

Pentanacci and Pentanacci-Lucas hybrid numbers

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Abstract

The aim of this study is to introduce the Pentanacci and Pentanacci-Lucas hybrid numbers. Then, some properties with respect to these special numbers and relations between them are obtained. In addition, matrix formulations, generating functions, Binet-like formulas and some summation formulas of them are examined.

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1. Introduction

The hybrid numbers which come to exist by means of complex, dual and hyperbolic numbers are introduced by Özdemir in [16] and denoted by the notation $\mathbb{K}$. The set of hybrid numbers is determined as:

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$$\mathbb{K} = \{a + b\mathbf{i} + c\varepsilon + d\mathbf{h} \mid \mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \mathbf{i}\mathbf{h} = -\mathbf{h}\mathbf{i} = \varepsilon + \mathbf{i}; a, b, c, d \in \mathbb{R}\},$$

where 1, $\mathbf{i}$, ε and $\mathbf{h}$ represent real, complex, dual and hyperbolic units, respectively. These units are defined as

$$1 \leftrightarrow (1, 0, 0, 0), \mathbf{i} \leftrightarrow (0, 1, 0, 0), \varepsilon \leftrightarrow (0, 0, 1, 0), \mathbf{h} \leftrightarrow (0, 0, 0, 1)$$

and called as hybrid units. $\mathbb{K}$ is a non-commutative ring according to the addition and multiplication, [16]. More information about the hybrid numbers can be found in the studies [4, 16–18].

On the other side, special numbers have high importance and impact on lots of disciplines such as mathematics, architecture, engineering, music, art, photography, painting etc. There have been so many investigations in the literature that interest the sequences such as Fibonacci, Lucas, Pell, Tribonacci, Tetranacci, Gibonacci and the others, [6, 18, 11–13]. The n th Pentanacci number P_n (A001591 in [21]) and n th Pentanacci-Lucas number L_n (A074048 in [21]) are defined as follows:

$$P_n = P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5}, n \geq 5, \quad (1.1)$$

$$L_n = L_{n-1} + L_{n-2} + L_{n-3} + L_{n-4} + L_{n-5}, n \geq 5, \quad (1.2)$$

with the initial conditions $P_0 = 0, P_1 = 1, P_2 = 1, P_3 = 2, P_4 = 4$ and $L_0 = 5, L_1 = 1, L_2 = 3, L_3 = 7, L_4 = 15$, respectively. Detailed and elaborative information about these numbers can be found in [5, 20–26].

In literature, most studies have been content to special hybrid numbers and their properties. In [28, 33], Horadam hybrid numbers are examined, and also Pell and Pell-Lucas hybrid numbers, and Jacobsthal and Jacobsthal-Lucas hybrid numbers are studied in [30, 31], respectively. In [32], special spacelike hybrid numbers, in [3] generalized Fibonacci hybrid numbers and their properties are obtained. Moreover, [2] is a study which is related k -Pell hybrid numbers. In [9, 10], a new generalization of Fibonacci hybrid and Lucas hybrid numbers, and Horadam hybrid numbers are studied. Generalization of Fibonacci hybrid numbers is also examined in [34]. Additionally, generalized hybrid Tribonacci numbers, Tribonacci and Tribonacci-Lucas hybrid numbers, the hybrid number of Padovan and some properties of Van Der Laan hybrid sequence are examined in [15, 19, 35, 36], respectively. Hybrid Mersenne numbers [14], generalized Mersenne hybrid numbers [29] and hybridization of triangular numbers [1] are studied. Also, generalized Tetranacci hybrid numbers are presented in [27].

In this study, we examine the Pentanacci and Pentanacci-Lucas hybrid numbers. This study organized as 4 sections. In Section 1 and Section 2, we present a brief information related to hybrid, Pentanacci and Pentanacci-Lucas numbers. Then, in Section 3, we determine the Pentanacci and Pentanacci-Lucas hybrid numbers, present new properties and relations regarding them. Additionally, we obtain generating functions, Binet-like formulas, matrix formulations and some summation properties. Finally, we give conclusions in Section 4.

2. Basic Concepts

Let us give some fundamental properties of hybrid numbers [16–18], Pentanacci and Pentanacci-Lucas numbers [5, 20–26]:

Let $Z = a_0 + b_0\mathbf{i} + c_0\varepsilon + d_0\mathbf{h}$, $Y = a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h} \in \mathbb{K}$. Any two hybrid numbers are equal if and only if components of them are equal. Addition operation is commutative and associative. Multiplication operation is not commutative but it is associative. The addition and subtraction of two hybrid numbers are determined as $Z \pm Y = (a_0 \pm a_1) + (b_0 \pm b_1)\mathbf{i} + (c_0 \pm c_1)\varepsilon + (d_0 \pm d_1)\mathbf{h}$, [16]. Also, by means of the rules of multiplication of hybrid numbers given in Table 1, the multiplication of two hybrid numbers can be given as follows:

$$ZY = (a_0 + b_0\mathbf{i} + c_0\varepsilon + d_0\mathbf{h})(a_1 + b_1\mathbf{i} + c_1\varepsilon + d_1\mathbf{h}).$$

The conjugate of Z is given as: $\bar{Z} = a_0 - b_0\mathbf{i} - c_0\varepsilon - d_0\mathbf{h}$. Here, $S_z = a_0$ is scalar part and $\bar{V}_z = b_0\mathbf{i} + c_0\varepsilon + d_0\mathbf{h}$ is vector part of Z . So, $Z = S_z + V_z$ and $\bar{Z} = S_z - \bar{V}_z$ can be seen. The character of the hybrid number is denoted by $C(Z)$ and defined by, [16]:

$$C(Z) = \bar{Z}Z = Z\bar{Z} = a_0^2 + (b_0 - c_0)^2 - c_0^2 - d_0^2. \quad (2.1)$$

The characters of Z can be classified as: Z is spacelike, timelike, lightlike if $C(Z) < 0$, $C(Z) > 0$, $C(Z) = 0$, respectively. The norm of Z is defined as: $\|Z\| = \sqrt{|C(Z)|} = \sqrt{|a_0^2 + (b_0 - c_0)^2 - c_0^2 - d_0^2|}$. The inverse of Z is given by, [16]:

$$Z^{-1} = \frac{\bar{Z}}{C(Z)}, \quad \|Z\| \neq 0. \quad (2.2)$$

Also, the hybrid vector of Z is defined as: $\xi_Z = ((b_0 - c_0), c_0, d_0)$ and we have, [16]:

$$C_{\xi}(Z) = -(b_0 - c_0)^2 + c_0^2 + d_0^2. \quad (2.3)$$

The types of the hybrid number Z are given by: Z is elliptic (comlike), hyperbolic (hyperlike), parabolic (duallike) if $C_{\xi}(Z) < 0$, $C_{\xi}(Z) > 0$, $C_{\xi}(Z) = 0$, respectively. Besides, norm of the hybrid vector of Z is denoted by $N(Z)$ and determined by, [16]:

$$N(Z) = \sqrt{|C_{\xi}(Z)|}. \quad (2.4)$$

On the other hand, Pentanacci and Pentanacci-Lucas numbers have the recurrence relations as in equations (1.1) and (1.2), respectively. By extending the recurrence relations of the Pentanacci and Pentanacci-Lucas numbers to negative subscripts, we have the followings for all $n \geq 1$, [22–26]:

$$\begin{aligned} P_{-n} &= P_{-n+5} - P_{-n+1} - P_{-n+2} - P_{-n+3} - P_{-n+4}, \\ L_{-n} &= L_{-n+5} - L_{-n+1} - L_{-n+2} - L_{-n+3} - L_{-n+4}. \end{aligned}$$

