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A4-graph for the twisted group $^3D_4$ (3)

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Abstract

Assume that G is a finite group and $X = t^G$ where t is a non-identity element with $t^3 = 1$. The simple graph with a node set being X such that $a, b \in X$, are adjacent if ab^{-1} is an involution element, is called the A4-graph, and designated by $A_4(G, X)$. In this article, the construction of $A_4(G, X)$ is analyzed, where G is the twisted group of Lie type $^3D_4(3)$.

Subject Classification: *20D05, 68R01, 05C40, 05C69.*

Keywords: *Twisted groups, A4-graph, Group action, Connectivity.*

1. Introduction

One of the most beneficial approaches to investigating a group structure is by examining the action of the group on a graph. This method has proven its effectiveness in many recent studies, see for example [2, 3 and 5]. Suppose that $X = t^G$ where G is a finite group and t in G satisfy $1 \neq t$ and $t^3 = 1$. Aubad in [1] has established a graph containing the node set X where vertices $a, b \in X$, are adjacent if ab^{-1} is an involution element. This simple graph is defined as an A4-graph, and is specified as $A_4(G, X)$. The construction of $A_4(^3D_4(2), Y)$ where Y is a conjugacy class in $^3D_4(2)$ with representative of order equal to 3 has been analyzed in full detail as mentioned in [1]. The A4-graphs for tits group T and Mathieu group

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M_{20} are provided with full information in [8]. Moreover, analyzing the complete sets for the simple Mathieu groups are provided in [4].

In this article, we will let G to be the exceptional group of Lie type ${}^3D_4(3)$ and $X = t^G$ such that $1 \neq t$ and $t^3 = 1$. Hence, in G, X being a G-conjugacy class of representative t. The purpose of this study is to characterize various A4-graph features for the group G. The work entails describing the discs structure and calculating the diameters for the A4-graph.

Take an element x in X and i be a positive integer. Thus the i^{th} disc of x, is represented as $\Delta_i(x)$, and define as a set of all elements of X that has distance from x equal to i. It is important to note that G is transitive on A4-graph vertices and that G is embedded in the group of A4-graph automorphisms once G acts on X by conjugation. Moreover, the diameter of the A4-graph may be specified by Diam $A_4(G, X)$.

The structure of this article is as follows. In section 2, we present overall findings and give a tools to analyse the A4-graph. Section 3 gives a computational analysis of the A4-graph of the twisted group of Lie type ${}^3D_4(3)$. Finally, in section 4 we establish conclusions and set the future vision.

2. A4-Graph General Properties

The A4-graph offers several useful characteristics, such as simple, undirected, and regular graphs. Also, in the A4-graph, any two adjacent vertices generated the alternating group of degree 4. For the disc structures of the A4-graph, we should note that $\Delta_i(t)$ breaks out into a collection of specific $C_G(t)$-orbits. Moreover, $z \in \Delta_1(t)$ only if tz has order 3. First we will use the set below as a starting point:

$$X_C = \{z \in X \mid tz \in X\} \tag{1}$$

For any G-class C, where certain $C_G(t)$-orbits of the class X are combined to form the non-empty set X_C. To determine whether or not the i^{th} disc contains nodes inside the set X_C, it is critical to comprehend how the set X_C decomposes into $C_G(t)$- orbits. The following formula yields the size of the set X_C:

$$|X_C| = \frac{|G|}{|C_G(t)||C_G(w)|} \sum_{i=1}^{r} \frac{{}_i(w)\,{}_i(t)\,\overline{{}_i(t)}}{{}_i(1)} \tag{2}$$

Where $w \in C$ and χ_r, for $i = 1, .., r$ are the complex irreducible characters of G. As a result, using the computer mathematical programming GAP [7], particularly the function "***Class Multiplication Coefficient***", we can efficiently compute the size of X_C. We deal with the A4-graph for ${}^3D_4(3)$ computationally in this study, and we utilize the Online Atlas [6] to get a 26572-point permutation representation inside GAP. The major focus of our plan is on the computational approach indicated by using GAP and the Online Atlas. For the notation in this paper, we may use the notation mentioned in [1] and [4].

3. Main Results

We fully describe the A4-graph disc structures for the exceptional group of Lie type ${}^3D_4(3)$ throughout this section. There are only four ${}^3D_4(3)$-conjugacy classes with representative has order 3 namely 3A, 3B, 3C, and 3D in ${}^3D_4(3)$ and this classes can be offered by using GAP. Thus we deal with four different A4- graphs as we can see in the below cases.

