labeling of a graph. Let $G=(V,E)$ be a graph and let $f:E\to\{1,2,\cdots,|E|\}$ be a bijection. For each vertex $u\in V(G)$, the weight $w(u)=\sum_{e\in E(u)}f(e)$, where $E(u)$ is the set of edges incident to u. If $w(u)\neq w(v)$ for any two distinct vertices u and $v\in V(G)$, then f is called an antimagic labeling of G. A graph G is called antimagic if G has an antimagic labeling. Let $G=(V,E)$ be a graph of order p and size q having no isolated vertices. A bijection $f:E\to\{1,2,\cdots,|E|\}$ is called a local antimagic labeling if for all $uv\in E$ we have $w(u)\neq w(v)$, where $w(u)=\sum_{e\in E(u)}f(e)$. A graph G is local antimagic if G has a local antimagic labeling.

The concept of graph coloring and the chromatic number is very well-studied in graph theory [3]. More various topics have been studied in literature. For example recently in [7] the authors obtained chromatic numbers of some families of graphs. Motivated by chromatic number, several related parameters are introduced and studied by several authors. Recently, Nima Ghanbari & Saeid Alikhani[6] considered another variation of the corona of two graphs(called the neighbourhood corona of two graphs) and studied its total dominator chromatic number. Also the stability and bondage number of the total dominator chromatic number of certain graphs were studied.

The local antimagic chromatic number $\chi_{la}(G)$ is first introduced and defined by Arumugam et al.[1] to be the minimum number of colors taken over all colorings of G induced by local antimagic labeling of G. Among others they proved that the local antimagic chromatic number of cycles on n vertices is 3 and the local antimagic chromatic number of the complete graph on p vertices is p. More recently, Premalatha et al.[5] obtained the local antimagic chromatic number for corona product $P_n\circ K_m$, where the *corona product* $G\circ H$ of two graphs G and H is the graph obtained by taking $|V(G)|$ copies of H and joining the i-th vertex of G to all the vertices in the i-th copy of H.

Bacă et al.[2] investigated the local antimagic chromatic number for the disjoint union of multiple copies of a graph G, denoted by mG, $m\geq 1$. They proved that $\chi_{la}(mG)\leq\chi_{la}(G)$, where G is an equally 2-edge colorable graph and let f be a local vertex antimagic edge labeling of G

that uses $\chi_{la}(G)$ colors with every edge of G, it holds the condition given in Theorem 1[2]. Using this result, they obtained several results. Also, they got lower and upper bounds for the union of m copies of trees and forests. They obtained an upper bound for a local antimagic chromatic number for the disjoint union of arbitrary graphs, and they obtained the exact local antimagic chromatic number for the union of m copies of even cycles(for odd cycles upper bound only), paths, stars and complete graphs $K_n, n \equiv 1 \pmod 4$.

In this paper, we study the local chromatic number for generalized friendship graph of complete graphs and cycles.

2. Generalized Friendship Graph of Complete Graphs

The graph consisting of n triangles with a common vertex c is called a *friendship graph.* The graphs we study in this paper is defined as follows.

Definition 2.1 : Let $H_1, H_2, \ldots, H_n$ be the graphs on number of vertices $p_1, p_2, \cdots, p_n$ respectively, where $p_1 \le p_2 \le p_3, \ldots, \le p_n$. Then the graph $F_{p_1, p_2, p_3, \ldots, p_n}(H_1, H_2, \ldots, H_n)$ consisting of graphs $H_i, 1 \le i \le n,$ with a common vertex c is called a *generalized friendship graph* on $H_1, H_2, \ldots, H_n$ respectively. If $H_1 \cong H_2 \cong \ldots \cong H_n \cong H$ and $p_1 = p_2 = \ldots = p_n = p$ then the resulting graph is called the *generalized friendship graph of H* and is denoted by $F_p^n(H)$.

If $H \cong C_3$ then the generalized friendship graph $F_3^n(C_3)$ is a friendship graph usually defined in literature. Note that $F_p^n(K_p)$ is the generalized friendship graph of the complete graph K_p and we have the following.

Theorem 2.2 : *For the generalized friendship graph $F_p^n(K_p)$ of complete graph on p vertices, where p is odd and $n \ge 1$, then $\chi_{la}(F_p^n(K_p)) = p$.*

Proof : Let $V(F_p^n(K_p)) = \{c \cup v_i^j, 1 \le i \le p_i - 1, 1 \le j \le n\}$ be the vertex set of $F_p^n(K_p)$ with order $n(p_i - 1) + 1$, where $p_1 = p_2 = \ldots = p_n = p$ is odd and let $E(F_p^n(K_p)) = \{cv_i^j \cup v_i^j v_{i+k}^j, 1 \le i \le p-1, 1 \le j \le n, 1 \le k \le p-2\}$ be the edge set of $F_p^n(K_p)$ with size $|E| = q = \frac{np(p-1)}{2}$. Let c be the central vertex of $F_p^n(K_p)$. Define the edge labeling $f_1 : E(F_p^n(K_p)) \to \{1, 2, \ldots, |E|\}$ by

