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Complex intuitionistic fuzzy soft lattice ordered group associated with ℓ-ideal

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Abstract

In this paper, we study the concept of complex intuitionistic fuzzy soft convex ℓ-group ($\mathcal{CIFSCL}$-$\mathcal{G}$). We examine some of its innate algebraic properties theoretically. Moreover, we define the notion of complex intuitionistic fuzzy soft ℓ-ideal ($\mathcal{CIFSL}$-$\mathcal{I}$). We analyze the relationship between the $\mathcal{CIFSCL}$-$\mathcal{G}$, $\mathcal{CIFSL}$-$\mathcal{I}$ structures and corresponding structures in complex intuitionistic fuzzy soft lattice ordered group ($\mathcal{CIFSL}$-$\mathcal{G}$).

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1. Introduction

Rosenfeld [19] applied the concept of fuzzy set theory [25] to group theory and then established the notion of fuzzy subgroups. Swamy et al. [22] introduced the notion of fuzzy prime ideals of rings and then Naseem Ajmal [2] introduced the idea of fuzzy lattices. Then Saibaba [20] studied the concept of L-fuzzy ℓ-groups and L-fuzzy ℓ-ideals, where L is a complete lattice. The convex subgroup plays a crucial role in the

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analysis of positive cones of an ordered group and compatible orders, mainly in characterizing the ℓ-subgroups generated by a subsemigroup. So, Saibaba [21] defined the concepts of L-fuzzy convex sub ℓ-groups. Many researchers have worked on various fuzzy extensions with lattice theory [4, 11, 12, 13, 24 23].

Buckley [6], in 1989, integrated the ideas of complex numbers and fuzzy numbers under the topic fuzzy complex numbers. This led to the origination of the complex fuzzy set (CFS) model by Ramot et al. [18]. Then the development and extension of the CFS theory to group theory such as complex fuzzy group [1], complex fuzzy soft group [9] are introduced. Alkouri et al. [3] initiated the concept of complex intuitionistic fuzzy sets (CIFSs), which is generalized from the concept of CFS by adding the non-membership term and also defined the relations on CIFSs. M. Gulzar et al. [10] introduced the concept of complex intuitionistic fuzzy subgroups, which is generated by two intuitionistic fuzzy subgroups. Kumar et al. [8] initiated the distance and entropy measures on complex intuitionistic fuzzy soft sets (CIFSSs). Quek et al. [14] studied the algebraic structures pertaining to groups on CIFSSs environment. In the recent past, the extensions of CIFSs and CIFSSs have been developed under various uncertain environments. Recently, the extension of the CIFSS theory to ℓ-group is introduced which is known as complex intuitionistic fuzzy soft lattice ordered group ($\mathcal{CIFSL}$-$\mathcal{G}$) [16, 17]. Further, some new operations on $\mathcal{CIFSL}$-$\mathcal{G}$ and CIFS-COPRAS method are presented in [15].

The main contribution of this work includes:

Many research works on CIFS theory are growing tremendously. In the aforementioned studies, we can find algebraic properties of convex ℓ-subgroup and ℓ-ideal over a fuzzy sets, whereas the incorporation of convex ℓ-subgroup and ℓ-ideal of a $\mathcal{CIFSL}$-$\mathcal{G}$ have yet to be explored. In this paper, we develop the theory of complex intuitionistic fuzzy soft convex ℓ-group ($\mathcal{CIFSCL}$-$\mathcal{G}$) and complex intuitionistic fuzzy soft ℓ-ideal ($\mathcal{CIFSL}$-$\mathcal{I}$) as a continuation of $\mathcal{CIFSL}$-$\mathcal{G}$ which was introduced in [16, 17]. In the context of this paper, the concept of $\mathcal{CIFSL}$-$\mathcal{G}$, convex ℓ-subgroup and ℓ-ideal are used as a base to introduce the theory of $\mathcal{CIFSCL}$-$\mathcal{G}$ and $\mathcal{CIFSL}$-$\mathcal{I}$.

This paper is organized as follows: Section 2 contains some relevant basic concepts. Section 3 provides the notion of $\mathcal{CIFSCL}$-$\mathcal{G}$ and also explore the algebraic results of $\mathcal{CIFSCL}$-$\mathcal{G}$. Section 4 defines the $\mathcal{CIFSL}$-$\mathcal{I}$ and studies its algebraic properties.

2. Preliminaries

Definition 2.1 : [7] A system $G = (G, \circ, \leq)$ is called a lattice ordered group (ℓ - group), if

(i) $(G, \circ)$ is a group,

(ii) $(G, \leq)$ is a lattice,

(iii) $x \leq y \Rightarrow a \circ x \circ b \leq a \circ y \circ b,$ for all $a, b, x, y \in G.$

Definition 2.2 : [7] If x is an element of ℓ-group G, then $|x|$ is said to be the absolute value of x and is defined by $|x| = x \vee x^{-1}$. Any element x of G can be represented as $x = x^{+}x^{-}$, where $x^{+} = x \vee e_G$ is a positive part of x and $x^{-} = x \wedge e_G$ is a negative part of x and e_G is the identity element of G.

Definition 2.3 : [7] A subgroup H of a ℓ- group G is called a convex ℓ-subgroup if and only if $|x| \leq |u| \Rightarrow x \in H,$ for every $u \in H, x \in G.$

Definition 2.4 : [7] A normal convex ℓ- subgroup of ℓ- group is called an lattice ordered ideal (ℓ- ideal).

