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Computing Y-index for some special graph

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Abstract

The topological index has transformed the chemical structure into a numeric number and they make a connection between chemistry and mathematics. In this paper, we compute Y-index for some special graphs in which it was named m^k-graph and Jahangir graph. Also, we compute this topological index for subdivided m^k-graph of path graph, comb graph, ladder graph, and triangular ladder graph. Finally, we calculate Y-index for another kind of special graph in which it was named Jahangir graph.

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1. Introduction

We recommend [5], [8], [9] and [11] for general symbols and ideas in graph theory. In this paper, we have assumed the simple graphs, connected with no directed or weighted edges. For instance, $G = (V, E)$, we assume the number of vertices and edges are p, q and denote by $|V| = p$, $|E| = q$. The symbol $d_G(u)$ represents the degree of a vertex u which is the number of vertices is adjacent to u. There are a lot of topological indices that are studied. In Article [7], you can see how to calculate these types of indices to obtain some chemical properties of materials. Among these topological indices, the most significant in which scientists were

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interested in studying, is Zagreb indices that correlate with molecules and their chemical properties. Gutman and Trinajsti [8] introduced the First and Second Zagreb index which were determined as

$$M_1(G)=\sum_{v\in V(G)} d_G^2(v)=\sum_{uv\in E(G)}[d_G(u)+d_G(v)], M_2(G)=\sum_{uv\in E(G)} d(u)d(v).$$

In [10], F- index was introduced and defined as

$$F(G)=\sum_{u\in V(G)} d^3(u)=\sum_{uv\in E(G)}(d^2(u)+d^2(v)).$$

In [1], Zheng and Li instituted the first general Zagreb index of a graph as follows

$$M_1^{\alpha+1}(G)=\sum_{v\in V(G)} d_G^{\alpha+1}(v)=\sum_{uv\in E(G)}(d_G^{\alpha}(u)+d_G^{\alpha}(v)) \text{ for } \alpha\in R.$$

In [3], Alameria and et al named the first general Zagreb index, Y-Index where $\alpha=3.$ Therefor, the $Y-$index of a graph was defined as

$$Y(G)=\sum_{u\in V(G)} d_G^4(v)=\sum_{uv\in E(G)}[d_G^3(u)\ +d_G^3(v)].$$

Also in [4], Alameria and et al defined the $Y-$coindex as

$$\overline{Y}(G)\ =\sum_{u\notin V(G)} d_G^4(v)=\sum_{uv\notin E(G)}[d_G^3(u)\ +d_G^3(v)].$$

In this article the relationship among the Y-index and its complement index, Y-co index was given and some properties of this index for special graph like complete graph K_n, cycle graph C_n and complete bipartite graph $K_{m,n}$ was computed.

In [2], Ayache and Alameri have defined a new kind of graph which was called m^k-graph and was shown by $m^k(G)$. This type of graph obtained from a simple graph G by k successive iterations of a special kind of construction. Consider an integer $m\geq 2,$ if we have m copies of graph G such as $G_1,G_2,\ldots,G_m$ in which each vertex u of a copy G_i is adjacent to the corresponding vertex v in another copy G_j then the graph mG is created. The complete graph K_m is a mG graph when the graph G has one vertex. In the following way, we consider a non-negative integer k and like to define

$$m^0(G)=G,\ \mathrm{m}^1(G)=mG,\ \mathrm{m}^2(G)=m(m(G)),\ldots,m^{k+1}(G)=m(m^k(G)).$$

Let as before $G=(V,E)$ in which $|V|=p,\ |E|=q.$ By calculating the total number of vertices and edges in m^k-graph one can obtain

$|V(m^k(G)| = m^k p,\ |E((m^k(G)| = m^k q + \binom{m}{2} pkm^{k-1}$. Also we define the degree of each vertex in $m^k(G)$, $d_{m^k(G)} = d_G(v) + k(m-1)$.

Another definition we like to present is subdivision graph which was derived from a graph *G*. Subdivision graph, which is shown by $S(G)$, If we omit the edge uv in graph *G* and add a new vertex *w* and two new edge *uw* and *wv* in original graph then we have $S(G)$. So $V(S) = V(G) \cup E(G)$.

Now we describe another kind of graph which was named Jahangir graph. The Jahangir graph is a connected and semi-regular graph and is showed by $J_{n,m}$. Let for all $n \geq 2$ and $m \geq 3$, Jahangir Graph $J_{n,m} = (V, E)$ in which $|V| = nm+1$, $|E| = m(n+1)$, is a graph consisting a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.

You can see [6] to get more acquainted with this type of graph.

Wang and et al. computed Weiner index and Hosoya Polynomial of $J_{2,m}, J_{3,m}$ and $J_{4,m}$ in [12].