Indeed, the recurrence relations (1.1) and (1.2) are valid $\forall n \in \mathbb{Z}$. Some values of the Pentanacci and Pentanacci-Lucas numbers are presented in Table 2.

The following relations between Pentanacci and Pentanacci-Lucas numbers are satisfied $\forall n \in \mathbb{Z}$, [25]:

$$\begin{cases} L_n = -P_{n+3} + 7P_{n+1} - 2P_n - P_{n-1}, & (2.5a) \\ L_n = -P_{n+2} + 6P_{n+1} - 3P_n - 2P_{n-1} - P_{n-2}, & (2.5b) \\ L_n = 3P_{n+4} - 7P_{n+2} - 9P_n - 8P_{n-1} - 4P_{n-2}, & (2.5c) \end{cases}$$

$$\begin{cases} P_n = \frac{1}{1198}(27L_{n+3} + 18L_{n+2} + 3L_{n+1} - 22L_n + 136L_{n-1}), & (2.6a) \\ P_n = \frac{1}{1198}(45L_{n+3} - 15L_{n+1} - 40L_n + 118L_{n-1} - 18L_{n-2}). & (2.6b) \end{cases}$$

Table 2
Some Values of the Pentanacci and Pentanacci-Lucas Numbers

n	...	-6	-5	-4	-3	-2	-1	0	1	2	3	4	5	6	...
P_n	...	0	-1	1	0	0	0	0	1	1	2	4	8	16	...
L_n	...	-7	9	-1	-1	-1	-1	5	1	3	7	15	31	57	...

Moreover, characteristic equation of the Pentanacci and Pentanacci-Lucas numbers is $x^5 - x^4 - x^3 - x^2 - x - 1 = 0$ and the roots of this equation are, [23]:

$$\begin{cases} r_1 &= 1.96595, \\ r_2 &= -0.678351 + 0.458536i, \\ r_3 &= -0.678351 - 0.458536i = \bar{r}_2, \\ r_4 &= 0.195377 + 0.848854i, \\ r_5 &= 0.195377 - 0.848854i = \bar{r}_4. \end{cases}$$

By using these roots, Binet-like formulas of P_n and L_n are given in [23], respectively:

$$\begin{aligned} P_n &= \frac{r_1^{n+3}}{(r_1 - r_2)(r_1 - r_3)(r_1 - r_4)(r_1 - r_5)} + \frac{r_2^{n+3}}{(r_2 - r_1)(r_2 - r_3)(r_2 - r_4)(r_2 - r_5)} \\ &+ \frac{r_3^{n+3}}{(r_3 - r_1)(r_3 - r_2)(r_3 - r_4)(r_3 - r_5)} + \frac{r_4^{n+3}}{(r_4 - r_1)(r_4 - r_2)(r_4 - r_3)(r_4 - r_5)} \\ &+ \frac{r_5^{n+3}}{(r_5 - r_1)(r_5 - r_2)(r_5 - r_3)(r_5 - r_4)} \end{aligned} \quad (2.7)$$

and

$$L_n = r_1^n + r_2^n + r_3^n + r_4^n + r_5^n. \quad (2.8)$$

Also, considering the study [7], in [25] matrix formulations for Pentanacci and Pentanacci-Lucas numbers are studied by using the following 5×5 matrix:

$$\begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

3. Pentanacci and Pentanacci-Lucas Hybrid Numbers

In this section, we present the Pentanacci and Pentanacci-Lucas hybrid numbers, and obtain some properties about them. We give classifications regarding characters and types of these special hybrid numbers. Additionally, we obtain the generating functions, Binet-like

formulas, matrix formulations and summation formulas with respect to these numbers.

Definition 3.1: The n th Pentanacci hybrid number $\tilde{P}_n$ and n th Pentanacci-Lucas hybrid number $\tilde{L}_n$ are determined as follows, $\forall n \in \mathbb{N}$:

$$\tilde{P}_n = P_n + P_{n+1}\mathbf{i} + P_{n+2}\varepsilon + P_{n+3}\mathbf{h}, \quad (3.1)$$

$$\tilde{L}_n = L_n + L_{n+1}\mathbf{i} + L_{n+2}\varepsilon + L_{n+3}\mathbf{h}, \quad (3.2)$$

and $\forall n \in \mathbb{Z}^+$, Pentanacci and Pentanacci-Lucas hybrid numbers with negative subscripts are defined by:

$$\tilde{P}_{-n} = P_{-n} + P_{-n+1}\mathbf{i} + P_{-n+2}\varepsilon + P_{-n+3}\mathbf{h}, \quad (3.3)$$

$$\tilde{L}_{-n} = L_{-n} + L_{-n+1}\mathbf{i} + L_{-n+2}\varepsilon + L_{-n+3}\mathbf{h}, \quad (3.4)$$

where $\mathbf{i}^2 = -1, \varepsilon^2 = 0, \mathbf{h}^2 = 1, \mathbf{i}\mathbf{h} = -\mathbf{h}\mathbf{i} = \varepsilon + \mathbf{i}$ (see Table 1), also P_n, L_n, P_{-n} and L_{-n} are the n th Pentanacci, n th Pentanacci-Lucas, $-n$ th Pentanacci and $-n$ th Pentanacci-Lucas number, respectively.

Note that, we can say that the equalities (3.1) and (3.2) are valid $\forall n \in \mathbb{Z}$. We shall denote the set of the Pentanacci and Pentanacci-Lucas hybrid numbers by $\tilde{\mathbb{P}}$ and $\tilde{\mathbb{L}}$, respectively.