Definition 2.5 : [5] If G and H are ℓ - groups and $\varphi : G \to H$ is a group homomorphism from G into H such that $(x \vee y)\varphi = x\varphi \vee y\varphi$, $(x \wedge y)\varphi = x\varphi \wedge y\varphi$, for all $x, y \in G$, then φ is called an ℓ - homomorphism or an homomorphism of ℓ - groups

Definition 2.6: [8] Let E be a set of parameters, $CIFS(U)$ denotes the collection of all complex intuitionistic fuzzy sets on U and $\tilde{F}$ be a function from E to $CIFS(U)$. Then the set of ordered pairs $\{(\varepsilon, \tilde{F}(\varepsilon)) : \varepsilon \in E, \tilde{F}(\varepsilon) \in CIFS(U)\}$, denoted by $(\tilde{F}, E)$, is a complex intuitionistic fuzzy soft set $(CIFSS)$ on U. For all $\varepsilon \in E$,

$$\tilde{F}(\varepsilon) = \{(x, \mu_{\tilde{F}(\varepsilon)}(x), \nu_{\tilde{F}(\varepsilon)}(x)) : x \in U\} = \{(x, r_{\tilde{F}(\varepsilon)}(x)e^{iw_{\tilde{F}(\varepsilon)}(x)}, \tau_{\tilde{F}(\varepsilon)}(x)e^{i\psi_{\tilde{F}(\varepsilon)}(x)}) : x \in U\}$$

where $\mu_{\tilde{F}(\varepsilon)} : U \to \{a : a \in \mathbb{C}, |a| \leq 1\}$ and $\nu_{\tilde{F}(\varepsilon)} : U \to \{a' : a' \in \mathbb{C}, |a'| \leq 1\}$, $|\mu_{\tilde{F}(\varepsilon)}(x) + \nu_{\tilde{F}(\varepsilon)}(x)| \leq 1.$

Definition 2.7 : [14] Let $(\widetilde{F}, E) \in$ CIFSS(U) and S be a non-null proper subset of U. Then $(\widetilde{F}, E)$ is called characteristic over S, if $\mu_0 = \{\mu_{\tilde{F}(a)} : a \in E\}$ and $\nu_0 = \{\nu_{\tilde{F}(a)} : a \in E\}$, in which

$$\mu_0(x) = \begin{cases} re^{iw}, & x \in S \\ 1 \sim re^{iw}, & x \in U - S \end{cases} \text{ and } \nu_0(x) = \begin{cases} \tau e^{i\psi}, & x \in S \\ 1 \sim \tau e^{i\psi}, & x \in U - S, \end{cases}$$

where $re^{iw} \geq \tau e^{i\psi}$, $w+\psi \in \{2\pi, 4\pi\}$ and $1 \sim re^{iw} = (1-r)e^{iw'}$ in which $w' = \begin{cases} 2\pi - w, & w<2\pi \\ w, & w=2\pi \end{cases}$.

2.1 *Complex Intuitionistic Fuzzy Soft Lattice Ordered Group*

Throughout this paper $G = G(.,\vee,\wedge)$ is a lattice ordered group, $\mathcal{O}_1 = \{x \in \mathbb{C} \,|\, |x| \leq 1\}$ and

(i) $\mu \geq \nu$, if both $r \geq \tau$ and $w \geq \psi$ and

(ii) $\mu \leq \nu$, if both $r \leq \tau$ and $w \leq \psi$, where $\mu = re^{iw}$ and $\nu = \tau e^{i\psi}$, with $r, \tau \in [0,1]$ and $w, \psi \in (0, 2\pi]$.

Definition 2.8 : [16] Let $(\tilde{F}, E) \in CIFSS$ over G Then $(\tilde{F}, E)$ is said to be a complex intuitionistic fuzzy soft lattice ordered group $(\mathcal{CIFSL}\text{-}\mathcal{G})$ over G if for all $a \in \rho(\tilde{F}, E)$, and $x, y, e_G \in G$,

(i) $\mu_{\tilde{F}(a)}(xy^{-1}) \geq \mu_{\tilde{F}(a)}(x) \wedge \mu_{\tilde{F}(a)}(y)$,

(ii) $\nu_{\tilde{F}(a)}(xy^{-1}) \leq \nu_{\tilde{F}(a)}(x) \vee \nu_{\tilde{F}(a)}(y)$,

(iii) $\mu_{\tilde{F}(a)}(x \vee e_G) \geq \mu_{\tilde{F}(a)}(x)$,

(iv) $\nu_{\tilde{F}(a)}(x \vee e_G) \leq \nu_{\tilde{F}(a)}(x)$,

where $\rho(\tilde{F}, E) = \{a \in E : \tilde{F}(a) \text{ is non-null}\}$ is the support set of $(\tilde{F}, E)$ and e_G is the identity element of G.

Definition 2.9 : [14] Let $(\tilde{F}, E) \in CIFSS(U)$ and $\alpha, \beta \in \mathcal{O}_1$. The (α, β) - level set of $(\tilde{F}, E)$, denoted as $(\tilde{F}, E)_{(\alpha,\beta)}$, is a soft set on U and is defined by $(\tilde{F}, E)_{(\alpha,\beta)} = \{(a, \tilde{F}_{(\alpha,\beta)}(a)) \,|\, a \in E, \tilde{F}_{(\alpha,\beta)}(a) \in P(U)\}$, where $\tilde{F}_{(\alpha,\beta)}(a) = \{x \in U \,|\, \mu_{\tilde{F}(a)}(x) \geq \alpha, \nu_{\tilde{F}(a)}(x) \leq \beta\}$, for all $a \in E$.