$$f_1(cv_i^j) = \begin{cases} q - n\left[(i-1)p - \dfrac{i(i-1)}{2}\right] + 1 - j, & i \equiv 1,2 \,(\text{mod } 4), 1 \le j \le n, \\ q - n\left[(i-1)p - \dfrac{i(i-1)}{2}\right] - n + j, & i \equiv 0,3 \,(\text{mod } 4), 1 \le j \le n \end{cases}$$

$$f_1(v_i^j v_{i+k}^j) = \begin{cases} q - n\left[\displaystyle\sum_{r=1}^{i-1}(p-r)\right] - n(k+1) + j, & 1 \le i \le p-3, 1 \le k \le p-1-i, \\ & k \text{ is odd and } 1 \le j \le n, \\ q - n\left[\displaystyle\sum_{r=1}^{i-1}(p-r)\right] - nk + 1 - j, & 1 \le i \le p-3, 1 \le k \le p-1-i, \\ & k \text{ is even and } i = p-2, \\ & k = 1, 1 \le j \le n \end{cases}$$

Clearly, f_1 admits a local antimagic labeling of $F_p^n(K_p)$ with weights are as follows

$$w_1(c) = \begin{cases} \displaystyle\sum_{j=1}^{n}(A(q+1-j)) - n\sum_{i=1}^{p-1}\left[(i-1)p - \frac{i(i-1)}{2}\right], & j \equiv 1,2 \,(\text{mod } 4) \\ \displaystyle\sum_{j=1}^{n}(B(q-n+j)) - n\sum_{i=1}^{p-1}\left[(i-1)p - \frac{i(i-1)}{2}\right], & j \equiv 0,3 \,(\text{mod } 4). \end{cases}$$

$$w_1(v_i^j) = \begin{cases} C + \displaystyle\sum_{k=1}^{i-1} f(v_i^j v_{i-k}^j), & i \equiv 1,2 \,(\text{mod } 4), 1 \le j \le n \\ D + \displaystyle\sum_{k=1}^{i-1} f(v_i^j v_{i-k}^j), & i \equiv 0,3 \,(\text{mod } 4), 1 \le j \le n. \end{cases}$$

where

$$A = \begin{cases} \dfrac{p-1}{2}, & p \equiv 1 \,(\text{mod } 4) \\ \dfrac{p+1}{2}, & p \equiv 3 \,(\text{mod } 4) \end{cases}$$

$$B = \begin{cases} \dfrac{p-1}{2}, & p \equiv 1 \,(\text{mod } 4) \\ \dfrac{p-3}{2}, & p \equiv 3 \,(\text{mod } 4) \end{cases}$$

$$C=\begin{cases}(p-i)q-(p-1-i)n\left[\sum_{r=1}^{i-1}(p-r)\right]-n\left[(i-1)p-\frac{i(i-1)}{2}\right]-(j-1)-\\ \qquad\sum_{k\equiv 1,3\,(\mathrm{mod}\,4)}((k+1)n+j-2)-\sum_{k\equiv 0,2\,(\mathrm{mod}\,4)}(nk+j-1)\end{cases}$$

$$D=\begin{cases}(p-i)q-(p-1-i)n\left[\sum_{r=1}^{i-1}(p-r)\right]-n\left[(i-1)p-\frac{i(i-1)}{2}\right]-(j+n)-\\ \qquad\sum_{k\equiv 1,\,3\,(\mathrm{mod}\,4)}(kn-j-1)-\sum_{k\equiv 0,2\,(\mathrm{mod}\,4)}(n(k+1)-j)\end{cases}$$

Thus $\chi_{la}(F_p^n(K_p))\leq p.$ Since $\chi(K_p)=p,$ it follows that we get $\chi_{la}(F_p^n(K_p))\geq p$ and hence $\chi_{la}(F_p^n(K_p))=p.$ □

Example 2.3 : The generalized friendship graph $F_7^3(K_7)$ admits a local vertex antimagic labeling with 7 colors. The following three matrices gives the vertex weights of $F_7^3(K_7)$. Here $w_1(c)=$ the sum of all 6^{th} column entries of the j^{th} matrix $=486,\ 1\leq j\leq 3.$ The vertex weight $w_1(v_i^j)$ is the i^{th} row sum of the j^{th} matrix $=$ $w_1(v_1^j)=327,\ w_1(v_2^j)=249,\ w_1(v_3^j)=195,$ $w_1(v_4^j)=159,\ w_1(v_5^j)=135$ and $w_1(v_6^j)=117,\ 1\leq j\leq 3.$

$$\begin{bmatrix}58&57&52&51&46&63\\58&40&39&34&33&45\\57&40&27&22&21&28\\52&39&27&15&10&16\\51&34&22&15&4&9\\46&33&21&10&4&3\end{bmatrix}\begin{bmatrix}59&56&53&50&47&62\\59&41&38&35&32&44\\56&41&26&23&20&29\\53&38&26&14&11&17\\50&35&23&14&5&8\\47&32&20&11&5&2\end{bmatrix}\begin{bmatrix}60&55&54&49&48&61\\60&42&37&36&31&43\\55&42&25&24&19&30\\54&37&25&13&12&18\\49&36&24&13&6&7\\48&31&19&12&6&1\end{bmatrix}$$