3.1 *Discs Structure of $A_4({}^3D_4(3), 3A)$*

The first important result about $A_4({}^3D_4(3), 3A)$ is that the A4 graph is disconnected. Furthermore, let $t \in$ 3A then 3A = t^G have size 53144. The action of $C_G(t)$ on 3A produces 6 orbits. These orbits are separated into X_C sets as follows [1A(1), 3A(1, 168), 3C(13608), 4A(19683), 6A(19683)].

3.2 *Discs Structure of $A_4({}^3D_4(3), 3B)$*

Similar to above the graph $A_4({}^3D_4(3), 3B)$ is disconnected and if $t \in$3B the class 3B = t^G has the size 43524936. There are 446 $C_G(t)$-orbits in the next a complete list of X_C sets [1A(1), 3A(8, 216), 3B(1, 8, 9^{24}, 216^{37} , 5832), 3C(216^{28}, 5832), 3D(216^{51}, 5832), 4A(17496^2), 4B(19683, 157464^2), 6B(19683), 6C(17496^{12}, 157464^2), 6D(17496^{14}, 157464^2), 7AC (19683), 7D(472392), 9A(17496^{12}), 9B(17496^{52}, 472392), 12A(17496^{24}), 12B(157464^2), 13AF (19683), 13GK (472392), 14AC (19683), 21DF (157464^4, 472392), 26AF (19683), 28AF (157464^2, 19683), 39AF (157464^2), 39GL (157464^2, 472392), 42AC (157464^4, 472392^2), 52AF (472392^2), 73AR (472392), 78AF (157464^4), 84AF (157464^2)].

3.3 *Discs Structure of $A_4({}^3D_4(3), 3C)$*

Let $t \in$3C then the class 3C = t^G has the size 120530592 and the action by conjugation of $C_G(t)$ on 3C generates 990 orbits. The $A_4({}^3D_4(3), 3C)$ in this case is connected with Diam $A_4({}^3D_4(3), 3C) = 4$. In the following, we will explain how the X_C sets are distributed on the different i^{th} discs of the graph $A_4(^3D_4(3), 3C)$: $\Delta_1(t)$ = [3C(6561)], , $\Delta_2(t)$ = [6A(13122), 6B(170586), 6C(3159, 28431, 170586^4), 7D(170586^8), 8A(13122^2, 170586^4), 8B(170586^6), 13GH(13122^2), 21AC(28431, 85293, 170586^2), 21DF(170586^9), 26AC(170586^3), 26DF(170586^2), 93AC(85293), 39DF(85293, 170586), 42AC(28431, 56862^2, 85293, 170586^3), 52AC(170586^2), 56AC(170586^8), 73AE(170586), 73FJ(170586^6), OR(170586^2), 78AC(28431, 56862^2, 170586^2), 78DF(28431), 104JL(170586^2)], $\Delta_3(t)$ = [2A(6561), 3A(6), 3B(78^3, 2106), 3C(39^6, 78^6, 2106, 13122^4), 3D(39^6, 78^9, 2106), 4A(486^2, 13122^3), 4B(28431, 56862^2, 17058^3), 6A(486^2, 13122^3), 6B(56862^2, 170586^3), 6C(3159^6), 6D(3159^6, 6318^2, 56862^2, 170586^8), 7D(170586^{28}), 8A(13122^6, 170586^8), 8B(170586^{14}), 9A(6318^{12}), 9B(6318^{30}, 170586^{16}), 12A(3159^{12}, 6318^2, 170586^6), 12B(56862^2, 170586^6), 13AF(28431), 13GH(13122^3, 170586^4), 14AC(56862^2, 85293^2, 170586^2), 21AC(170586^2), 21DF(56862^4, 170586^7), 26AC(56862^2), 26DF(56862^2, 170586), 28AF(28431, 56862^2, 170586^3), 39AC(56862^2, 170586), 39DF(56862^2), 39GL(56862^2, 170586), 42AC(170586^3), 52AC(85293), 52DF(85293, 170586^2), 56AC(170586^{12}), 56DF(170586^{20}), 73AE(170586^8), 73FJ(170586^3), 73KN(170586^9), 73OR(170586^7), 78AC(170586^4), 78DF(56862^2, 170586^6), 84AF(56862^2, 170586^6), 104AI(170586^2)], $\Delta_4(t)$ = [1A(1), 3A(2), 3B(39^{12}, 78^{12}), 3C(2, 78^{10}), 3D(78^6), 4A(3159)].