3. Complex Intuitionistic Fuzzy Soft Convex Lattice Ordered Group

In this section, we study the complex intuitionistic fuzzy soft convex lattice ordered group and attain its relevant properties.

Definition 3.1 : Let $(\tilde{F}, E) \in CIFSS$ over G. Then $(\tilde{F}, E)$ is said to be a complex intuitionistic fuzzy soft convex lattice ordered group $(\mathcal{CIFSCL}\text{-}\mathcal{G})$ of G if for every $a \in E$ and $x, y \in G$,

(i) $\mu_{\tilde{F}(a)}(xy^{-1}) \geq \mu_{\tilde{F}(a)}(x) \wedge \mu_{\tilde{F}(a)}(y)$ and $\nu_{\tilde{F}(a)}(xy^{-1}) \leq \nu_{\tilde{F}(a)}(x) \vee \nu_{\tilde{F}(a)}(y)$,

(ii) If $|x| \le |y|$ implies $\mu_{\tilde{F}(a)}(x) \ge \mu_{\tilde{F}(a)}(y)$ and $\nu_{\tilde{F}(a)}(x) \le \nu_{\tilde{F}(a)}(y)$.

Proposition 3.2 : A $(\tilde{F},E) \in CIFSS$ of a ℓ - group G is a $\mathcal{CIFSCL}$ - $\mathcal{G}$ of G if and only if for each $a \in E$ and for arbitrary $\alpha, \beta \in \mathcal{O}_1$, with $\tilde{F}_{(\alpha,\beta)}(a) \ne \varnothing$, $\tilde{F}_{(\alpha,\beta)}(a)$ is a convex ℓ - subgroup of G.

Example 1 : Consider the ℓ - group $\mathbb{R}\times\mathbb{R}$ under addition with the ordered relation $\le$ is defined as $(x,y) \le (u,v) \Leftrightarrow x = u, y = v$ or $x \le u, y < v$.

Now, we define the $CIFSS$ $(\tilde{F},E)$ over the ℓ - group $\mathbb{R}\times\mathbb{R}$ by

$$\mu_{\tilde{F}(a)}(x,y) = \begin{cases} 0.7e^{i2\pi(0.5)}, & \text{if } y = 0 \\ 0.6e^{i2\pi(0.3)}, & \text{otherwise,} \end{cases} \quad \nu_{\tilde{F}(a)}(x,y) = \begin{cases} 0.3e^{i2\pi(0.5)}, & \text{if } y = 0 \\ 0.4e^{i2\pi(0.7)}, & \text{otherwise} \end{cases}$$

$$\mu_{\tilde{F}(b)}(x,y) = \begin{cases} 0.6e^{i2\pi(0.5)}, & \text{if } y = 0 \\ 0.4e^{i2\pi(0.3)}, & \text{otherwise,} \end{cases} \quad \nu_{\tilde{F}(b)}(x,y) = \begin{cases} 0.4e^{i2\pi(0.5)}, & \text{if } y = 0 \\ 0.6e^{i2\pi(0.7)}, & \text{otherwise,} \end{cases}$$

for $E = \{a,b\}$ and $(x,y) \in \mathbb{R}\times\mathbb{R}$. The non-empty (α,β) - level set of $(\tilde{F},E)$ is defined by

$$(\tilde{F}, E)_{(\alpha,\beta)} = \{(a, \tilde{F}_{(\alpha,\beta)}(a)) \mid a \in E, \tilde{F}_{(\alpha,\beta)}(a) \in P(\mathbb{R}\times\mathbb{R})\},$$

where $\tilde{F}_{(\alpha,\beta)}(a) = \{(x,0) \in \mathbb{R}\times\mathbb{R} \mid \mu_{\tilde{F}(a)}(x,0) \ge \alpha, \nu_{\tilde{F}(a)}(x,0) \le \beta\}$ for $a \in E$ and $\alpha, \beta \in \mathcal{O}_1$ with $\alpha = \beta = 0.5e^{i2\pi(0.5)}$. It is easy to verify that the $\tilde{F}_{(\alpha,\beta)}(a)$ forms a convex ℓ - subgroup of $\mathbb{R}\times\mathbb{R}$. By Proposition 3.2, we get $(\tilde{F},E) \in \mathcal{CIFSCL}$ - $\mathcal{G}$.

Proposition 3.3 : Let $(\tilde{F},E) \in \mathcal{CIFSCL}$ - $\mathcal{G}$ of G. Then for $x,y \in G$, $|x| \le |y|$ implies $\mu_{\tilde{F}(a)}(x) \ge \mu_{\tilde{F}(a)}(y)$ and $\nu_{\tilde{F}(a)}(x) \le \nu_{\tilde{F}(a)}(y)$, for all $a \in E$ if and only if for $x,y \in G$,

(i) $\mu_{\tilde{F}(a)}(x \vee y) \ge \mu_{\tilde{F}(a)}(x) \wedge \mu_{\tilde{F}(a)}(y)$ and $\nu_{\tilde{F}(a)}(x \vee y) \le \nu_{\tilde{F}(a)}(x) \vee \nu_{\tilde{F}(a)}(y)$,

(ii) $\mu_{\tilde{F}(a)}(x \wedge y) \ge \mu_{\tilde{F}(a)}(x) \vee \mu_{\tilde{F}(a)}(y)$ and $\nu_{\tilde{F}(a)}(x \wedge y) \le \nu_{\tilde{F}(a)}(x) \wedge \nu_{\tilde{F}(a)}(y)$.