Example 2.4 : The generalized friendship graph $F_5^4(K_5)$ admits a local vertex antimagic labeling with 5 colors. The following four matrices gives the vertex weights of $F_5^4(K_5)$. Here $w_1(c)=$ the sum of all the 4^{th} column entries of the j^{th} matrix $=296,\ 1\leq j\leq 4.$ The vertex weight $w_1(v_i^j)$ is the i^{th} row sum of the j^{th} matrix $=w_1(v_1^j)=130,\ w_1(v_2^j)=90,\ w_1(v_3^j)=66$ and $w_1(v_4^j)=50,\ 1\leq j\leq 4.$

$$\begin{bmatrix}33&32&25&40\\33&17&16&24\\32&17&8&9\\25&16&8&1\end{bmatrix}\begin{bmatrix}34&31&26&39\\34&18&15&23\\31&18&7&10\\26&15&7&2\end{bmatrix}\begin{bmatrix}35&30&27&38\\35&19&14&22\\30&19&6&11\\27&14&6&3\end{bmatrix}\begin{bmatrix}36&29&28&37\\36&20&13&21\\29&20&5&12\\28&13&5&4\end{bmatrix}$$

Theorem 2.5 : *For the generalized friendship graph* $F_p^n(K_p)$ *of complete graph on* $p \geq 4$ *vertices, where p is even and* $n \geq 1$ *is odd, we have* $\chi_{la}(F_p^n(K_p)) = p$.

Proof : Let $V(F_p^n(K_p)) = \{c \cup v_i^j, 1 \leq i \leq p-1, 1 \leq j \leq n\}$ be the vertex set of $(F_p^n(K_4))$ with order $n(p-1)+1$, where p is even and $n \geq 1$ is odd. Let $E(F_p^n(K_p)) = \{cv_i^j \cup v_i^j v_{i+k}^j, 1 \leq i \leq p-1, 1 \leq j \leq n, 1 \leq k \leq p-2\}$ be the edge set of $F_p^n(K_p)$ with size $|E| = q = \frac{np(p-1)}{2}$. Let c be the central vertex of $F_p^n(K_p)$.

Define the edge labeling $f_2 : E(F_p^n(K_p)) \to \{1, 2, ..., |E|\}$ by

$$f_2(cv_i^j) = \begin{cases} A - \left(\frac{j-1}{2}\right), & j \text{ is odd}, 1 \leq i \leq \frac{p-2}{2} \\ A - \left(\frac{n-1}{2}\right) - \left(\frac{j-1}{2}\right), & j \text{ is odd}, i = \frac{p-2}{2} + 2 \\ A - n + \left(\frac{j-1}{2}\right) + 1, & j \text{ is odd}, \frac{p-2}{2} + 3 \leq i \leq p-1 \\ A - \left(\frac{n-1}{2}\right) - \left(\frac{j}{2}\right), & j \text{ is even}, 1 \leq i \leq \frac{p-2}{2} \\ A - \left(\frac{j}{2}\right) + 1, & j \text{ is even}, i = \frac{p-2}{2} + 2 \\ A - \left(\frac{n-1}{2}\right) + \left(\frac{j}{2}\right), & j \text{ is even}, \frac{p-2}{2} + 3 \leq i \leq p-1 \\ A - n + j, & 1 \leq j \leq n, i = \frac{p-2}{2} + 1 \end{cases}$$

$$f_2(v_1^j v_{1+k}^j) = \begin{cases} q - (n+1) - \left(\frac{j+1}{2}\right), & k = 1, j \text{ is odd} \\ q - (n-1) - \left(\frac{j}{2}\right), & k = 1, j \text{ is even} \\ q - \left[kn + \left(\frac{n+1}{2}\right)\right] - \left(\frac{j}{2}\right), & 2 \leq k \leq p-2, k \text{ is even and } j \text{ is even} \\ q - kn - \left(\frac{j-1}{2}\right), & 2 \leq k \leq p-2, k \text{ is even and } j \text{ is odd} \\ q - \left[kn + \left(\frac{n-1}{2}\right)\right] + \left(\frac{j}{2}\right), & 3 \leq k \leq p-3, k \text{ is odd and } j \text{ is even} \\ q - (k+1)n + \left(\frac{j+1}{2}\right), & 3 \leq k \leq p-3, k \text{ is odd and } j \text{ is odd} \end{cases}$$

$$f_2(v_i^j v_{i+k}^j) = \begin{cases} B-(i+k)n+\left(\frac{j+1}{2}\right), & j \text{ is odd, } k \text{ is odd}, 2 \le i \le \frac{p}{2}-1, \\ & 1 \le k \le p-1-2i \\ B-(i+k-1)n-\left(\frac{j-1}{2}\right), & j \text{ is odd, } k \text{ is even}, 2 \le i \le \frac{p}{2}-1, \\ & 1 \le k \le p-1-2i \\ B-(i+k-1)n-\left(\frac{j}{2}\right)-2, & j \text{ is even, } k \text{ is odd}, 2 \le i \le \frac{p}{2}-1, \\ & 1 \le k \le p-1-2i \\ B-(i+k)n-\left(\frac{j}{2}\right)+3, & j \text{ is even, } k \text{ is even}, 2 \le i \le \frac{p}{2}-1, \\ & 1 \le k \le p-1-2i \end{cases}$$