Example 3.1 : Let give the some values of $\tilde{P}_n$ and $\tilde{L}_n$, $\forall n \in \mathbb{Z}$:

$$\begin{array}{ll} \tilde{P}_0 = \mathbf{i} + \varepsilon + 2\mathbf{h}, & \tilde{L}_0 = 5 + \mathbf{i} + 3\varepsilon + 7\mathbf{h}, \\ \tilde{P}_1 = 1 + \mathbf{i} + 2\varepsilon + 4\mathbf{h}, & \tilde{L}_1 = 1 + 3\mathbf{i} + 7\varepsilon + 15\mathbf{h}, \\ \tilde{P}_2 = 1 + 2\mathbf{i} + 4\varepsilon + 8\mathbf{h}, & \tilde{L}_2 = 3 + 7\mathbf{i} + 15\varepsilon + 31\mathbf{h}, \\ \tilde{P}_3 = 2 + 4\mathbf{i} + 8\varepsilon + 16\mathbf{h}, & \tilde{L}_3 = 7 + 15\mathbf{i} + 31\varepsilon + 57\mathbf{h}, \\ \tilde{P}_4 = 4 + 8\mathbf{i} + 16\varepsilon + 31\mathbf{h}, & \tilde{L}_4 = 15 + 31\mathbf{i} + 57\varepsilon + 113\mathbf{h}, \\ \tilde{P}_5 = 8 + 16\mathbf{i} + 31\varepsilon + 61\mathbf{h}, & \tilde{L}_5 = 31 + 57\mathbf{i} + 113\varepsilon + 223\mathbf{h}, \\ \vdots & \vdots \end{array}$$

and

$$\begin{array}{ll} \tilde{P}_{-1} = \varepsilon + \mathbf{h}, & \tilde{L}_{-1} = -1 + 5\mathbf{i} + \varepsilon + 3\mathbf{h}, \\ \tilde{P}_{-2} = \mathbf{h}, & \tilde{L}_{-2} = -1 - \mathbf{i} + 5\varepsilon + \mathbf{h}, \\ \tilde{P}_{-3} = 0, & \tilde{L}_{-3} = -1 - \mathbf{i} - \varepsilon + 5\mathbf{h}, \\ \tilde{P}_{-4} = 1, & \tilde{L}_{-4} = -1 - \mathbf{i} - \varepsilon - \mathbf{h}, \end{array}$$

$$\begin{aligned}\tilde{P}_{-5} &= -1 + \mathbf{i}, & \tilde{L}_{-5} &= 9 - \mathbf{i} - \varepsilon - \mathbf{h}, \\ \tilde{P}_{-6} &= -\mathbf{i} + \varepsilon, & \tilde{L}_{-6} &= -7 + 9\mathbf{i} - \varepsilon - \mathbf{h}, \\ &\vdots & &\vdots\end{aligned}$$

Now, let introduce the algebraic properties of $\tilde{P}_n, \tilde{P}_m \in \tilde{\mathbb{P}}, \forall n, m \in \mathbb{Z}$. Let $\tilde{P}_n = P_n + P_{n+1}\mathbf{i} + P_{n+2}\varepsilon + P_{n+3}\mathbf{h}$ and $\tilde{P}_m = P_m + P_{m+1}\mathbf{i} + P_{m+2}\varepsilon + P_{m+3}\mathbf{h}$. The scalar part of $\tilde{P}_n : S_{\tilde{P}_n} = P_n$. The vector part of $\tilde{P}_n : \tilde{V}_{\tilde{P}_n} = P_{n+1}\mathbf{i} + P_{n+2}\varepsilon + P_{n+3}\mathbf{h}$. The conjugate of $\tilde{P}_n$:

$$\overline{\tilde{P}_n} = P_n - P_{n+1}\mathbf{i} - P_{n+2}\varepsilon - P_{n+3}\mathbf{h} \quad (3.5)$$

where $\overline{\tilde{P}_n} = S_{\tilde{P}_n} - \tilde{V}_{\tilde{P}_n}$. The equality, addition, subtraction, multiplication by scalar and hybridian multiplication are described by:

$$\tilde{P}_n = \tilde{P}_m \Leftrightarrow P_n = P_m, P_{n+1} = P_{m+1}, P_{n+2} = P_{m+2}, P_{n+3} = P_{m+3}, \quad (3.6)$$

$$\tilde{P}_n \pm \tilde{P}_m = (P_n \pm P_m) + (P_{n+1} \pm P_{m+1})\mathbf{i} + (P_{n+2} \pm P_{m+2})\varepsilon + (P_{n+3} \pm P_{m+3})\mathbf{h}, \quad (3.7)$$

$$c\tilde{P}_n = cP_n + cP_{n+1}\mathbf{i} + cP_{n+2}\varepsilon + cP_{n+3}\mathbf{h}, c \in \mathbb{R}, \quad (3.8)$$

$$\tilde{P}_n \tilde{P}_m = (P_n + P_{n+1}\mathbf{i} + P_{n+2}\varepsilon + P_{n+3}\mathbf{h})(P_m + P_{m+1}\mathbf{i} + P_{m+2}\varepsilon + P_{m+3}\mathbf{h}). \quad (3.9)$$

Note that, the above algebraic properties are valid for $\tilde{L}_n$, as well.

Theorem 3.1 : Let $\tilde{P}_n, \tilde{P}_{-n} \in \tilde{\mathbb{P}}$ and $\tilde{L}_n, \tilde{L}_{-n} \in \tilde{\mathbb{L}}$. Then, the following recurrence relations can be given:

$$\tilde{P}_n = \tilde{P}_{n-1} + \tilde{P}_{n-2} + \tilde{P}_{n-3} + \tilde{P}_{n-4} + \tilde{P}_{n-5}, n \geq 5, \quad (3.10)$$

$$\tilde{L}_n = \tilde{L}_{n-1} + \tilde{L}_{n-2} + \tilde{L}_{n-3} + \tilde{L}_{n-4} + \tilde{L}_{n-5}, n \geq 5, \quad (3.11)$$

with the initial conditions $\tilde{P}_0, \tilde{P}_1, \tilde{P}_2, \tilde{P}_3, \tilde{P}_4$ and $\tilde{L}_0, \tilde{L}_1, \tilde{L}_2, \tilde{L}_3, \tilde{L}_4$ (see in Example 3.1). Also, the recurrence relations for these numbers with negative subscripts are as follows:

$$\tilde{P}_{-n} = \tilde{P}_{-n+5} - \tilde{P}_{-n+1} - \tilde{P}_{-n+2} - \tilde{P}_{-n+3} - \tilde{P}_{-n+4}, n \geq 1, \quad (3.12)$$

$$\tilde{L}_{-n} = \tilde{L}_{-n+5} - \tilde{L}_{-n+1} - \tilde{L}_{-n+2} - \tilde{L}_{-n+3} - \tilde{L}_{-n+4}, n \geq 1. \quad (3.13)$$

Proof : We get the recurrence relation (3.10) by using equations (1.1), (3.1):

$$\begin{aligned}
\tilde{P}_{n-1} + \tilde{P}_{n-2} + \tilde{P}_{n-3} + \tilde{P}_{n-4} + \tilde{P}_{n-5} &= P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4} + P_{n-5} \\
&\quad + (P_n + P_{n-1} + P_{n-2} + P_{n-3} + P_{n-4})\mathbf{i} \\
&\quad + (P_{n+1} + P_n + P_{n-1} + P_{n-2} + P_{n-3})\mathbf{e} \\
&\quad + (P_{n+2} + P_{n+1} + P_n + P_{n-1} + P_{n-2})\mathbf{h} \\
&= P_n + P_{n+1}\mathbf{i} + P_{n+2}\mathbf{e} + P_{n+3}\mathbf{h} \\
&= \tilde{P}_n.
\end{aligned}$$

By the same way, the recurrence relations (3.11), (3.12), (3.13) can be shown. $\square$

Note that, the recurrence relations (3.10) and (3.11) are valid $\forall n \in \mathbb{Z}$.