Proposition 3.4 : Let $(\tilde{F},E) \in CIFSS(G)$ be a characteristic over S, where S is a non- null proper subset of G. If $(\tilde{F},E) \in \mathcal{CIFSCLG}(G)$, then S is a convex ℓ - subgroup of G.

Proof : It is easy to prove that S is a subgroup of G. We have to prove that S is convex ℓ - subgroup of G. Let $y \in S$ and $x \in G$ with $|x| \le |y|$. Then by Definition 2.7, there exist $\alpha = re^{i\omega}, \beta = \tau e^{i\psi} \in \mathcal{O}_1$ with

$re^{iw} \geq \tau e^{i\psi}$ such that $\mu_{\tilde{F}(a)}(y) = \mu_0(y) = \alpha$ and $\nu_{\tilde{F}(a)}(y) = \nu_0(y) = \beta$, for all $a \in E$. Since, $(\tilde{F}, E) \in \mathcal{CIFSCL}\text{-}\mathcal{G}(G)$, then $\mu_{\tilde{F}(a)}(x) \geq \mu_{\tilde{F}(a)}(y) = \alpha$ and $\nu_{\tilde{F}(a)}(x) \leq \nu_{\tilde{F}(a)}(y) = \beta$.

To prove: S is convex ℓ- subgroup of G. (i.e.,) To prove: $\mu_{\tilde{F}(a)}(x) = \alpha$, $\nu_{\tilde{F}(a)}(x) = \beta$ implies $x \in S$.

(a) Suppose $x \in S$, then the proof is obvious. Suppose $x \notin S$, then $\mu_{\tilde{F}(a)}(x) = 1 \sim \alpha$ implies $\alpha \leq 1 \sim \alpha$, since $\mu_{\tilde{F}(a)}(x) \geq \alpha$. This implies $|1 \sim \alpha| \geq |\alpha|$ and therefore $|1 \sim \alpha| = |\alpha| \Rightarrow 1 - r = r$. Then, we have $1 \sim \alpha = re^{iw'}$.

Case 1 : Suppose $w = 2\pi$, then by Definition 2.7, we get $w' = w$. Hence $1 \sim \alpha = re^{iw} = \alpha$ implies that $\mu_{\tilde{F}(a)}(x) = \alpha$.

Case 2 : Suppose $w < 2\pi$, then by Definition 2.7, we get $w' = 2\pi - w$. Since $re^{iw} \geq \tau e^{i\psi}$ and $w + \psi = 2\pi$, then $w \geq \psi$ and $w \geq \pi$.

Now, $re^{i(2\pi - w)} = 1 \sim \alpha \geq \alpha = re^{iw}$ implies that $2\pi - w \geq w \Rightarrow \pi \geq w$. Thus, $w = \pi$ and therefore $w = w' = \pi$ and $1 \sim \alpha = re^{iw} = \alpha$. Hence $\mu_{\tilde{F}(a)}(x) = \alpha$.

(b) Suppose $x \in S$, then the proof is obvious. Suppose $x \notin S$, then $\nu_{\tilde{F}(a)}(x) = 1 \sim \beta$ implies $\beta \geq 1 \sim \beta$, since $\nu_{\tilde{F}(a)}(x) \leq \beta$. This implies $|1 \sim \beta| \leq |\beta|$ and therefore $|1 \sim \beta| = |\beta| \Rightarrow 1 - \tau = \tau$. Then, we have $1 \sim \beta = \tau e^{i\psi'}$.

Case 1 : Suppose $\psi = 2\pi$, then by Definition 2.7, we get $\psi' = \psi$. Hence $1 \sim \beta = \tau e^{i\psi} = \beta$ implies that $\nu_{\tilde{F}(a)}(x) = \beta$.

Case 2 : Suppose $\psi < 2\pi$, then by Definition 2.7, we get $\psi' = 2\pi - \psi$. Since $re^{iw} \geq \tau e^{i\psi}$ and $w + \psi = 2\pi$, then $w \geq \psi$ and $\psi \leq \pi$.

Now, $\tau e^{i(2\pi - \psi)} = 1 \sim \beta \leq \beta = \tau e^{i\psi}$ implies that $2\pi - \psi \leq \psi \Rightarrow \pi \leq \psi$. Thus, $\psi = \pi$ and therefore $\psi = \psi' = \pi$ and $1 \sim \beta = \tau e^{i\psi} = \beta$. Hence $\nu_{\tilde{F}(a)}(x) = \beta$.

Therefore, $x \in S$, whenever $|x| \leq |y|$ with $x \in G$ and $y \in S$. Hence, S is a convex ℓ- subgroup of G. □

Remark 1 : The converse of Proposition 3.4 is not true.