3.4 *Discs Structure of $A_4(^3D_4(3), 3D)$*

The class 3D has a size of 223842528 and for $t \in$3D the action by conjugation of $C_G(t)$ on 3D produces 3006 orbits. Also, the A4-graph, in this case, is connected with Diam $A4(^3D_4(3), 3D) = 4$. Next we separate the sets X_C into the A4-graph discs:$\Delta_1(t)$ = [3D(6561)], $\Delta_2(t)$ = [4A(13122), 4B(91854), 6D(1701, 15309, 91854^4), 7D(13122^2), 8A(91854^6), 8B(13122^2, 91854^4), 9B(91854^4), 12AB(91854^2), 13GK(91854), 21AC(15309, 45927), 21DF(30618^2), 26DF(91854), 28AC(91854^2), 28DF(91854), 28GH(91854^4), 39AC(45927), 39DF(45927, 91854), 39GI(91854^6), 42AC(15309, 45927, 91854^{11}), 52AC(91854^8), 52DF(91854^{11}), 56AC(91854^8), 73AC(91854^4), 73DF(91854^8), 73GI(91854^6), 73JL(91854^9), 73MO(91854^5), 73PR(91854), 78AC(15309, 91854^6), 78DF(15309), 84AC(91854^2), 84DF(91854^4), 104AC(91854^8), 104JL(91854^4)], $\Delta_3(t)$ = [2A(6561), 3B(42^3, 1134), 3C(21^6, 42^{20}, 1134), 3D(6, 21^6, 42^{16}, 1134, 13122^4), 4A(3402^2, 13122^2), 4B(13122, 15309, 30618^2, 91854), 6A(3402^2, 13122^4), 6B(30618^2, 91854^4), 6C(1701^4, 3402^{26}, 30618^2, 91854^8), 6D(1701^6, 3402^{26}), 7D(13122^9, 91854^{25}), 8A(91854^{14}), 8B(13122^6, 91854^8), 9A(3402^{12}), 9B(3402^{77}, 91854^{12}), 12A(1701^{12}, 3402^{50}, 91854^4), 12B(13122^4, 30618^2), 13AF(15309), 13GK(91854^8), 14AC(30618^2,

45927^2, 91854^5), 21AC(91854^7), 21DF(30618^2, 91854^{49}), 26AC(30618^2, 91854^6), 26DF(30618^2, 91854^5), 28AC(15309, 30618^2, 91854^4), 28DF(15309, 30618^2, 91854^5), 28GH(91854^{16}), 39AC(30618^2, 91854^4), 39DF(30618^2, 91854^3), 39GI(30618^2, 91854^{10}), 39JL(30618^2, 91854^{16}), 42AC(30618^2, 91854^{19}), 52AC(45927, 91854^{12}), 52DF(45927, 91854^9), 56AC(91854^{48}), 56DF(91854^{56}), 73AC(91854^{32}), 73DF(91854^{28}), 73GI(91854^{30}), 73JL(91854^{27}), 73MO(91854^{31}), 73PR(91854^{35}), 78AC(30618^2, 91854^{24}), 78DF(30618^2, 91854^{30}), 84AC(30618^2, 91854^{28}), 84DF(30618^2, 91854^{26}), 104AC(91854^{12}), 104DI(91854^{20}), 104JKL(91854^{16})], $\Delta_4(t)$ = [1A(1), 3A(2, 42), 3B(21^{12}, 42^{60}), 3C(42^{31}), 3D(2, 6^2, 42^{57}), 4A(1701), 6C(1701^2), 9B(3402)].

The following theorem can be formed from the aforementioned findings:

Theorem 1: *Suppose that $G \cong {}^3D_4(3)$ then we have the following results:*

1. *$A_4({}^3D_4(3), 3A)$ and $A_4({}^3D_4(3), 3B)$ are totally disconnected*.
2. *Diam $A_4({}^3D_4(3), 3C) = 4$. And for $t \in 3C$ we have $|\Delta_1(t)| = 6561$ with 1 $C_G(t)$-orbits, $|\Delta_2(t)| = 28611306$ with 200 $C_G(t)$-orbits, $|\Delta_3(t)| = 91906908$ with 744 $C_G(t)$-orbits, and $|\Delta_4(t)| = 5816$ with 44 $C_G(t)$-orbits.*
3. *Diam $A_4({}^3D_4(3), 3D) = 4$. And for $t \in 3D$ we have $|\Delta_1(t)| = 6561$ with 1 $C_G(t)$-orbits, $|\Delta_2(t)| = 33884892$ with 395 $C_G(t)$-orbits, $|\Delta_3(t)| = 189936042$ with 2439 $C_G(t)$-orbits, and $|\Delta_4(t)| = 15032$ with 170 $C_G(t)$-orbits.*

4. Discussion and Conclusion

Study alternating group A4 inside the ${}^3D_4(3)$ has been acquired. For this target, we have analyzed the A4-graph by using computational approaches. In addition, the work that has been achieved, opens the door to studying more complicated groups such as Monster Group M and Baby Monster Group B.

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