Theorem 3.2: $\forall n \in \mathbb{Z}$, the following relations can be given between Pentanacci and Pentanacci-Lucas hybrid numbers as below:

$$\tilde{L}_n = -\tilde{P}_{n+3} + 7\tilde{P}_{n+1} - 2\tilde{P}_n - \tilde{P}_{n-1}, \quad (3.14a)$$

$$\tilde{L}_n = -\tilde{P}_{n+2} + 6\tilde{P}_{n+1} - 3\tilde{P}_n - 2\tilde{P}_{n-1} - \tilde{P}_{n-2}, \quad (3.14b)$$

$$\tilde{L}_n = 3\tilde{P}_{n+4} - 7\tilde{P}_{n+2} - 9\tilde{P}_n - 8\tilde{P}_{n-1} - 4\tilde{P}_{n-2}, \quad (3.14c)$$

$$\tilde{P}_n = \frac{1}{1198} (27\tilde{L}_{n+3} + 18\tilde{L}_{n+2} + 3\tilde{L}_{n+1} - 22\tilde{L}_n + 136\tilde{L}_{n-1}), \quad (3.15a)$$

$$\tilde{P}_n = \frac{1}{1198} (45\tilde{L}_{n+3} - 15\tilde{L}_{n+1} - 40\tilde{L}_n + 118\tilde{L}_{n-1} - 18\tilde{L}_{n-2}). \quad (3.15b)$$

Proof: We obtain equation (3.14a) by using equations (2.5a), (3.1), (3.2), (3.7) and (3.8):

$$\begin{aligned}
-\tilde{P}_{n+3} + 7\tilde{P}_{n+1} - 2\tilde{P}_n - \tilde{P}_{n-1} &= -P_{n+3} + 7P_{n+1} - 2P_n - P_{n-1} \\
&\quad + (-P_{n+4} + 7P_{n+2} - 2P_{n+1} - P_n)\mathbf{i} \\
&\quad + (-P_{n+5} + 7P_{n+3} - 2P_{n+2} - P_{n+1})\mathbf{e} \\
&\quad + (-P_{n+6} + 7P_{n+4} - 2P_{n+3} - P_{n+2})\mathbf{h} \\
&= L_n + L_{n+1}\mathbf{i} + L_{n+2}\mathbf{e} + L_{n+3}\mathbf{h} \\
&= \tilde{L}_n.
\end{aligned}$$

The properties (3.14b)-(3.15b) can be easily shown by the same way. $\square$

It should be noted that, in following theorems and corollaries, all of the features that applied to Pentanacci hybrid numbers can also be

applied to Pentanacci-Lucas hybrid numbers, so we omit them for the sake of brevity. Only Binet-like formulas and generating functions are given separately.

Theorem 3.3 : $\forall n \in \mathbb{Z}$, the following properties related to Pentanacci hybrid numbers are satisfied:

- (i)
$$\left. \begin{aligned} & \dagger \tilde{P}_n - \mathbf{i} \tilde{P}_{n+1} - \varepsilon \tilde{P}_{n+2} - \mathbf{h} \tilde{P}_{n+3} \\ & \ddagger \tilde{P}_n - \tilde{P}_{n+1} \mathbf{i} - \tilde{P}_{n+2} \varepsilon - \tilde{P}_{n+3} \mathbf{h} \end{aligned} \right\} = P_n + P_{n+2} - 2P_{n+3} - P_{n+6},$$
- (ii)
$$C(\tilde{P}_n) = P_n^2 + (P_{n+1} - P_{n+2})^2 - P_{n+2}^2 - P_{n+3}^2,$$
- (iii)
$$\tilde{P}_n + \bar{\tilde{P}}_n = 2P_n,$$
- (iv)
$$\tilde{P}_n^2 = 2\tilde{P}_n P_n - \bar{\tilde{P}}_n \tilde{P}_n.$$

Proof :

- (i) $\dagger$ From Table 1, and equation (3.1), we get:

$$\begin{aligned} \tilde{P}_n - \mathbf{i} \tilde{P}_{n+1} - \varepsilon \tilde{P}_{n+2} - \mathbf{h} \tilde{P}_{n+3} &= P_n + P_{n+1} \mathbf{i} + P_{n+2} \varepsilon + P_{n+3} \mathbf{h} \\ &\quad - \mathbf{i}(P_{n+1} + P_{n+2} \mathbf{i} + P_{n+3} \varepsilon + P_{n+4} \mathbf{h}) \\ &\quad - \varepsilon(P_{n+2} + P_{n+3} \mathbf{i} + P_{n+4} \varepsilon + P_{n+5} \mathbf{h}) \\ &\quad - \mathbf{h}(P_{n+3} + P_{n+4} \mathbf{i} + P_{n+5} \varepsilon + P_{n+6} \mathbf{h}) \\ &= P_n + P_{n+2} - 2P_{n+3} - P_{n+6}. \end{aligned}$$

And the other equality $\ddagger$ of (i) can be shown by the same way.

- (ii) With the help of Table 1 and equations (3.1), (3.5), we obtain:

$$\begin{aligned} C(\tilde{P}_n) &= \tilde{P}_n \bar{\tilde{P}}_n \\ &= (P_n + P_{n+1} \mathbf{i} + P_{n+2} \varepsilon + P_{n+3} \mathbf{h})(P_n - P_{n+1} \mathbf{i} - P_{n+2} \varepsilon - P_{n+3} \mathbf{h}) \\ &= P_n^2 + P_{n+1}^2 - 2P_{n+1} P_{n+2} - P_{n+3}^2 \\ &= P_n^2 + (P_{n+1} - P_{n+2})^2 - P_{n+2}^2 - P_{n+3}^2. \end{aligned}$$

- (iii) With the help of equations (3.1), (3.5) and (3.7), we have:

$$\begin{aligned} \tilde{P}_n + \bar{\tilde{P}}_n &= (P_n + P_{n+1} \mathbf{i} + P_{n+2} \varepsilon + P_{n+3} \mathbf{h}) + (P_n - P_{n+1} \mathbf{i} - P_{n+2} \varepsilon - P_{n+3} \mathbf{h}) \\ &= 2P_n. \end{aligned}$$