Let $G = \mathbb{R}(+, \vee, \wedge)$ be a ℓ-group and $S = \mathbb{Z}(+, \vee, \wedge)$ be a convex ℓ-subgroup of G. Let $(\tilde{F}, E)$ be characteristic over S, where $E = \{a, b\}$. Define the $(\tilde{F}, E)$ over $\mathbb{R}$ by

$$\mu_{\tilde{F}(a)}(x)=\begin{cases}1e^{i2\pi}, & \text{if } x\in\mathbb{Z}\\ 0e^{i2\pi}, & \text{if } x\notin\mathbb{Z}\end{cases},\quad \nu_{\tilde{F}(a)}(x)=\begin{cases}0e^{i2\pi}, & \text{if } x\in\mathbb{Z}\\ 1e^{i2\pi}, & \text{if } x\notin\mathbb{Z}\end{cases},$$

$$\mu_{\tilde{F}(b)}(x)=\begin{cases}0.7e^{i\pi}, & \text{if } x\in\mathbb{Z}\\ 0.3e^{i\pi}, & \text{if } x\notin\mathbb{Z}\end{cases},\quad \nu_{\tilde{F}(b)}(x)=\begin{cases}0.3e^{i\pi}, & \text{if } x\in\mathbb{Z}\\ 0.7e^{i\pi}, & \text{if } x\notin\mathbb{Z}\end{cases},$$

where $E=\{a,b\}$. Clearly, $(\tilde{F},E)\notin\mathcal{CIFSCL}\text{-}\mathcal{G}$ of $\mathbb{R}$. Since, for $x(=5.5)$, $y(=8)\in\mathbb{R}\quad\Rightarrow|5.5|\le|8|$. But $\mu_{\tilde{F}(a)}(5.5)=0e^{i2\pi}<1e^{i2\pi}=\mu_{\tilde{F}(a)}(8)$ and $\nu_{\tilde{F}(a)}(5.5)=1e^{i2\pi}>0e^{i2\pi}=\nu_{\tilde{F}(a)}(8)$. So, $(\tilde{F},E)$ is not $\mathcal{CIFSCL}\text{-}\mathcal{G}$ of $\mathbb{R}$.

Remark 2 : Let $(\tilde{F},E)\in CIFSS(G)$ be a characteristic over S, where S is a non- null proper subset of G. Then $(\tilde{F},E)\in\mathcal{CIFSL}\text{-}\mathcal{G}(G)$ if and only if S is a ℓ - subgroup.

Definition 3.5 : Let $\phi:U_1\to U_2$ and $\varphi:E_1\to E_2$ be two functions, where E_1 and E_2 are the parameter sets for $CIFSS$s over the universal sets U_1 and U_2, respectively. Then the pair (ϕ,φ) is called a complex intuitionistic fuzzy soft function.

Definition 3.6 : Let $(\tilde{F}_1,E_1)$, $(\tilde{F}_2,E_2)\in CIFSS$ over U_1 and U_2, respectively and let (ϕ,φ) be a complex intuitionistic fuzzy soft function. Then

(i) The image of $(\tilde{F}_1,E_1)$ under (ϕ,φ), denoted by $(\phi,\varphi)(\tilde{F}_1,E_1)$, is $CIFSS$ over U_2, defined by $(\phi,\varphi)(\tilde{F}_1,E_1)=(\phi(\tilde{F}_1),\varphi(E_1))$ such that

$$\mu_{\phi(\tilde{F}_1)(a_2)}(y)=\begin{cases}\bigvee_{\phi(x)=y}\ \bigvee_{\varphi(a_1)=a_2}\mu_{\tilde{F}_1(a_1)}(x), & \text{if } \phi^{-1}(y)\neq\varnothing\\ 0 & \text{Otherwise}\end{cases}\quad\text{and}$$

$$\nu_{\phi(\tilde{F}_1)(a_2)}(y)=\begin{cases}\bigwedge_{\phi(x)=y}\ \bigwedge_{\varphi(a_1)=a_2}\nu_{\tilde{F}_1(a_1)}(x), & \text{if } \phi^{-1}(y)\neq\varnothing\\ 0, & \text{Otherwise}\end{cases}$$

where $a_2\in\varphi(E_1)$ and $y\in U_2$.

(ii) The pre-image of $(\tilde{F}_2,E_2)$ under (ϕ,φ), denoted by $(\phi,\varphi)^{-1}(\tilde{F}_2,E_2)$, is $CIFSS$ over U_1, defined by $(\phi,\varphi)^{-1}(\tilde{F}_2,E_2)=(\phi^{-1}(\tilde{F}_2),\varphi^{-1}(E_2))$ such that

$$\mu_{\phi^{-1}(\tilde{F}_2)(a_1)}(x)=\mu_{\tilde{F}_1(\varphi(a_1))}\phi(x)\text{ and }\nu_{\phi^{-1}(\tilde{F}_2)(a_1)}(x)=\nu_{\tilde{F}_1(\varphi(a_1))}\phi(x)$$

where $a_1\in\varphi^{-1}(E_2)$ and $x\in U_1$.

If ϕ is a ℓ- homomorphism from G_1 to G_2, then (ϕ,φ) is called a complex intuitionistic fuzzy soft lattice ordered group homomorphism ($\mathcal{CIFS}$ ℓ*-homomorphism*), where G_1 and G_2 are ℓ- groups. If ϕ and φ are injective (surjective), then (ϕ,φ) is said to be injective (surjective).