In this paper, we compute $Y-$index for $m^k(G)$ and its subdivided graph of path graph, comb graph, ladder graph and triangular graph. Also, we compute $Y-$index for Jahangir graph.

2. Y-index for m^k-graph

Later, Ayache and Alameri studied on m^k-graph and found explicit formulas of some topological indices such as Wiener index, Schultz index for this kind of graph. In this section, we want to compute the $Y-$index for this kind of graph.

Theorem 1 : *Let consider a graph $G = (V, E)$ in which $|V| = p$, $|E| = q$. Then the $Y-$index of $m^k(G)$ is*

$$Y(m^kG) = m^k[Y(G) + 4k(m-1)F(G) + 6k^2(m-1)^2 M_1(G) + 4k^3(m-1)^3 q + k^4(m-1)^4 p].$$

Proof : By considering the definition of $Y-$index we have

$$Y(m^kG) = \sum_{v \in V(m^kG)} d^4_{m^kG}(v) = m^k \sum_{v \in V(G)} (d_G(v) + k(m-1))^4$$
$$= m^k[Y(G) + 4k(m-1)F(G) + 6k^2(m-1)^2 M_1(G) + 4k^3(m-1)^3 q + k^4(m-1)^4 p].$$

So the desired result is achieved.

We achieved the following corollary if we consider $k = 1$.

Corollary 1:

$$Y(mG) = mY(G) + 8\binom{m}{2}F(G) + 12\binom{m}{2}(m-1)M_1(G) + 8\binom{m}{2}(m-1)^2 q$$
$$+2\binom{m}{2}(m-1)^3 p.$$

In the following, we desire to calculate $Y-$index of m^k-graph of some kind of graphs like cube graph, complete graph and cycle graph. By the definition of m^k-graph, if G has only one vertex and $m=2$, then we have cube graph Q_k of dimension k and also by the definition of m^k-graph, mG is the complete graph K_m, if G has only one vertex.

Example 1 : By considering the definition of m^k-graph, cube graph, complete graph and cycle graph, we achieved these results:

1) $Y(Q_k) = k^4 2^k$.
2) $Y(K_m) = m(m-1)^4$.
3) $Y(mC_n) = 16mn + 32mn(m-1) + 24mn(m-1)^2 + 4mn(m-1)^3 + mn(m-1)^4$.

3. Y-index for subdivided m^k-graph of path and comb graph

In this section, we want to calculate $Y-$index for a graph in which was derived from graph G. We defined the subdivision graph. Now we desire to compute $Y-$index for subdivided m^k-graph of path graph and comb graph. Let as before the number of copies is showed by m and the number of vertices is showed by n.

Theorem 2 : *For $k=1$, consider the graph G is a graph of subdivided m^k-graph of path graph. Then the $Y-$index of this graph is*

$$Y(G) = 2m^5 + m(n-2)(m+1)^4 + 16\left(m(n-1) + \binom{m}{2}n\right).$$

Proof : At first we give the vertex partition of subdivided m^k-graph generation by P_n for $k=1$.

Its vertex set can be partitioned into three sets E_1, E_2 and E_3 such that

$E_1 = \{u | d_G(u) = 2\}$, $|E_1| = m(n-1) + \binom{m}{2} n$, & $E_2 = \{u | d_G(u) = m\}$, $|E_2| = 2m$,
$E_3 = \{u | d_G(u) = m+1\}$, $|E_3| = m(n-2)$.

Therefore,

$$Y(G) = \sum_{v \in V(G)} d_G^4(v) = \left(m(n-1) + \binom{m}{2} n \right) 2^4 + 2m.m^4 + m(n-2)(m+1)^4$$
$$= 2m^5 + m(n-2)(m+1)^4 + 16\left(m(n-1) + \binom{m}{2} n \right).$$

Theorem 3 : *For* $k = 1$, *consider the graph G is a graph of subdivided* m^k*-graph of comb graph. Then the* $Y-$*index of this graph is*

$$Y(G) = nm^5 + 2m(m+1)^4 + m(n-2)(m+2)^4 + 16\left(m(2n-1) + 2\binom{m}{2} n \right).$$

Proof : At first we give the vertex partition of subdivided m^k-graph generation by comb graph for $k = 1$.

Its vertex set can be partitioned into four sets E_1, E_2, E_3 and E_4 such that

$E_1 = \{u | d_G(u) = 2\}$, $|E_1| = m(2n-1) + 2\binom{m}{2} n$, &
$E_2 = \{u | d_G(u) = m\}$, $|E_2| = mn$, &
$E_3 = \{u | d_G(u) = m+1\}$, $|E_3| = 2m$, &
$E_4 = \{u | d_G(u) = m+2\}$, $|E_4| = m(n-2)$.