(iv) Through using the property at part (iii), we get:

$$\tilde{P}_n^2 = \tilde{P}_n \tilde{P}_n = \tilde{P}_n (2P_n - \bar{\tilde{P}}_n) = 2\tilde{P}_n P_n - \tilde{P}_n \bar{\tilde{P}}_n.$$

□

Theorem 3.4 : $\forall n, m \in \mathbb{Z}$, the following properties related to Pentanacci hybrid numbers are given:

- (i) $\tilde{P}_n \bar{\tilde{P}}_m - \bar{\tilde{P}}_n \tilde{P}_m = 2(P_{n+1}P_m - P_n P_{m+1})\mathbf{i} + 2(P_{n+2}P_m - P_n P_{m+2})\varepsilon$
 $+ 2(P_{n+3}P_m - P_n P_{m+3})\mathbf{h},$
- (ii) $\tilde{P}_n \bar{\tilde{P}}_m + \bar{\tilde{P}}_n \tilde{P}_m = 2[P_n P_m + P_{n+1}P_{m+1} - P_{n+1}P_{m+2} - P_{n+2}P_{m+1} - P_{n+3}P_{m+3}]$
 $+ (P_{n+3}P_{m+1} - P_{n+1}P_{m+3})\mathbf{i}$
 $+ (P_{n+2}P_{m+3} + P_{n+3}P_{m+1} - P_{n+1}P_{m+3} - P_{n+3}P_{m+2})\varepsilon$
 $+ (P_{n+1}P_{m+2} - P_{n+2}P_{m+1})\mathbf{h},$
- (iii) $\tilde{P}_n \tilde{P}_m - \bar{\tilde{P}}_n \bar{\tilde{P}}_m = 2(P_n P_{m+1} + P_{n+1}P_m)\mathbf{i} + 2(P_n P_{m+2} + P_{n+2}P_m)\varepsilon$
 $+ 2(P_n P_{m+3} + P_{n+3}P_m)\mathbf{h},$
- (iv) $\tilde{P}_n \tilde{P}_m + \bar{\tilde{P}}_n \bar{\tilde{P}}_m = 2[P_n P_m - P_{n+1}P_{m+1} + P_{n+1}P_{m+2} + P_{n+2}P_{m+1} + P_{n+3}P_{m+3}]$
 $+ (P_{n+1}P_{m+3} - P_{n+3}P_{m+1})\mathbf{i}$
 $+ (P_{n+1}P_{m+3} - P_{n+2}P_{m+3} - P_{n+3}P_{m+1} + P_{n+3}P_{m+2})\varepsilon$
 $+ (P_{n+2}P_{m+1} - P_{n+1}P_{m+2})\mathbf{h}.$

Proof :

(i) By using equations (3.1) and (3.5), we get as follows:

$$\begin{aligned} & \tilde{P}_n \bar{\tilde{P}}_m - \bar{\tilde{P}}_n \tilde{P}_m \\ &= (P_n + P_{n+1}\mathbf{i} + P_{n+2}\varepsilon + P_{n+3}\mathbf{h})(P_m - P_{m+1}\mathbf{i} - P_{m+2}\varepsilon - P_{m+3}\mathbf{h}) \\ & \quad - (P_n - P_{n+1}\mathbf{i} - P_{n+2}\varepsilon - P_{n+3}\mathbf{h})(P_m + P_{m+1}\mathbf{i} + P_{m+2}\varepsilon + P_{m+3}\mathbf{h}) \\ &= 2(P_{n+1}P_m - P_n P_{m+1})\mathbf{i} + 2(P_{n+2}P_m - P_n P_{m+2})\varepsilon \\ & \quad + 2(P_{n+3}P_m - P_n P_{m+3})\mathbf{h}. \end{aligned}$$

By using the same manner, we can show easily the other properties.

□

Thanks to study [16], we can give Theorem 3.5, Theorem 3.6 and Corollary 3.7 as follows:

Theorem 3.5 : *The matrix form of Pentanacci hybrid numbers is as follows:*

$$\varphi(\tilde{P}_n) = \begin{pmatrix} P_n + P_{n+2} & P_{n+1} - P_{n+2} + P_{n+3} \\ -P_{n+1} + P_{n+2} + P_{n+3} & P_n - P_{n+2} \end{pmatrix}. \quad (3.16)$$

Proof : In [16], the matrix form of a hybrid number $Z = a_0 + b_0\mathbf{i} + c_0\mathbf{\varepsilon} + d_0\mathbf{h}$ represented by $\varphi(Z)$ and written as:

$$\varphi(Z) = \begin{pmatrix} a_0 + c_0 & b_0 - c_0 + d_0 \\ -b_0 + c_0 + d_0 & a_0 - c_0 \end{pmatrix},$$

where

$$\varphi(1) \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \varphi(\mathbf{i}) \leftrightarrow \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, \varphi(\mathbf{\varepsilon}) \leftrightarrow \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix}, \varphi(\mathbf{h}) \leftrightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}.$$

By the same manner, we complete the proof:

$$\begin{aligned} \varphi(\tilde{P}_n) &= P_n \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + P_{n+1} \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} + P_{n+2} \begin{pmatrix} 1 & -1 \\ 1 & -1 \end{pmatrix} + P_{n+3} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \\ &= \begin{pmatrix} P_n + P_{n+2} & P_{n+1} - P_{n+2} + P_{n+3} \\ -P_{n+1} + P_{n+2} + P_{n+3} & P_n - P_{n+2} \end{pmatrix}. \end{aligned}$$

□

Theorem 3.6 : If $\tilde{P}_n$ is Pentanacci hybrid number, $\forall n \in \mathbb{Z}$, the following properties hold:

- (i) $C(\tilde{P}_n) = \det(\varphi(\tilde{P}_n))$.
- (ii) $C_\varepsilon(\tilde{P}_n) = \frac{(tr(\varphi(\tilde{P}_n)))^2 - 4\det(\varphi(\tilde{P}_n))}{4} = -(P_{n+1} - P_{n+2})^2 + P_{n+2}^2 + P_{n+3}^2$.
- (iii) $\tilde{P}_n^{-1} = \frac{\overline{\tilde{P}_n}}{C(\tilde{P}_n)}, \|\tilde{P}_n\| \neq 0$.
- (iv) $N(\tilde{P}_n) = \sqrt{|C_\varepsilon(\tilde{P}_n)|}$.

- (v) The characters of $\tilde{P}_n$ are classified as: $\tilde{P}_n$ is spacelike, timelike, lightlike if $C(\tilde{P}_n) < 0$, $C(\tilde{P}_n) > 0$, $C(\tilde{P}_n) = 0$, respectively.
- (vi) The types of the $\tilde{P}_n$ are classified as: $\tilde{P}_n$ is elliptic, hyperbolic, parabolic if $C_\xi(\tilde{P}_n) < 0$, $C_\xi(\tilde{P}_n) > 0$, $C_\xi(\tilde{P}_n) = 0$, respectively.