Definition 3.7 : Let $(\tilde{F}_1,E_1)\in\mathcal{CIFSL}$-$\mathcal{G}$ of G_1 and (ϕ,φ) be complex intuitionistic fuzzy soft function from G_1 to G_2. Then $(\tilde{F}_1,E_1)$ is said to be (ϕ,φ)- invariant if

$$\phi(x)=\phi(y)\Rightarrow \mu_{\tilde{F}1(a_1)}(x)=\mu_{\tilde{F}1(a_1)}(y)\,(i.e.)\,r_{\tilde{F}1(a_1)}(x)=r_{\tilde{F}1(a_1)}(y), w_{\tilde{F}1(a_1)}(x)$$
$$= w_{\tilde{F}1(a_1)}(y) \text{ and}$$
$$\nu_{\tilde{F}1(a_1)}(x)=\nu_{\tilde{F}1(a_1)}(y)\,(i.e.)\,\tau_{\tilde{F}1(a_1)}(x)=\tau_{\tilde{F}1(a_1)}(y), \psi_{\tilde{F}1(a_1)}(x)=\psi_{\tilde{F}1(a_1)}(y).$$

for all $x,y\in G_1$, $a_1\in E_1$.

Proposition 3.8 : Let $(\tilde{F}_1,E_1)$, $(\tilde{F}_2,E_2)\in\mathcal{CIFSCL}$-$\mathcal{G}$ over G_1 and G_2, respectively. Let (ϕ,φ) be a onto $\mathcal{CIFS}$ ℓ*-homomorphism* from G_1 to G_2. Then $(\phi,\varphi)(\tilde{F}_1,E_1)$ is $\mathcal{CIFSCL}$-$\mathcal{G}$ over G_2, if $(\tilde{F}_1,E_1)$ is (ϕ,φ)- invariant.

Proof : We can easily prove the condition (i) in Definition 3.1. For each $a_2\in\varphi(E_1)$, $x_2,y_2\in G_2$, there exists $x_1,y_1\in G_1$ such that $\phi(x_1)=x_2$ and $\phi(y_1)=y_2$. Let $|\phi(x_1)|\le|\phi(y_1)|$, for $\phi(x_1),\phi(y_1)\in G_2$. Since ϕ is ℓ- homomorphism, $|\phi(x_1)|\le|\phi(y_1)|\Rightarrow\phi(|x_1|)\le\phi(|y_1|)\Rightarrow\phi(|y_1|)=\phi(|x_1|)\vee\phi(|y_1|)=\phi(|x_1|\vee|y_1|)$.

We know that, $(\tilde{F}_1,E_1)$ is $(\phi,\varphi)-$invariant, $\mu_{\tilde{F}1(a_1)}(|y_1|)=\mu_{\tilde{F}1(a_1)}(|x_1|\vee|y_1|)$ (1)

Since, $|x_1|\vee|y_1|\ge|x_1|$ and $(\tilde{F}_1,E_1)\in\mathcal{CIFSCL}$-$\mathcal{G}$. Then $\mu_{\tilde{F}1(a_1)}(|x_1|\vee|y_1|)\le\mu_{\tilde{F}1(a_1)}(x_1)$, this implies $\mu_{\tilde{F}1(a_1)}(|y_1|)\le\mu_{\tilde{F}1(a_1)}(x_1)$ [by (1)]. Therefore $\mu_{\tilde{F}1(a_1)}(y_1)\le\mu_{\tilde{F}1(a_1)}(x_1)$. Similarly, we can get $\nu_{\tilde{F}1(a_1)}(y_1)\ge\nu_{\tilde{F}1(a_1)}(x_1)$. Now, $\mu_{\phi(\tilde{F}1)(a_2)}(x_2)=\bigvee_{\phi(x_1)=x_2}\bigvee_{\varphi(a_1)=a_2}\mu_{\tilde{F}1(a_1)}(x_1)\ge\bigvee_{\phi(y_1)=y_2}\bigvee_{\varphi(a_1)=a_2}\mu_{\tilde{F}1(a_1)}(y_1)=\mu_{\phi(\tilde{F}1)(a_2)}(y_2)$. Similarly, we can get $\nu_{\phi(\tilde{F}1)(a_2)}(x_2)\le\nu_{\phi(\tilde{F}1)(a_2)}(y_2)$. Hence, $(\phi,\varphi)(\tilde{F}_1,E_1)$ is $\mathcal{CIFSCL}$-$\mathcal{G}$ over G_2. This completes the proof. □

Proposition 3.9 : Let $(\tilde{F}_1,E_1)$, $(\tilde{F}_2,E_2)\in\mathcal{CIFSCL}$-$\mathcal{G}$ over G_1 and G_2, respectively. Let (ϕ,φ) be a onto $\mathcal{CIFS}$ ℓ*-homomorphism* from G_1 to G_2. Then $(\phi,\varphi)^{-1}(\tilde{F}_2,E_2)$ is $\mathcal{CIFSCL}$-$\mathcal{G}$ over G_1.