$$f_2(v_i^j v_{p-i+k}^j) = \begin{cases} C-\left(\frac{n-1}{2}\right)+(i-k)n-\left(\frac{j-1}{2}\right), & k=2, j \text{ is odd}, k+1 \le i \le \frac{p}{2} \\ C+(i-k-1)n+\left(\frac{j+1}{2}\right), & k \text{ is even, } j \text{ is odd}, 4 \le k \le p-3, \\ & k+1 \le i \le \frac{p}{2}+\frac{k-2}{2} \\ C+(i-k)n-\left(\frac{j-1}{2}\right), & k \text{ is odd, } j \text{ is odd}, 1 \le k \le p-3, \\ & k+1 \le i \le \frac{p}{2}+\frac{k-1}{2} \\ C+(i-2)n+1-\left(\frac{j}{2}\right), & k=2, j \text{ is even}, k+1 \le i \le \frac{p}{2} \\ C-\left(\frac{n-1}{2}\right)+(i-k)n-\left(\frac{j}{2}\right), & k=1,3, \ j \text{ is even}, \\ & k+1 \le i \le \frac{p}{2}+1 \\ C+\left(\frac{n+1}{2}\right)+(i-k)+\left(\frac{j-2}{2}\right), & k \text{ is even, } j \text{ is even}, 4 \le k \le p-3, \\ & k+1 \le i \le \frac{p}{2}+\frac{k-2}{2} \\ C+\left(\frac{n-1}{2}\right)-\left(\frac{j-2}{2}\right), & k \text{ is odd, } j \text{ is even}, 4 \le k \le p-3, \\ & k+1 \le i \le \frac{p}{2}+\frac{k-1}{2} \end{cases}$$

$$f_2(v_i^j v_{p-i}^j) = q-n\left[1+\sum_{r=1}^{i}(p-r-1)\right]+j, 1 \le i \le \frac{p}{2}-1, 1 \le j \le n.$$

where

$$A = q - n\left[\sum_{r=1}^{i-1}(p-r)\right]$$

$$B = q - n\left[\sum_{r=1}^{i-1}(p-r-1)\right]$$

$$C = q - n\left[\sum_{r=1}^{i}(p-r)\right]$$

Then the vertex weights are

$$w_2(c) = \sum_{i=1}^{p-1} f_2(cv_i^j), 1 \le j \le n$$
$$w_2(v_1^j) = f_2(cv_1^j) + \sum_{k=1}^{p-2} f_2(v_1^j v_{1+k}^j), 1 \le j \le n$$
$$w_2(v_i^j) = \sum_{k=1}^{i-1} f_2(v_{i-k}^j v_i^j) + f_2(cv_i^j) + \sum_{k=1}^{p-2} f_2(v_i^j v_{i+k}^j), 2 \le i \le p-1, 1 \le j \le n.$$

Thus $\chi_{la}(F_p^n(K_p)) \le p.$ Since $\chi(K_p) = p,$ it follows that we get $\chi_{la}(F_p^n(K_p)) \ge p$ and hence $\chi_{la}(F_p^n(K_p)) = p.$ □

Example 2.6 : The generalized friendship graph $F_6^5(K_6)$ admits a local vertex antimagic labeling with 6 colors. The following five matrices gives the vertex weights of $F_6^5(K_6)$. Here $w_2(c) =$ the sum of all 5^{th} column entries of the j^{th} matrix $= 825,$ $1 \le j \le 5.$ The vertex weight $w_2(v_i^j)$ is the i^{th} row sum of the j^{th} matrix $=$ $w_2(v_1^j) = 315,$ $w_2(v_2^j) = 230,$ $w_2(v_3^j) = 175,$ $w_2(v_4^j) = 140,$ $w_2(v_5^j) = 115,$ $1 \le j \le 5.$

$$\begin{bmatrix} 68 & 65 & 56 & 51 & 75 \\ 68 & 41 & 36 & 35 & 50 \\ 65 & 41 & 25 & 18 & 26 \\ 56 & 36 & 25 & 10 & 13 \\ 51 & 35 & 18 & 10 & 1 \end{bmatrix} \begin{bmatrix} 70 & 62 & 59 & 52 & 72 \\ 70 & 44 & 37 & 32 & 47 \\ 62 & 44 & 22 & 20 & 27 \\ 59 & 37 & 22 & 7 & 15 \\ 52 & 32 & 20 & 7 & 4 \end{bmatrix} \begin{bmatrix} 67 & 64 & 57 & 53 & 74 \\ 67 & 42 & 38 & 34 & 49 \\ 64 & 42 & 24 & 17 & 28 \\ 57 & 38 & 24 & 9 & 12 \\ 53 & 34 & 17 & 9 & 2 \end{bmatrix}$$

$$\begin{bmatrix} 69 & 61 & 60 & 54 & 71 \\ 69 & 45 & 39 & 31 & 46 \\ 61 & 45 & 21 & 19 & 29 \\ 60 & 39 & 21 & 6 & 14 \\ 54 & 31 & 19 & 6 & 5 \end{bmatrix} \begin{bmatrix} 66 & 63 & 58 & 55 & 73 \\ 66 & 43 & 40 & 33 & 48 \\ 63 & 43 & 23 & 16 & 30 \\ 58 & 40 & 23 & 8 & 11 \\ 55 & 33 & 16 & 8 & 3 \end{bmatrix}$$