Proof :

- (i) It is obvious using part (ii) of Theorem 3.3 and equation (3.16).
- (ii) The proof is clear by the properties (2.3) and (3.16).
- (iii) It can be seen by the properties (2.2), (3.5) and part (ii) of Theorem 3.3.
- (iv) It is clear from equation (2.4) and part (ii) of Theorem 3.6.
- (v) Considering equation (2.1), taking into account character classification the hybrid number (Section 2) and part (ii) of Theorem 3.3, it is obvious.
- (vi) Using equation (2.3), considering type classification of the hybrid number (Section 2) and part (ii) of Theorem 3.6, it is clear.

□

In the light of the above theorem, we can give the following corollary:

Corollary 3.1 : *If $\tilde{P}_n$ is Pentanacci hybrid number, $\forall n \in \mathbb{Z}$, the following outcomes can be given:*

- (i) $\|\tilde{P}_n\| = \sqrt{|\det(\varphi(\tilde{P}_n))|}$.
- (ii) *The inverse of $\tilde{P}_n$ exists if and only if $\det(\varphi(\tilde{P}_n)) \neq 0$.*
- (iii) *Lightlike Pentanacci hybrid numbers have no inverse.*
- (iv) $\Delta_{\varphi(\tilde{P}_n)} = (tr(\varphi(\tilde{P}_n)))^2 - 4\det(\varphi(\tilde{P}_n))$ is equal to discriminant of the characteristic polynomial of a 2×2 real matrix $\varphi(\tilde{P}_n)$. Hence,

$$C_\xi(\tilde{P}_n) = \frac{\Delta_{\varphi(\tilde{P}_n)}}{4}.$$
- (v) The type of a spacelike Pentanacci hybrid number is certainly hyperbolic. The type of a lightlike Pentanacci hybrid number is either parabolic or hyperbolic. Also, the type of a timelike Pentanacci hybrid number can be hyperbolic, parabolic or elliptic.

- (vi) The characters and types of Pentanacci hybrid number¹ $\tilde{P}_n$ are subject to the trace and determinant of $\varphi(\tilde{P}_n)$. Hence, if $\Delta_{\varphi(\tilde{P}_n)} < 0, \Delta_{\varphi(\tilde{P}_n)} > 0, \Delta_{\varphi(\tilde{P}_n)} = 0$, then $\tilde{P}_n$ is elliptic, hyperbolic, parabolic, respectively. Additionally, if $\det(\varphi(\tilde{P}_n)) < 0, \det(\varphi(\tilde{P}_n)) > 0, \det(\varphi(\tilde{P}_n)) = 0$, then $\tilde{P}_n$ is spacelike, timelike, lightlike, respectively.

Now, we shall present the following matrix properties.

Theorem 3.7: $\forall n \in \mathbb{Z}^+$, the following matrix properties are obtained:

$$(i) \quad \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}^n \begin{pmatrix} \tilde{P}_0 \\ \tilde{P}_1 \\ \tilde{P}_2 \\ \tilde{P}_3 \\ \tilde{P}_4 \end{pmatrix} = \begin{pmatrix} \tilde{P}_n \\ \tilde{P}_{n+1} \\ \tilde{P}_{n+2} \\ \tilde{P}_{n+3} \\ \tilde{P}_{n+4} \end{pmatrix},$$

$$(ii) \quad \begin{pmatrix} \tilde{P}_0 & \tilde{P}_1 & \tilde{P}_2 & \tilde{P}_3 & \tilde{P}_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}^n = \begin{pmatrix} \tilde{P}_n & \tilde{P}_{n+1} & \tilde{P}_{n+2} & \tilde{P}_{n+3} & \tilde{P}_{n+4} \end{pmatrix},$$

$$(iii) \quad \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \tilde{P}_4 & \tilde{P}_3 & \tilde{P}_2 & \tilde{P}_1 & \tilde{P}_0 \\ \tilde{P}_3 & \tilde{P}_2 & \tilde{P}_1 & \tilde{P}_0 & \tilde{P}_{-1} \\ \tilde{P}_2 & \tilde{P}_1 & \tilde{P}_0 & \tilde{P}_{-1} & \tilde{P}_{-2} \\ \tilde{P}_1 & \tilde{P}_0 & \tilde{P}_{-1} & \tilde{P}_{-2} & \tilde{P}_{-3} \\ \tilde{P}_0 & \tilde{P}_{-1} & \tilde{P}_{-2} & \tilde{P}_{-3} & \tilde{P}_{-4} \end{pmatrix} = \begin{pmatrix} \tilde{P}_{n+4} & \tilde{P}_{n+3} & \tilde{P}_{n+2} & \tilde{P}_{n+1} & \tilde{P}_n \\ \tilde{P}_{n+3} & \tilde{P}_{n+2} & \tilde{P}_{n+1} & \tilde{P}_n & \tilde{P}_{n-1} \\ \tilde{P}_{n+2} & \tilde{P}_{n+1} & \tilde{P}_n & \tilde{P}_{n-1} & \tilde{P}_{n-2} \\ \tilde{P}_{n+1} & \tilde{P}_n & \tilde{P}_{n-1} & \tilde{P}_{n-2} & \tilde{P}_{n-3} \\ \tilde{P}_n & \tilde{P}_{n-1} & \tilde{P}_{n-2} & \tilde{P}_{n-3} & \tilde{P}_{n-4} \end{pmatrix},$$

¹ As an example, $\tilde{P}_n$ is spacelike hyperbolic $\forall n \in \mathbb{N}$, $\tilde{P}_{-6}$ is timelike elliptic and $\tilde{P}_{-3}$ is lightlike parabolic.

$$\begin{aligned}
 \text{(iv)} \quad & \begin{pmatrix} \tilde{P}_0 & \tilde{P}_1 & \tilde{P}_2 & \tilde{P}_3 & \tilde{P}_4 \\ \tilde{L}_0 & \tilde{L}_1 & \tilde{L}_2 & \tilde{L}_3 & \tilde{L}_4 \end{pmatrix} \begin{pmatrix} 0 & 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 & 1 \\ 0 & 0 & 0 & 1 & 1 \end{pmatrix}^n \\
 &= \begin{pmatrix} \tilde{P}_n & \tilde{P}_{n+1} & \tilde{P}_{n+2} & \tilde{P}_{n+3} & \tilde{P}_{n+4} \\ \tilde{L}_n & \tilde{L}_{n+1} & \tilde{L}_{n+2} & \tilde{L}_{n+3} & \tilde{L}_{n+4} \end{pmatrix} \\
 \text{(v)} \quad & \begin{pmatrix} 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \tilde{P}_4 & \tilde{L}_4 \\ \tilde{P}_3 & \tilde{L}_3 \\ \tilde{P}_2 & \tilde{L}_2 \\ \tilde{P}_1 & \tilde{L}_1 \\ \tilde{P}_0 & \tilde{L}_0 \end{pmatrix} = \begin{pmatrix} \tilde{P}_{n+4} & \tilde{L}_{n+4} \\ \tilde{P}_{n+3} & \tilde{L}_{n+3} \\ \tilde{P}_{n+2} & \tilde{L}_{n+2} \\ \tilde{P}_{n+1} & \tilde{L}_{n+1} \\ \tilde{P}_n & \tilde{L}_n \end{pmatrix}.
 \end{aligned}$$