Proof : We can easily prove the condition (i) in Definition 3.1. For each $a_1 \in \varphi^{-1}(E_2)$, $x_1, y_1 \in G_1$, there exists $x_2, y_2 \in G_2$ such that $\phi^{-1}(x_2) = x_1$ and $\phi^{-1}(y_2) = y_1$. Let $|x_1| \le |y_1|$, for $x_1, y_1 \in G_1$. This implies $|\phi(x_1)| \le |\phi(y_1)|$. Since, (ϕ, φ) is a onto $\mathcal{CIFS}$ ℓ*-homomorphism*, $\mu_{\phi^{-1}(\tilde{F}_1)(a_1)}(x_1) = \mu_{\tilde{F}_1(\varphi(a_1))}(\phi(x_1)) \ge \mu_{\tilde{F}_1(\varphi(a_1))}(\phi(y_1)) = \mu_{\phi^{-1}(\tilde{F}_1)(a_1)}(y_1)$. Similarly, $\nu_{\phi^{-1}(\tilde{F}_1)(a_1)}(x_1) \le \nu_{\phi^{-1}(\tilde{F}_1)(a_1)}(y_1)$. Hence, $(\phi, \varphi)^{-1}(\tilde{F}_2, E_2)$ is $\mathcal{CIFSCL}$-$\mathcal{G}$ over G_1.

□

4. Complex Intuitionistic Fuzzy Soft Lattice Ordered Ideal

In this section, we extend complex intuitionistic fuzzy soft lattice ordered ideal and its related characteristics.

Definition 4.1 : Let $(\tilde{F}, E) \in CIFSS$ over G. Then $(\tilde{F}, E)$ is said to be a complex intuitionistic fuzzy soft lattice ordered ideal $(\mathcal{CIFSL}$-$\mathcal{I})$ of G if for every $a \in E$, and $x, y \in G$,

(i) $\mu_{\tilde{F}(a)}(xy^{-1}) \ge \mu_{\tilde{F}(a)}(x) \wedge \mu_{\tilde{F}(a)}(y)$ and $\nu_{\tilde{F}(a)}(xy^{-1}) \le \nu_{\tilde{F}(a)}(x) \vee \nu_{\tilde{F}(a)}(y)$,

(ii) $\mu_{\tilde{F}(a)}(xy) = \mu_{\tilde{F}(a)}(yx)$ and $\nu_{\tilde{F}(a)}(xy) = \nu_{\tilde{F}(a)}(yx)$,

(iii) If $|x| \le |y|$ implies $\mu_{\tilde{F}(a)}(x) \ge \mu_{\tilde{F}(a)}(y)$ and $\nu_{\tilde{F}(a)}(x) \le \nu_{\tilde{F}(a)}(y)$.

In other words, a normal complex intuitionistic fuzzy soft convex lattice ordered group of G is a $\mathcal{CIFSL}$-$\mathcal{I}$ of G.

Remark 3 : Every $\mathcal{CIFSL}$-$\mathcal{I}$ is an $\mathcal{CIFSL}$-$\mathcal{G}$. But the converse is not true.
In the example of Remark 1, we can easily get $(\tilde{F}, E)$ is $\mathcal{CIFSL}$-$\mathcal{G}$, but not $\mathcal{CIFSL}$-$\mathcal{I}$.

Proposition 4.2 : A $(\tilde{F}, E) \in CIFSS$ of a ℓ - group G is a $\mathcal{CIFSL}$-$\mathcal{I}$ of G if and only if for each $a \in E$ and for arbitrary $\alpha, \beta \in \mathcal{O}_1$, with $\tilde{F}_{(\alpha,\beta)}(a) \ne \emptyset$, $\tilde{F}_{(\alpha,\beta)}(a)$ is a ℓ - ideal of G.

Proof : We can easily prove from the Proposition 3.2 and Definition 4.1. □

Definition 4.3 : [15] Let $(\tilde{F}_1, E_1)$ and $(\tilde{F}_2, E_2) \in \mathcal{CIFSL}$-$\mathcal{G}(G)$. Then their sum $(\tilde{F}_1, E_1) \oplus (\tilde{F}_2, E_2)$ is defined as: $(\tilde{F}_1, E_1) \oplus (\tilde{F}_2, E_2) = \{(c, (\tilde{F}_1 \oplus \tilde{F}_2)(c)) : c = (a, b) \in E_1 \times E_2\}$, where

$$\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(x)=\sup_{x=p\vee q}\{min\{r_{\tilde{F}1(a)}(p),r_{\tilde{F}2(b)}(q)\}e^{i\,min\{w_{\tilde{F}1(a)}(p),w_{\tilde{F}2(b)}(q)\}}\} \text{ and}$$

$$\nu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(x)=\inf_{x=p\vee q}\{max\{\tau_{\tilde{F}1(a)}(p),\tau_{\tilde{F}2(b)}(q)\}e^{i\,max\{\psi_{\tilde{F}1(a)}(p),\psi_{\tilde{F}2(b)}(q)\}}\}, \text{ for all } x\in G.$$

Proposition 4.4 : The sum of two $\mathcal{CIFSL}$ - $\mathcal{I}$ of G is again a $\mathcal{CIFSL}$ - $\mathcal{I}$ of G.