Example 2.7 : The generalized friendship graph $F_{10}^{3}(K_{10})$ admits a local vertex antimagic labeing with 10 colors. The following three matrices gives the vertex weights of $F_{10}^{3}(K_{10})$. Here $w_2(c) =$ the sum of all the 9^{th} column entries of the j^{th} matrix $= 1458,\ 1 \le j \le 3.$ The vertex weight $w_2(v_i^j)$ is the i^{th} row sum of the j^{th} matrix $= w_2(v_1^j) = 1098,\ w_2(v_2^j) = 903,\ w_2(v_3^j) = 750,$ $w_2(v_4^j) = 633,\quad w_2(v_5^j) = 546,\ w_2(v_6^j) = 483,\ w_2(v_7^j) = 438,\quad w_2(v_8^j) = 405,$ $w_2(v_9^j) = 378,\ 1 \le j \le 3.$

$$\begin{bmatrix} 131 & 129 & 124 & 123 & 118 & 117 & 112 & 109 & 135 \\ 131 & 103 & 102 & 97 & 96 & 91 & 88 & 87 & 108 \\ 129 & 103 & 79 & 78 & 73 & 70 & 69 & 65 & 84 \\ 124 & 102 & 79 & 58 & 55 & 54 & 50 & 48 & 63 \\ 123 & 97 & 78 & 58 & 42 & 38 & 36 & 31 & 43 \\ 118 & 96 & 73 & 55 & 42 & 27 & 22 & 21 & 29 \\ 117 & 91 & 70 & 54 & 38 & 27 & 15 & 10 & 16 \\ 112 & 88 & 69 & 50 & 36 & 22 & 15 & 6 & 7 \\ 109 & 87 & 65 & 48 & 31 & 21 & 10 & 6 & 1 \end{bmatrix}$$

$$\begin{bmatrix} 132 & 127 & 126 & 121 & 120 & 115 & 114 & 110 & 133 \\ 132 & 105 & 100 & 99 & 94 & 93 & 89 & 85 & 106 \\ 127 & 105 & 81 & 76 & 75 & 71 & 67 & 66 & 82 \\ 126 & 100 & 81 & 60 & 56 & 52 & 51 & 46 & 61 \\ 121 & 99 & 76 & 60 & 40 & 39 & 34 & 33 & 44 \\ 120 & 94 & 75 & 56 & 40 & 25 & 24 & 19 & 30 \\ 115 & 93 & 71 & 52 & 39 & 25 & 13 & 12 & 18 \\ 114 & 89 & 67 & 51 & 34 & 24 & 13 & 4 & 9 \\ 110 & 85 & 66 & 46 & 33 & 19 & 12 & 4 & 3 \end{bmatrix}$$

$$\begin{bmatrix} 130 & 128 & 125 & 122 & 119 & 116 & 113 & 111 & 134 \\ 130 & 104 & 101 & 98 & 95 & 92 & 90 & 86 & 107 \\ 128 & 104 & 80 & 77 & 74 & 72 & 68 & 64 & 83 \\ 125 & 101 & 80 & 59 & 57 & 53 & 49 & 47 & 62 \\ 122 & 98 & 77 & 59 & 41 & 37 & 35 & 32 & 45 \\ 119 & 95 & 74 & 57 & 41 & 26 & 23 & 20 & 28 \\ 116 & 92 & 72 & 53 & 37 & 26 & 14 & 11 & 17 \\ 113 & 90 & 68 & 49 & 35 & 23 & 14 & 5 & 8 \\ 111 & 86 & 64 & 47 & 32 & 20 & 11 & 5 & 2 \end{bmatrix}$$

Theorem 2.8 : *For the generalized friendship graph* $F_4^n(K_4)$ *of complete graph on* 4 *vertices, where* $n \geq 1$ *is even, then* $\chi_{la}(F_4^n(K_4)) = 4.$

Proof : Let $V(F_4^n(K_4)) = \{c \cup v_i^j, 1 \leq i \leq 3, 1 \leq j \leq n\}$ be the vertex set of $F_4^n(K_4)$ with order $3n+1,$ where n is even and let $E(F_p^n(K_4)) = \{cv_i^j \cup v_i^j v_{i+k}^j, 1 \leq i \leq 3, 1 \leq j \leq n, 1 \leq k \leq 2\}$ be the edge set of $F_4^n(K_4)$ with size $|E| = q = 6n.$ Let c be the central vertex of $F_4^n(K_4).$

Define the edge labeling $f_3 : E(F_p^n(K_4)) \to \{1, 2, \ldots, 6n\}$ by

$$f_3(cv_i^j) = \begin{cases} 4p - i + 1, & i = 1, 1 \leq j \leq n \\ 2j - 1, & i = 2, 1 \leq j \leq n \\ 3p - i + 1, & i = 3, 1 \leq j \leq n \end{cases}$$

$$f_3(v_i^j v_{i+k}^j) = \begin{cases} 6p - i + 1, & i = 1, k = 1, 1 \leq j \leq n \\ 2j, & i = 1, k = 2, 1 \leq j \leq n \\ 5p - i + 1, & i = 2, k = 1, 1 \leq j \leq n. \end{cases}$$