Proof:

- (i) Let use mathematical induction on n . If we take $n = 1, 2, 3$, then results are obviously true. Let assume that it is true $\forall n \in \mathbb{Z}^+$. Namely;

$$\begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}^n \begin{pmatrix} \tilde{P}_0 \\ \tilde{P}_1 \\ \tilde{P}_2 \\ \tilde{P}_3 \\ \tilde{P}_4 \end{pmatrix} = \begin{pmatrix} \tilde{P}_n \\ \tilde{P}_{n+1} \\ \tilde{P}_{n+2} \\ \tilde{P}_{n+3} \\ \tilde{P}_{n+4} \end{pmatrix}.$$

Using this fact and equation (3.10), we prove that it is true $\forall n+1 \in \mathbb{Z}^+$:

$$\begin{aligned}
 & \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}^{n+1} \begin{pmatrix} \tilde{P}_0 \\ \tilde{P}_1 \\ \tilde{P}_2 \\ \tilde{P}_3 \\ \tilde{P}_4 \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix}^n \begin{pmatrix} \tilde{P}_0 \\ \tilde{P}_1 \\ \tilde{P}_2 \\ \tilde{P}_3 \\ \tilde{P}_4 \end{pmatrix} \\
 &= \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 1 & 1 & 1 & 1 & 1 \end{pmatrix} \begin{pmatrix} \tilde{P}_n \\ \tilde{P}_{n+1} \\ \tilde{P}_{n+2} \\ \tilde{P}_{n+3} \\ \tilde{P}_{n+4} \end{pmatrix} = \begin{pmatrix} \tilde{P}_{n+1} \\ \tilde{P}_{(n+1)+1} \\ \tilde{P}_{(n+1)+2} \\ \tilde{P}_{(n+1)+3} \\ \tilde{P}_{(n+1)+4} \end{pmatrix}
 \end{aligned}$$

The other parts of the theorem can be easily obtained by using the same manner. $\square$

Theorem 3.8 : (Generating Function). Let $\tilde{P}_n \in \tilde{\mathbb{P}}$ and $\tilde{L}_n \in \tilde{\mathbb{L}}$. The following generating functions can be given for $\tilde{P}_n$ and $\tilde{L}_n$, respectively:

$$G(x) = \frac{\tilde{P}_0 + (\tilde{P}_1 - \tilde{P}_0)x + (\tilde{P}_2 - \tilde{P}_1 - \tilde{P}_0)x^2 + (\tilde{P}_3 - \tilde{P}_2 - \tilde{P}_1 - \tilde{P}_0)x^3 + (\tilde{P}_4 - \tilde{P}_3 - \tilde{P}_2 - \tilde{P}_1 - \tilde{P}_0)x^4}{1 - x - x^2 - x^3 - x^4 - x^5}, \quad (3.17)$$

$$H(x) = \frac{\tilde{L}_0 + (\tilde{L}_1 - \tilde{L}_0)x + (\tilde{L}_2 - \tilde{L}_1 - \tilde{L}_0)x^2 + (\tilde{L}_3 - \tilde{L}_2 - \tilde{L}_1 - \tilde{L}_0)x^3 + (\tilde{L}_4 - \tilde{L}_3 - \tilde{L}_2 - \tilde{L}_1 - \tilde{L}_0)x^4}{1 - x - x^2 - x^3 - x^4 - x^5}. \quad (3.18)$$

Proof : Assume that the function

$$G(x) = \sum_{n=0}^{\infty} \tilde{P}_n x^n = \tilde{P}_0 + \tilde{P}_1 x + \tilde{P}_2 x^2 + \dots + \tilde{P}_n x^n + \dots$$

be generating function of the Pentanacci hybrid numbers. Multiplying both sides by x, x^2, x^3, x^4, x^5 , we have:

$$\begin{aligned} xG(x) &= \tilde{P}_0 x + \tilde{P}_1 x^2 + \tilde{P}_2 x^3 + \dots + \tilde{P}_n x^{n+1} + \dots \\ x^2 G(x) &= \tilde{P}_0 x^2 + \tilde{P}_1 x^3 + \tilde{P}_2 x^4 + \dots + \tilde{P}_n x^{n+2} + \dots \\ x^3 G(x) &= \tilde{P}_0 x^3 + \tilde{P}_1 x^4 + \tilde{P}_2 x^5 + \dots + \tilde{P}_n x^{n+3} + \dots \\ x^4 G(x) &= \tilde{P}_0 x^4 + \tilde{P}_1 x^5 + \tilde{P}_2 x^6 + \dots + \tilde{P}_n x^{n+4} + \dots \\ x^5 G(x) &= \tilde{P}_0 x^5 + \tilde{P}_1 x^6 + \tilde{P}_2 x^7 + \dots + \tilde{P}_n x^{n+5} + \dots \end{aligned}$$

If the necessary calculations are taken and equation (3.10) are used, we get:

$$\begin{aligned} (1 - x - x^2 - x^3 - x^4 - x^5)G(x) &= \tilde{P}_0 + (\tilde{P}_1 - \tilde{P}_0)x + (\tilde{P}_2 - \tilde{P}_1 - \tilde{P}_0)x^2 \\ &\quad + (\tilde{P}_3 - \tilde{P}_2 - \tilde{P}_1 - \tilde{P}_0)x^3 \\ &\quad + (\tilde{P}_4 - \tilde{P}_3 - \tilde{P}_2 - \tilde{P}_1 - \tilde{P}_0)x^4. \end{aligned}$$

Therefore, we obtain equation (3.17). Equation (3.18) is obtained similarly. $\square$