Therefore,

$$Y(G) = \sum_{v \in V(G)} d_G^4(v) = \left(m(2n-1) + 2\binom{m}{2} n \right) 2^4 + mn.m^4 + 2m(m+1)^4$$
$$+ m(n-2)(m+2)^4$$
$$= \mathrm{nm}^5 + 2m(m+1)^4 + m(n-2)(m+2)^4 + 16\left(m(2n-1) + 2\binom{m}{2} n \right).$$

4. Y-index for subdivided m^k-graph of ladder graph and triangular graph

In this section, we want to calculate Y-index for a graph in which was derived from graph G. We defined the subdivision graph. Now we desire to compute $Y-$index for subdivided m^k-graph of ladder graph and triangular graph. Let as before the number of copies is showed by m and the number of vertices is showed by n.

Theorem 4 : *For* $k=1$, *consider the graph G is a graph of subdivided* m^k*-graph of ladder graph. Then the* $Y-$*index of this graph is*

$$Y(G)=4m.(m+1)^4+2m(n-2)(m+2)^4+16\left(m(3n-2)+2\binom{m}{2}n\right).$$

Proof : At first we give the edge partition and vertex partition of subdivided m^k-graph generation by ladder graph for $k=1$.

Its vertex set can be partitioned into three sets E_1, E_2 and E_3 such that

$$E_1=\left\{u\middle|d_G(u)=2\right\},\quad \left|E_1\right|=m(3n-2)+2\binom{m}{2}n,\quad \&$$

$$E_2=\left\{u\middle|d_G(u)=m+1\right\},\ \left|E_2\right|=4m,\ \&$$

$$E_3=\left\{u\middle|d_G(u)=m+2\right\},\ \left|E_3\right|=2m(n-2).$$

Therefore,

$$Y(G)=\sum_{v\in V(G)}d_G^4(v)=\left(m(3n-2)+2\binom{m}{2}n\right)2^4+4m(m+1)^4+2m(n-2)(m+2)^4$$

$$=\ 4m(m+1)^4+2m(n-2)(m+2)^4+16\left(m(3n-2)+2\binom{m}{2}n\right).$$

Theorem 5 : *For* $k=1$, *consider the graph G is a graph of subdivided* m^k*-graph of the triangular ladder graph. Then the* $Y-$*index of this graph is*

$$Y(G)=m^4\left(m(4n-3)+2\binom{m}{2}n\right)+2m(m+1)^4+2m(m+2)^4+2m(n-2)(m+3)^4.$$

Proof : At first we give the vertex partition of subdivided m^k-graph generation by comb graph for $k=1$.

Its vertex set can be partitioned into four sets E_1, E_2, E_3 and E_4 such that

$$E_1 = \{u | d_G(u) = m\}, \ |E_1| = m(4n-3) + 2\binom{m}{2}n, \quad \&$$
$$E_2 = \{u | d_G(u) = m+1\}, \ |E_2| = 2m, \ \&$$
$$E_3 = \{u | d_G(u) = m+2\}, \ |E_3| = 2m, \quad \&$$
$$E_4 = \{u | d_G(u) = m+3\}, \ |E_4| = 2m(n-2).$$

Therefore,

$$Y(G) = \sum_{v \in V(G)} d_G^4(v) = \left(m(4n-3) + 2\binom{m}{2}n \right) m^4 + 2m(m+1)^4 + 2m(m+2)^4$$
$$+ 2m(n-2)(m+3)^4.$$

5. Y-index for Jahangir graph

In this section, we compute Y-index for Jahangir graph. The Jahangir graph is a connected and semi-regular graph and is showed by $J_{n,m}$. Let for all $n \geq 2$ and $m \geq 3$, Jahangir Graph $J_{n,m} = (V, E)$ in which $|V| = nm+1, |E| = m(n+1)$, is a graph consisting a cycle C_{nm} with one additional vertex which is adjacent to m vertices of C_{nm} at distance n to each other on C_{nm}.

Theorem 6 : *Let consider a graph $J_{2,m} = (V, E)$ in which $|V| = 2m+1$, $|E| = 3m$. Then the $Y-$index of $J_{2,m}$ is $Y(J_{2,m}) = m^4 + 97m$.*

Proof : At first, we give the vertex partition of this kind of Jahangir graph. Its vertex set can be partitioned into three sets E_1, E_2 and E_3 such that

$$E_1 = \{u | d_G(u) = m\}, \ |E_1| = 1 \quad \& \quad E_2 = \{u | d_G(u) = 3\}, \ |E_2| = m,$$
$$E_3 = \{u | d_G(u) = 2\}, \ |E_3| = m.$$

So

$$Y(J_{2,m}) = \sum_{v \in V(J_{2,m})} d_{J_{2,m}}^4(v) = 1.m^4 + m.3^4 + m.2^4 = m^4 + 97m.$$

