

***I*-prime fuzzy submodules**

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Abstract

In this paper, we introduce a new generalization of prime F.S.M (P.F.S), called *I*- prime F.S.M and *I* be any ideal of a commutative ring *R* also, we study some properties of it and giving some characterizations of *I*-P.F.S. We establish the situation of *I*-P.F.Ss in F.M of the *R*-module and, we show that *I*-P.F.S and P.F.S are coincident under certain conditions.

Subject Classification: 03B52, 03E72, 08A72.

Keywords: Prime fuzzy submodule, Weakly prime fuzzy submodule and *I*-prime submodule.

1. Introduction

All through this paper, *R* is a nonzero identity commutative ring and *I* will be a proper ideal of *R*, and *X* will be a unitary *R*-module. Lotfi A. Zadeh proposed the fuzzy set concept in 1965 to mathematically describe ambiguity in its most abstract form, and to address such issues, he assigned a grade of participation to each member from a particular

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set. In [1] introduced the concept of the fuzzy subset. In [2] presented the idea of fuzzy ideals of the ring and prime fuzzy ideals of a ring. In [3] introduced the concept of F.S.M and P.F.S. Prime ideals assume a focal part in commutative ring theory. The idea of I -prime submodules, or the arrangement of a prime submodule of R -module X , has recently been investigated by [4]. In this paper, we introduce and study some basic facts about I -P.F.S of the R -module, and explore the fundamental properties and some theorems, and propositions [5,6].

2. Preliminaries

We will go through several well-known definitions of fuzzy ideals and F.S.Ms.

Definition 2.1 [1]: Let B be a nonempty set, $[0,1]$ is closed interval. A function from B into $[0,1]$ is a fuzzy set C in B (a fuzzy subset C of B).

Definition 2.2 [7]: Let $x_e : B \rightarrow [0,1]$ be a fuzzy subset of B , for all $x \in B$, $e \in [0,1]$ defined by $x_e(k) = \begin{cases} e & \text{if } b=x \\ 0 & \text{if } b \neq x \end{cases}$ for all $k \in B$. In B , x_e is known as a fuzzy singleton or fuzzy point.

Definition 2.3 [1]: In B , if K and H are two fuzzy sets, then:

- a) $K = H$ if and only if $K(x) = H(x)$ for all $x \in B$.
- b) $K \subseteq H$ if and only if $K(x) \leq H(x)$ for all $x \in B$.
- c) $(K \cap H)(x) = \min\{K(x), H(x)\}$ for all $x \in B$.
- d) $(K \cup H)(x) = \max\{K(x), H(x)\}$ for all $x \in B$, and $(K \cap H), (K \cup H)$ are fuzzy subsets in B .

Definition 2.4 [1]: Let $f : X \rightarrow N$ where X and N any sets, with S and H being fuzzy subsets of X and N , respectively. The fuzzy set in N is defined by:

$$f(S)(h) = \begin{cases} \sup\{S(d) | d \in f^{-1}(h)\} & \text{if } f^{-1}(h) \neq \emptyset \\ 0 & \text{otherwise} \end{cases} \text{ for all } h \in N \text{ and } f^{-1}(h) = \{m \in X, f(m) = h\}.$$

The image of S is designated by $f(S)$, the inverse image of H , designated by $f^{-1}(H)$ in the fuzzy set in X defined by: $f^{-1}(H)(x) = H(f(x))$ for all $x \in X$.

Definition 2.5 [8]: Let $f : X \rightarrow N$ where X and N any sets. A fuzzy subset S of X is called f -invariant if $S(a) = S(b)$ whenever $f(a) = f(b)$, where $a, b \in X$.

Definition 2.6 [9]: Let X be R -module. A fuzzy subset A of X is called F.M of R -module X if:

1. $A(c - k) \geq \min \{A(c), A(k)\}$ for all $c, k \in M$.
2. $A(r.c) \geq A(c)$ for all $r \in R$ and $c \in M$.
3. $A(0) = 1$ for all $c, k \in M$.

Definition 2.7 [9]: Let H and K are two F.Ms of R -module X . K referred to as an F.S.M of H , if $K \subseteq H$.

Definition 2.8 [9]: Let A_1 and A_2 are two F.Ms of R -modules X_1 and X_2 respectively. $f : A_1 \rightarrow A_2$ is called a fuzzy homomorphism if $f : X_1 \rightarrow X_2$ is R -homomorphism and $A_2(f(x)) = A_1(x)$, for each $x \in X_1$.

Definition 2.9 [10]: Let H be F.M of R -module X and Y is proper F.S.M of H . Then Y is called P.F.S of H if $r_e y_e \subseteq Y$ for fuzzy singleton r_e of R and $y_e \subseteq H$, then either $r_e \subseteq [Y : H]$ or $y_e \subseteq Y$.

Definition 2.10 [11]: A non-constant F.S.M Y of H of R -module X . Y is called weakly prime, if for fuzzy singleton r_e and s_e of R and $y_e \subseteq H$, such that $r_e.s_e.y_e \subseteq Y$, then either $r_e.y_e \subseteq Y$ or $s_e.y_e \subseteq Y$. [12]

3. I- Prime F.S.Ms

We present the possibility of I -P.F.S and analyze a portion of its fundamental attributes in this section.

Definition 3.1: Let H be a fuzzy module is short (F.M) of X and let K is a proper fuzzy submodule is short (F.S.M) of X is called I -P.F.S of X if $r_e m_e \subseteq K - IK$ for fuzzy singleton r_e of R and $m_e \subseteq H$ implies that either $m_e \subseteq K$ or $r_e \subseteq [K : H]$.