Clearly, f_3 is a local antimagic labeling of $F_4^n(K_4)$ with weights as follows

$$w_3(v_i) = \begin{cases} 10n + 2, & i = 1, 1 \leq j \leq n \\ 11n + 1, & i = 2, 1 \leq j \leq n \\ 8n + 2. & i = 3, 1 \leq j \leq n \end{cases}$$

$$w_3(c) = 2(7n+1).$$

Thus $\chi_{la}(F_4^n(K_4)) \leq 4.$ Since $\chi(K_4) = 4,$ it follows that we get $\chi_{la}(F_4^n(K_4)) \geq 4$ and hence $\chi_{la}(F_4^n(K_4)) = 4.$ □

Theorem 2.9 : *For the generalized friendship graph $F_6^n(K_6)$ of complete graph on 6 vertices, where $n \geq 1$ is even, then $\chi_{la}(F_6^n(K_6)) = 6.$*

Proof : Let $V(F_6^n(K_6)) = \{c \cup v_i^j, 1 \leq i \leq 5, 1 \leq j \leq n\}$ be the vertex set of $F_6^n(K_6)$ with order $5n+1,$ where n is even and let $E(F_6^n(K_6)) = \{cv_i^j \cup v_i^j v_{i+k}^j, 1 \leq i \leq 5, 1 \leq j \leq n, 1 \leq k \leq 4\}$ be the edge set of $F_6^n(K_6)$ with size $|E| = q = 15n.$ Let c be the central vertex of $F_6^n(K_6).$

Define the edge labeling $f_4 : E(F_6^n(K_6)) \to \{1, 2, \ldots, 15n\}$ by

$$f_4(v_i^j v_{i+k}^j) = \begin{cases} np+j, & i=1, k=1, 1 \leq j \leq n \\ n(p-2-j+1), & i=1, k=2, 1 \leq j \leq n \\ n\left[\frac{p(p-1)}{2}\right], & i=1, k=3, 1 \leq j \leq n \\ 2j, & i=1, k=4, 1 \leq j \leq n \\ pn-j+1, & i=2, k=1, 1 \leq j \leq n \\ 2j-1, & i=2, k=2, 1 \leq j \leq n \\ n\left[\frac{p(p-1)}{2}-1\right], & i=2, k=3, 1 \leq j \leq n \\ n(p-2)+j, & i=3, k=1, 1 \leq j \leq n \\ n(p+4)+2j-1, & i=3, k=2, 1 \leq j \leq n \\ n(2p+1)-j+1,, & i=4, k=1, 1 \leq j \leq n \end{cases}$$

$$f_4(cv_i^j) = \begin{cases} (p+3)n-j+1, & i=1, 1 \leq j \leq n \\ (p+4)n-j+1, & i=2, 1 \leq j \leq n \\ (p+2)n-j+1, & i=3, 1 \leq j \leq n \\ pn-j+1, & i=4, 1 \leq j \leq n \\ 2(pn-j+1), & i=5, 1 \leq j \leq n \end{cases}$$

Clearly, f_4 is a local antimagic labeling of $F_6^n(K_6)$ with weights as follows,

$$w_4(v_i^j) = \begin{cases} n(p^2-2)+3, & i=1, 1 \le j \le n \\ np^2+2, & i=2, 1 \le j \le n \\ n(p^2-p+2)+2, & i=3, 1 \le j \le n \\ np^2-n+2, & i=4, 1 \le j \le n \\ n(p^2+2p+1)+3, & i=5, 1 \le j \le n. \end{cases}$$

$$w_4(c) = np(p+1).$$

Thus $\chi_{la}(F_6^n(K_6)) \le 6.$ Since $\chi(K_6) = 6,$ it follows that we get $\chi_{la}(F_6^n(K_6)) \ge 6$ and hence $\chi_{la}(F_6^n(K_6)) = 6.$ □

3. Generalized Friendship Graph of Cycles

In this section we switch the study to the generalized friendship graph $F_p^n(C_p)$ of cycles C_p, where $p \ge 3.$

Observation 3.1 : Let $G \cong F_{p_i}(H_i),\ 1 \le i \le n$ be a generalized friendship graph, where H_i is isomorphic to a cycle on p_i vertices with p_i odd. Then $\chi_{la}(G) \ge 3.$

Proof : Since $\chi(C_p) = 3$ when *p* is odd, it follows that we get $\chi_{la}(G) \ge 3.$ □

Lemma 3.2 : *Let $G = F_p^n(C_p)$ be the generalized friendship graph of cycle on p vertices with p is even. Then $\chi_{la}(F_p^n(C_p)) \ge 3.$*

Proof : Suppose $\chi_{la}(F_p^n(C_p)) = 2,$ where *p* is even. Then there exits a local antimagic labeling *f* that induces a 2–coloring of $F_p^n(C_p).$ Let c_1 be the color of the central vertex *c* and $v_i^j,$ if *i* is odd, $1 \le i \le p,$ $1 \le j \le n$ and let c_2 be the color of the vertices $v_i^j,$ if *i* is even, $2 \le i \le p-1,$ $1 \le j \le n.$ Clearly, $\frac{np}{2}$ vertices receive a c_1 color and $\frac{np}{2}-1$ vertices receive a color c_2.