Theorem 7 : *Let consider a graph $J_{3,m} = (V, E)$ in which $|V| = 3m+1$, $|E| = 4m$. Then the $Y-$index of $J_{3,m}$ is $Y(J_{3,m}) = m^4 + 113m$.*

Proof : At first, we give the vertex partition of this kind of Jahangir graph. Its vertex set can be partitioned into three sets E_1, E_2 and E_3 such that

$$E_1 = \{u \mid d_G(u) = m\}, \quad |E_1| = 1, \quad \& \quad E_2 = \{u \mid d_G(u) = 3\}, \quad |E_2| = m,$$
$$E_3 = \{u \mid d_G(u) = 2\}, \quad |E_3| = 2m.$$

So

$$Y(J_{3,m}) = \sum_{v \in V(J_{3,m})} d^4_{J_{3,m}}(v) = 1m^4 + m3^4 + 2m2^4 = m^4 + 113m.$$

Theorem 8 : *Let consider a graph* $J_{4,m} = (V, E)$ *in which* $|V| = 4m+1, \ |E| = 5m.$ *Then the* $Y-$*index of* $J_{4,m}$ *is* $Y(J_{4,m}) = m^4 + 129m.$

Proof : At first, we give the vertex partition of this kind of Jahangir graph. Its vertex set can be partitioned into three sets E_1, E_2 and E_3 such that

$$E_1 = \{u \mid d_G(u) = m\}, \ |E_1| = 1, \ \& \ E_2 = \{u \mid d_G(u) = 3\}, \ |E_2| = m,$$
$$E_3 = \{u \mid d_G(u) = 2\}, \ |E_3| = 3m.$$

So

$$Y(J_{4,m}) = \sum_{v \in V(J_{4,m})} d^4_{J_{4,m}}(v) = 1.m^4 + m.3^4 + 3.m.2^4 = m^4 + 129m.$$

Theorem 9 : *Let consider a graph* $J_{5,m} = (V, E)$ *in which* $|V| = 5m+1, \ |E| = 6m.$ *Then the* $Y-$*index of* $J_{5,m}$ *is* $Y(J_{5,m}) = m^4 + 145m.$

Proof : At first, we give the vertex partition of this kind of Jahangir graph. Its vertex set can be partitioned into three sets E_1, E_2 and E_3 such that

$$E_1 = \{u \mid d_G(u) = m\}, \ |E_1| = 1, \ \& \ E_2 = \{u \mid d_G(u) = 3\}, \ |E_2| = m,$$
$$E_3 = \{u \mid d_G(u) = 2\}, \ |E_3| = 4m.$$

So

$$Y(J_{5,m}) = \sum_{v \in V(J_{5,m})} d^4_{J_{5,m}}(v) = 1.m^4 + m.3^4 + 4.m.2^4 = m^4 + 145m.$$

Finally, we compute the $Y-$index in general case of Jahangir graph, $J_{n,m}$ for all $n \geq 2$ and $m \geq 3$.

Theorem 10 : *Let consider a graph* $J_{n,m} = (V, E)$ *in which* $|V| = nm+1$, $|E| = m(n+1)$. *Then the* $Y-$ *index of* $J_{n,m}$ *is* $Y(J_{n,m}) = \mathrm{m}^4 + 97m + 16m(n-2)$.

Proof : At first, we give the edge partition of this kind of Jahangir graph. Its edge set can be partitioned into three sets E_1, E_2 and E_3 such that

$$E_1 = \left\{uv \middle| d(u) = 2 \ \& \ d(v) = 2\right\}, \ \ |E_1| = m(n-2), \ \&$$
$$E_2 = \left\{uv \middle| d(u) = 2 \ \& \ d(v) = 3\right\}, \ \ |E_2| = 2m, \ \&$$
$$E_3 = \left\{uv \middle| d(u) = 3 \ \& \ d(v) = m\right\}, \ \ |E_3| = m. \ \&$$

So

$$\begin{aligned} Y(J_{n,m}) &= \sum_{uv \in E(J_{n,m})} (d^3_{J_{n,m}}(u) + d^3_{J_{n,m}}(v)) = m(n-2)[2^3 + 2^3] \\ &+ 2m[2^3 + 3^3] + m[3^3 + m^3] \\ &= m^4 + 97m + 16m(n-2). \end{aligned}$$

6. Conclusion

In this paper, we consider a kind of graph that was named m^k-graph and compute a new topological index, $Y-$index, for this graph. Then, we try to find the $Y-$index for subdivided m^k-graph of a path graph, comb graph, ladder graph and triangular ladder graph. Finally, we compute Y-index for another kind of graph which is named Jahangir graph.

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