Remark and Example 3.2: Every prime and weakly P.F.S is I -P.F.S although this is not always the case, as we can see in the following example. The rings Z and Z_{12} as Z -module. Take $J = 4Z$ as the ideal of Z and $P = (4)$ is the submodule of Z_{12} generated by Then P is an I -P.S of Z_{12} .

Theorem 3.3: Let K be an I -Prime. Then K is prime if $[K : H]K \not\subseteq IK$.

Corollary 3.4: Let K be 0-P.F.S of H such that $[K : H]K \neq 0$. Then H is a P.F.S of H .

Theorem 3.5: Let H be an F.M of X and K is proper F.S.M of H . Then the following statements are equivalent:

1. K is I-P.F.S of H .
2. For $r_e \subseteq R - [K : H]$ then $[K :_H r_e] = K \cup [IK :_H r_e]$.
3. For $r_e \subseteq R - [K : H]$ then $[K :_H r_e] = [IK :_H r_e]$ or $[K :_H r_e] = K$.

Proof: $1 \rightarrow 2$

Let K be an I-prime. We take $r_e \subseteq R - [K : H]$ and $m_e \subseteq [K :_H r_e]$ for fuzzy singleton r_e of R and $m_e \subseteq H$, and so $r_e m_e \subseteq K$. If $r_e m_e \not\subseteq IK$, then K is I-prime and $m_e \subseteq K$. If $r_e m_e \subseteq IK$, then $m_e \subseteq [K :_H r_e]$.

$2 \rightarrow 3$ It's clear that.

$3 \rightarrow 1$

Let $r_e m_e \subseteq K - IK$. If $r_e \not\subseteq [K : H]$, then by hypothesis $[K :_H r_e] = [IK :_H r_e]$ or $[K :_H r_e] = K$, since $r_e m_e \not\subseteq IK$ and $m_e \subseteq [IK :_H r_e]$. But $m_e \subseteq [K :_H r_e]$ which means that $[K :_H r_e] \neq [IK :_H r_e]$. Hence $[K :_H r_e] = K$, and $m_e \subseteq K$. Therefore, K is I-P.F.S of H .

Corollary 3.6: Let K be I-P.F.S of H and $IK \subseteq [K : H]^2 K$. Then K is F-prime where $F = \bigcap_{n=1}^{\infty} [K : H]^n$.

Corollary 3.7: Let H be a fuzzy multiplication module is short (F.M.M) of X and K be an I-P.F.S of H . If K is not prime, then $K^2 \subseteq IK$.

Corollary 3.8: Let H be an F.M.M of X and K be an I-P.F.S of H . Then $K \subseteq \sqrt{IK}$ or $\sqrt{IK} \subseteq K$.

Theorem 3.9: Let H be an F.M of X and K be an F.S.M of H . Then K is I-P.F.S in H if and only if K / IK is weakly P.F.S in H / IK .

Proof: Suppose that K is I-prime in H . Let for fuzzy singleton r_e of R and $m_e \subseteq H$ within H / IK . Then $r_e m_e \subseteq K - IK$ implies $m_e \subseteq K$ or $r_e \subseteq [K : H]$, hence $r_e \subseteq [K : H] = [K / IK : H / IK]$ or $m_e + IK \subseteq K / IK$. This K / IK is weakly P.F.S in H / IK . Conversely, Suppose that K / IK is weakly P.F.S in H / IK such that $r_e m_e \subseteq K - IK$, then $0 \neq r_e m_e + IK = r_e (m_e + IK) \subseteq \frac{K}{IK}$ and so $(m_e + IK) \subseteq \frac{K}{IK}$ or $r_e \subseteq [K / IK : H / IK] = [K : H]$. Therefore, K is I-P.F.S of H .

Lemma 3.10: Let H be an F.M.M of X and K be an I-P.F.S of H such that $[K : H] \subseteq I$. Then $\sqrt{[IK : H]}K = IK$.

Proposition 3.11: Let H_1, H_2 are F.Ms of X and $H = H_1 \oplus H_2$. $K = K_1 \oplus K_2$ is I-P.F.S of H such that then K_1, K_2 are I-P.F.Ss in H_1 and H_2 , respectively.

Lemma 3.12: Let H be a fuzzy faithful multiplication module is short (F.F.M.M) of an R -module M . K be an F.S.M of H , P are finitely generated faithful multiplication is short (F.G.F.M) ideal of R . Then:

- i. $K = [PK : P]$.
- ii. If $K \subseteq PH$, then $[LK : P] = L[K : P]$ for any ideal's L of R .

Theorem 3.13: Let H be an F.F.M.M of X and K be an F.S.M of H , J are (F.G.F.M) ideal of R . Then H is I-P.F.S of JH if and only if $[K : J]$ is I-P.F.S in H .

Theorem 3.14: Let H be a fuzzy (F.G.F.M.M) of X and K be a proper F.S.M of H . Then the following statements are equivalents:

1. K is I-P.F.S in H .
2. $[K : H]$ is I-prime ideal in R .
3. $K = JH$ for some I-prime ideal J of R .

Theorem 3.15: Let H be a fuzzy (F.G.F.M.M) of X and K are proper F.S.M of H such that $I[K : H] = [IK : H]$. Then K is I-P.F.S in H if and only if for any two F.S.Ms A and B of H with $A.B \subseteq K$ and $A.B \not\subseteq IK$ implies that either $A \not\subseteq K$ or $B \not\subseteq K$.

Corollary 3.16: Let H be a fuzzy (F.G.F.M.M) of X , K is proper F.S.M of H such that $I[K : H] = [IK : H]$. Then K is I-P.F.S in H if and only if whenever for any fuzzy singleton r_e, s_e of R such that $r_e s_e \subseteq K - IK$ implies $r_e \subseteq [K : H]$ or $s_e \subseteq [K : H]$.

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Received February, 2022

Revised March, 2022