Proof : Let $(\tilde{F}_1,E_1)$ and $(\tilde{F}_2,E_2)\in\mathcal{CIFSL}$ - $\mathcal{I}(G)$. We can easily prove the conditions (i) and (ii) in Definition 4.1. Let $x,y\in G$ such that $|x|\le|y|$. So, $x^+=x\vee e_G\le|x|\le|y|\Rightarrow x^+\le|y|=|y_1\vee y_2|\le|y_1|\vee|y_2|$, where e_G is the identity element. There exist positive elements $x_1,x_3\le|y_1|$ and $x_2\le|y_2|$ such that $x^+=x_1\vee x_2\vee x_3$.

$$\begin{aligned}\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(y)&=\sup_{y=y_1\vee y_2}\{min\{r_{\tilde{F}1(a)}(y_1),r_{\tilde{F}2(b)}(y_2)\}e^{i\,min\{w_{\tilde{F}1(a)}(y_1),w_{\tilde{F}2(b)}(y_2)\}}\}\\&=\sup_{y=y_1\vee y_2}\{min\{\mu_{\tilde{F}1(a)}(y_1),\mu_{\tilde{F}2(b)}(y_2)\}\}\\&\le\sup_{x^+=x_1\vee x_2\vee x_3}\{min\{\mu_{\tilde{F}1(a)}(x_1),\mu_{\tilde{F}2(b)}(x_2),\mu_{\tilde{F}2(b)}(x_3)\}\}\\&\le\sup_{x^+=(x_1\vee x_2)\vee x_3}\{min\{\mu_{\tilde{F}1(a)}(x_1\vee x_2),\mu_{\tilde{F}2(b)}(x_3)\}\}\\&\le\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(x^+).\end{aligned}$$

Similarly, we can show that $\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(y)\le\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(((x^{-1})^+)^{-1})$. Now, $\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(y)\le\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(x^+)\wedge\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(((x^{-1})^+)^{-1})\le\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}((x^+)((x^{-1})^+)^{-1})$ $=\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(x)$, since $x=(x^+)((x^{-1})^+)^{-1}$. Therefore, $\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(y)\le\mu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(x)$, whenever $|x|\le|y|$. Similarly, we can prove that $\nu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(y)\ge\nu_{(\tilde{F}1\oplus\tilde{F}2)(c)}(x)$. Hence, $(\tilde{F}_1,E_1)\oplus(\tilde{F}_2,E_2)\in\mathcal{CIFSL}$ - $\mathcal{I}(G)$. □

Definition 4.5 : [15] Let $(\tilde{F}_1,E_1)$ and $(\tilde{F}_2,E_2)\in\mathcal{CIFSL}$ - $\mathcal{G}(G)$. Then their product $(\tilde{F}_1,E_1)\otimes(\tilde{F}_2,E_2)$ is defined as: $(\tilde{F}_1,E_1)\otimes(\tilde{F}_2,E_2)=\{(c,(\tilde{F}_1\otimes\tilde{F}_2)(c)):c=(a,b)\in E_1\times E_2\}$, where

$$\mu_{(\tilde{F}1\otimes\tilde{F}2)(c)}(x)=\sup_{x\le p\wedge q}\{min\{r_{\tilde{F}1(a)}(p),r_{\tilde{F}2(b)}(q)\}e^{i\,min\{w_{\tilde{F}1(a)}(p),w_{\tilde{F}2(b)}(q)\}}\} \text{ and}$$

$$\nu_{(\tilde{F}1\otimes\tilde{F}2)(c)}(x)=\inf_{x\le p\wedge q}\{max\{\tau_{\tilde{F}1(a)}(p),\tau_{\tilde{F}2(b)}(q)\}e^{i\,max\{\psi_{\tilde{F}1(a)}(p),\psi_{\tilde{F}2(b)}(q)\}}\}, \text{ for all } x\in G.$$

Proposition 4.6 : The product of two $\mathcal{CIFSL}\text{-}\mathcal{I}$ of G is again a $\mathcal{CIFSL}\text{-}\mathcal{I}$ of G.

Proof : Proof for this proposition is similar to Proposition 4.4. □

Proposition 4.7 : Let $(\tilde{F},E) \in CIFSS(G)$ be a characteristic over S, where S is a non- null proper subset of G. If $(\tilde{F},E) \in \mathcal{CIFSL}\text{-}\mathcal{I}(G)$, then S is an ℓ-ideal of G. But the converse is not true.

Proof : The proof is obvious from Proposition 3.4 and Remark 1. □

Proposition 4.8 : Let $(\tilde{F}_1,E_1)$, $(\tilde{F}_2,E_2) \in \mathcal{CIFSL}\text{-}\mathcal{I}$ over G_1 and G_2, respectively. Let (ϕ,φ) be a onto $\mathcal{CIFS}\ \ell\text{-}homomorphism$ from G_1 to G_2. Then

(i) $(\phi,\varphi)(\tilde{F}_1,E_1)$ is $\mathcal{CIFSL}\text{-}\mathcal{I}$ over G_2, if $(\tilde{F}_1,E_1)$ is (ϕ,φ)- invariant.
(ii) Then $(\phi,\varphi)^{-1}(\tilde{F}_2,E_2)$ is $\mathcal{CIFSL}\text{-}\mathcal{I}$ over G_1.

Proof : We can easily prove from the Propositions 3.8, 3.9 and Definition 4.1. □

5. Conclusion

In this work, we have introduced the theoretical definition of $\mathcal{CIFSCL}\text{-}\mathcal{G}$. The fundamental properties and structural characteristics of $\mathcal{CIFSCL}\text{-}\mathcal{G}$ were examined and also developed the $\mathcal{CIFSCL}\text{-}\mathcal{G}$ structures pertaining to characteristic CIFSSs. Furthermore, in this paper, we defined the notion of $\mathcal{CIFSL}\text{-}\mathcal{I}$. We established some operations and $\mathcal{CIFS}\ell$-homomorphism on $\mathcal{CIFSL}\text{-}\mathcal{I}$. In future, we can establish congruence relations over ℓ-group and its characterization theorems for $\mathcal{CIFSL}\text{-}\mathcal{I}$ to extend this work.

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