Therefore

$$\begin{aligned} 2\sum_{uv \in E(G)} f(e) &= \sum_{i=1}^{p} w(v_i) \\ 2\sum_{i=1}^{q} i &= \sum_{i=1}^{p} w(v_i), \text{where } f(e_i) = i, 1 \le i \le q \\ 2\left(\frac{q(q+1)}{2}\right) &= \left(\frac{np}{2}-1\right)c_1 + \frac{np}{2}c_2 \end{aligned}$$

$$\left(\frac{np}{2}-1\right)c_1+\frac{np}{2}c_2=np(np+1) \tag{1}$$

If $G \cong F_p^n(C_p)$, p is odd then by Observation 3.1, we get $\chi_{la}(F_p^n(C_p)) \geq 3$. Since the number of c_2 color is $\frac{np}{2}$, it follows that we get only one possible c_2-color is $q+1=np+1$ and hence $c_1=np+3+\frac{6}{np-2}$. Since c_1 is an integer, it follows that we get either $n=1$ and $p=8$ or $n=2$ and $p=4$. If $n=1$ and $p=8$ then $c_1=12$ and the graph $G^*=F_8^1(C_8)\cong C_8$. From Observation 3.1, we get $\chi_{la}(G^*)\geq 3$. If $n=2$ and $p=4$ then $c_1=12$. Let $G^{**}=F_4^2(C_4)$ be a graph and let $V(G^{**})=\{c,v_1^1,v_2^1,v_3^1,v_1^2,v_2^2,v_3^2\}$. Since $c_1=12$ and $c_2=9$, it follows that we get $w(v_1^2)=w(v_2^2)=12$ and hence the incident edge labels of the vertex v_1^2 and v_2^2 is $f(v_1^1v_2^1)=4$, $f(v_2^1v_3^1)=8$ and $f(v_1^2v_2^2)=5$, $f(v_2^2v_3^2)=7$. Thus the edges cv_1^1, cv_3^1, cv_1^2 and cv_3^2 with edge labels $\{1,2,3,6\}$ and hence $w(c)=12$. Since $c_2=9$ and $c_1=12$, it follows that atleast one vertex from $V^*=\{v_1^1,v_3^1,v_1^2,v_3^2\}$ cannot get the vertex weight is 9, which is a contradiction. Thus $\chi_{la}(F_p^2)\geq 3$ and hence $\chi_{la}(F_p^n)\geq 3$, where p is even. □

Theorem 3.3 : *Let $F_p^n(C_p)$ be the generalized friendship graph of cycle on $p\geq 3$ vertices, then $\chi_{la}(F_p^n(C_p))=3$.*

Proof : Let $G=F_p^n(C_p)$ be the generalized friendship graph of cycles on $p\geq 3$ vertices and let $V(G)=\{c\cup v_i^j, 1\leq i\leq p-1, 1\leq j\leq n\}$ and $E(G)=\{cv_1^j\cup cv_{p-1}^j\cup v_i^jv_{i+1}^j, 1\leq i\leq p-2, 1\leq j\leq n\}$ be the vertex set and edge set of G. Then $|V(G)|=n(p-1)+1$ and $|E(G)|=q=np$.

Define $f_5: E(G)\to\{1,2,\ldots,|E(G)|\}$ by

$$f_5(cv_i^j)=\begin{cases} j, & i=1, 1\leq j\leq n\\ n\left(\frac{p+2}{2}\right)-j+1, & i=p-1 \text{ is odd}, 1\leq j\leq n\\ n\left(\frac{p-1}{2}\right)+j, & i=p-1 \text{ is even}, 1\leq j\leq n\end{cases}$$

$$f_5(v_i^jv_{i+1}^j)=\begin{cases} n\left[\frac{2p-i+1}{2}\right]+1-j, & i \text{ is odd}, 1\leq i\leq p-3, 1\leq j\leq n\\ \frac{ni}{2}+j, & i \text{ is even}, 1\leq i\leq p-2, 1\leq j\leq n\end{cases}$$

Clearly f_5 is a local antimagic labeling of G and

$$w_5(c)=\begin{cases} n\left[n\left(\frac{p+2}{2}\right)+1\right], & p \text{ is even} \\ \left[\frac{n^2(p-1)}{2}\right]+n(n+1), & p \text{ is odd} \end{cases}$$

$$w_5(v_i^j)=\begin{cases} np+1, & i\equiv 1,3(\text{mod } 4) \text{ and } 1\le j\le n \\ n(p+1)+1, & i\equiv 0,2(\text{mod } 4) \text{ and } 1\le j\le n \end{cases}$$

Thus $\chi_{la}(G)\le 3$. From Observation 3.1 and Lemma 3.2, we get $\chi_{la}(F_p^n(C_p))\ge 3$ and hence $\chi_{la}(F_p^n(C_p))=3$. □

Theorem 3.4 : *The generalized friendship graph* $F_{p_1,p_2,p_3,\dots,p_n}$ *of cycles on* $p_1,p_2,p_3,\dots,p_n$ *vertices with* $p_1=3$ *and* $p_2=p_3=\ldots=p_n=p\ge 4$. *Then* $\chi_{la}(F_{p_1,p_2,p_3,\dots,p_n})=3$.