Theorem 3.9 : (Binet-like Formula). Let $\tilde{P}_n \in \tilde{\mathbb{P}}$ and $\tilde{L}_n \in \tilde{\mathbb{L}}$. $\forall n \in \mathbb{Z}$, the Binet-like formulas for $\tilde{P}_n$ and $\tilde{L}_n$ are given, respectively:

$$\begin{aligned}
\tilde{P}_n = & \frac{\hat{a}r_1^{n+3}}{(r_1-r_2)(r_1-r_3)(r_1-r_4)(r_1-r_5)} + \frac{\hat{b}r_2^{n+3}}{(r_2-r_1)(r_2-r_3)(r_2-r_4)(r_2-r_5)} \\
& + \frac{\hat{c}r_3^{n+3}}{(r_3-r_1)(r_3-r_2)(r_3-r_4)(r_3-r_5)} + \frac{\hat{d}r_4^{n+3}}{(r_4-r_1)(r_4-r_2)(r_4-r_3)(r_4-r_5)} \\
& + \frac{\hat{e}r_5^{n+3}}{(r_5-r_1)(r_5-r_2)(r_5-r_3)(r_5-r_4)} \quad (3.19)
\end{aligned}$$

and

$$\tilde{L}_n = \hat{a}r_1^n + \hat{b}r_2^n + \hat{c}r_3^n + \hat{d}r_4^n + \hat{e}r_5^n, \quad (3.20)$$

$$\begin{cases} \hat{a} = 1 + r_1 \mathbf{i} + r_1^2 \varepsilon + r_1^3 \mathbf{h}, \\ \hat{b} = 1 + r_2 \mathbf{i} + r_2^2 \varepsilon + r_2^3 \mathbf{h}, \\ \hat{c} = 1 + r_3 \mathbf{i} + r_3^2 \varepsilon + r_3^3 \mathbf{h}, \\ \hat{d} = 1 + r_4 \mathbf{i} + r_4^2 \varepsilon + r_4^3 \mathbf{h}, \\ \hat{e} = 1 + r_5 \mathbf{i} + r_5^2 \varepsilon + r_5^3 \mathbf{h}. \end{cases}$$

Proof: Through utilizing the properties (2.8) and (3.2), we get:

$$\begin{aligned}
\tilde{L}_n &= L_n + L_{n+1} \mathbf{i} + L_{n+2} \varepsilon + L_{n+3} \mathbf{h} \\
&= r_1^n + r_2^n + r_3^n + r_4^n + r_5^n + (r_1^{n+1} + r_2^{n+1} + r_3^{n+1} + r_4^{n+1} + r_5^{n+1}) \mathbf{i} \\
&\quad + (r_1^{n+2} + r_2^{n+2} + r_3^{n+2} + r_4^{n+2} + r_5^{n+2}) \varepsilon \\
&\quad + (r_1^{n+3} + r_2^{n+3} + r_3^{n+3} + r_4^{n+3} + r_5^{n+3}) \mathbf{h} \\
&= (1 + r_1 \mathbf{i} + r_1^2 \varepsilon + r_1^3 \mathbf{h}) r_1^n + (1 + r_2 \mathbf{i} + r_2^2 \varepsilon + r_2^3 \mathbf{h}) r_2^n \\
&\quad + (1 + r_3 \mathbf{i} + r_3^2 \varepsilon + r_3^3 \mathbf{h}) r_3^n + (1 + r_4 \mathbf{i} + r_4^2 \varepsilon + r_4^3 \mathbf{h}) r_4^n \\
&\quad + (1 + r_5 \mathbf{i} + r_5^2 \varepsilon + r_5^3 \mathbf{h}) r_5^n \\
&= \hat{a}r_1^n + \hat{b}r_2^n + \hat{c}r_3^n + \hat{d}r_4^n + \hat{e}r_5^n.
\end{aligned}$$

Hence, we completed the proof of equation (3.20). In the same manner, we can obtain equation (3.19) by using equations (2.7) and (3.1). $\square$

Now, considering the study [26], we present following theorems that include some summation properties. Mathematical induction can be used for the proofs.

Theorem 3.10 : Let $\tilde{P}_n \in \tilde{\mathbb{P}}$. $\forall n, m \in \mathbb{N}$, the following properties hold:

- (i) $\sum_{n=0}^m \tilde{P}_n = \frac{1}{4}(\tilde{P}_{m+5} - \tilde{P}_{m+3} - 2\tilde{P}_{m+2} - 3\tilde{P}_{m+1} - \tilde{P}_4 + \tilde{P}_2 + 2\tilde{P}_1 + 3\tilde{P}_0),$
- (ii) $\sum_{n=0}^m \tilde{P}_{2n} = \frac{1}{8}(-\tilde{P}_{2m+2} + 4\tilde{P}_{2m+1} + 5\tilde{P}_{2m} + 2\tilde{P}_{2m-1} + 3\tilde{P}_{2m-2} + \tilde{P}_4$
 $-4\tilde{P}_3 + 3\tilde{P}_2 - 2\tilde{P}_1 + 5\tilde{P}_0),$
- (iii) $\sum_{n=0}^m \tilde{P}_{2n+1} = \frac{1}{8}(3\tilde{P}_{2m+2} + 4\tilde{P}_{2m+1} + \tilde{P}_{2m} + 2\tilde{P}_{2m-1} - \tilde{P}_{2m-2} - 3\tilde{P}_4$
 $+ 4\tilde{P}_3 - \tilde{P}_2 + 6\tilde{P}_1 + \tilde{P}_0).$

Theorem 3.11 : Let $\tilde{P}_{-n} \in \tilde{\mathbb{P}}$. $\forall n, m \in \mathbb{Z}^+$, the following summation properties are satisfied:

- (i) $\sum_{n=1}^m \tilde{P}_{-n} = \frac{1}{4}(-\tilde{P}_{-m+4} + \tilde{P}_{-m+2} + 2\tilde{P}_{-m+1} + 3\tilde{P}_{-m} + \tilde{P}_4 - \tilde{P}_2 - 2\tilde{P}_1 - 3\tilde{P}_0),$
- (ii) $\sum_{n=0}^m \tilde{P}_{-2n} = \frac{1}{8}(-3\tilde{P}_{-2m+3} + 4\tilde{P}_{-2m+2} - \tilde{P}_{-2m+1} + 6\tilde{P}_{-2m} + \tilde{P}_{-2m-1}$
 $-\tilde{P}_4 + 4\tilde{P}_3 - 3\tilde{P}_2 + 2\tilde{P}_1 - 5\tilde{P}_0),$
- (iii) $\sum_{n=0}^m \tilde{P}_{-2n+1} = \frac{1}{8}(\tilde{P}_{-2m+3} - 4\tilde{P}_{-2m+2} + 3\tilde{P}_{-2m+1} - 2\tilde{P}_{-2m} - 3\tilde{P}_{-2m-1}$
 $+ 3\tilde{P}_4 - 4\tilde{P}_3 + \tilde{P}_2 - 6\tilde{P}_1 - \tilde{P}_0).$

4. Conclusions

Inspired by the studies that include special hybrid numbers in the literature, we investigate the Pentanacci and Pentanacci-Lucas hybrid numbers in this study. Then, we give some special properties, give classifications with respect to characters and types, constitute generating functions, Binet-like formulas, matrix formulations and summation formulas regarding to them. It is expected that the results obtained in this study will shed light on future hybrid number studies and will be an important tool for applications in several research areas.

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