Proof : Let $G=F_{p_1,p_2,p_3,\dots,p_n}$ be the generalized friendship graph of cycles on $p_1,p_2,p_3,\dots,p_n$ vertices with $p_1=3$ and $p_2=p_3=\ldots=p_n=p\ge 4$ is even, $|V(G)|=n(p-1)+3$ and $|E(G)|=q=np+3$. Let $V(G)=\{c,u_1,u_2\cup v_i^j,1\le i\le p-1,1\le j\le n\}$ be the vertex set and let $E(G)=\{cu_1,cu_2\cup u_1u_2\cup cv_1^j\cup cv_{p-1}^j\cup v_i^jv_{i+1}^j,1\le i\le p-2,1\le j\le n\}$ be the edge set of G.

Define $f:E(G)\to\{1,2,\dots,q=np+3\}$ by

$$f(u_1u_2)=1$$

$$f(cu_i)=\begin{cases} q-1, & i=1 \\ q, & i=2 \end{cases}$$

$$f(cv_i^j)=\begin{cases} \frac{p(j-1)+4}{2}, & i=1,1\le j\le n, p \text{ is even and } j,\ p \text{ is odd} \\ \frac{p(2n-j+1)+3}{2}, & i=1,1\le j\le n, j \text{ is even, } p \text{ is odd} \\ \frac{p(2n-j)+4}{2}, & i=p-1,1\le j\le n, p \text{ is even and } j \text{ is even, } p \text{ is odd} \\ \frac{pj+3}{2}, & i=p-1,1\le j\le n, j,\ p \text{ is odd} \end{cases}$$

$$f(v_i^j v_{i+1}^j) = \begin{cases} \frac{p(2n-j+1)-i+3}{2}, & i \text{ is odd, } p \text{ is even and } i,\ j \text{ is even, } p \text{ is odd,} \\ & 1 \le j \le n, 1 \le i \le p-2 \\ \frac{p(j-1)+i+4}{2}, & i,\ p \text{ is even and } i,\ p \text{ is odd, } j \text{ is even and} \\ & i \text{ is even, } j,\ p \text{ is odd}, 1 \le j \le n, 1 \le i \le p-2 \\ \frac{p(2n-j+1)-i-1}{2}, & i, j, p \text{ is odd}, 1 \le j \le n, 1 \le i \le p-2 \end{cases}$$

Clearly, f is a local antimagic labeling of G and the vertex weights of G as follows:

$$w(u_i) = \begin{cases} q, & i \text{ is odd} \\ q+1, & i \text{ is even} \end{cases}$$

$$w(v_i^j) = \begin{cases} q, & i \text{ is odd, } p \text{ is even and } i, j, p \text{ is odd and } i, \\ & j \text{ is even, } p \text{ is odd}, 1 \le j \le n, 1 \le i \le p-1 \\ q+1, & i, p \text{ is even and } i \text{ is even, } j,\ p \text{ is odd and } i, \\ & p \text{ is odd, } j \text{ is even}, 1 \le j \le n, 1 \le i \le p-1 \end{cases}$$

$$w(c) = \begin{cases} 2n(p+1) + n^2 p - n\left(\frac{p-4}{2}\right) + 5, & p \text{ is even} \\ 2n(p+1) + n^2 p - n\left(\frac{p-3}{2}\right) + 5, & p \text{ is odd} \end{cases}$$

Therefore $\chi_{la}(G) \le 3$. Since G contains a triangle, from Observation 3.1 and Lemma 3.2, we get $\chi_{la}(F_{p_1, p_2, p_3, \ldots, p_n}) \ge 3$ and hence $\chi_{la}(F_{p_1, p_2, p_3, \ldots, p_n}) = 3$. □

4. Concusion and Scope

In this paper, we obtain the local antimagic chromatic number of generalized friendship graph $F_p^n(K_p)$ of complete graphs on p vertices, when n is odd, $p \ge 3$ and when n is even, $p = 4, 6$. Also, we determine completely the local antimagic chromatic number of generalized friendship graph $F_p^n(C_p)$ of cycle on p vertices. Moreover, we prove that $\chi_{la}(F_{p_1, p_2, p_3, \ldots, p_n}) = 3$, where $p_1 = 3$ and $p_2 = p_3 = \ldots = p_n = p \ge 4$. This leads naturally the following problem for further study.

Problem 4.1 : Determine the local antimagic chromatic number of the generalized friendship graph $F_p^n(K_p)$ of complete graphs on $p \ge 8$ vertices, where p is even and n is even.

We suggest furthermore to study a closely related problem that calculating the local antimagic chromatic numbers of the disjoint union of regular graphs as in [2], in particular that of nK_p the disjoint union of n copies of complete graphs on p vertices.

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