

On simple generalized m -flat rings

Shaimaa H. Ahmad *

Raida D. Mahmood [†]

Farah H. Mohammed [§]

Zubaida M. Ibraheem [‡]

Department of Mathematics

College of Computer Sciences and Mathematics

University of Mosul

Mosul 41002

Iraq

Abstract

A ring $\mathfrak{N}$ is known as a right (left) SGmF – ring, if each simple right (left) $\mathfrak{N}$ -modules are Gm – Flat. In this paper, we investigate characterizations of SGmF – ring. And from the results:

1. If $\mathfrak{N}$ is a semi-prime, and right SGmF–ring, then the $C(\mathfrak{N})$ is regular ring.
2. If $\mathfrak{N}$ is SGmF – ring, with every ideal of $\mathfrak{N}$ is principal then:
 - a. $\mathfrak{N}$ is NJ iff $\mathfrak{N}$ is feckly reduced,
 - b. $\mathfrak{N}$ is m – regular ring, if $\mathfrak{N}$ is ZC ring.

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1. Introduction

In this research paper, any module is unital, and $\mathfrak{N}$ represents an “associative ring with identity”. The symbols $N(\mathfrak{N})$, $J(\mathfrak{N})$ and $C(\mathfrak{N})$ denoted the set of nilpotent elements, the Jacobson radical and the center of $\mathfrak{N}$, respectively. For any $a \in \mathfrak{N}$, $r(a)$, $(\ell(a))$ stands for the right (left) annihilator of a . $\mathfrak{N}$ is “regular (strongly regular) if, for every $a \in \mathfrak{N}$, there

* E-mail: shaymaahatim@uomosul.edu.iq (Corresponding Author)

[†] E-mail: raida.1961@uomosul.edu.iq

[§] E-mail: farahalkadoo@uomosul.edu.iq

[‡] E-mail: z.mohammed@uomosul.edu.iq

is $b \in \mathfrak{N}$ so that $a = aba$ ($a = a^2b$) [1]. In [2], the regular elements in non-commutative rings have been discussed for example, $M_2(Z_2)$ and $M_2(Z_3)$. Following Rege [1] a ring $\mathfrak{N}$ is known as «a left (right) SF – ring if each simple left (right) $\mathfrak{N}$ - modules is flat». The connections between regular rings and SF – rings are studied in many papers (for example, following [3],[4],[5]and[6]). Also, [7] studied some properties of w-regular and antifuzzy semigroup. A ring $\mathfrak{N}$ is known as feckly reduced if, $\mathfrak{N} / J(\mathfrak{N})$ is reduced ring [8]. $\mathfrak{N}$ is NI iff $N(\mathfrak{N})$ is an ideal of $\mathfrak{N}$ [9].

2. SGmF– Rings

Following [10], “If $\mathfrak{N}$ is a ring and I is any ideal of $\mathfrak{N}$, then $\mathfrak{N} / I$ is right (left) generalized m – flat (GmF – ring), if for any $a \in I$, there is $b \in I$ and any fixed natural number m such that $a^m = ba^m$ ($a^m = a^m b$)”.

Now, we give many definitions that follows

Definition 2.1: A ring $\mathfrak{N}$ is known as right SGmF (SSGmF), if any simple (simple singular) right $\mathfrak{N}$ - module is GmF – ring.

Remark: Every SF – ring is SGmF – ring, but the converse is not true

Example: Z_9 of integer modulo 9 is SG2F- ring, which is not SF – ring.

From now on, a SGmF – ring means right SGmF – ring unless otherwise stated.

Definition 2.2: A ring $\mathfrak{N}$ satisfy condition $(*)$ if, for every $a \in \mathfrak{N}$ there exists a fixed natural number n such that $\ell(a^n) \subseteq \ell(a)$.

Now, a condition that is both necessary and sufficient of SGmF – ring to implies to SF – ring.

Proposition 2.3: Let $\mathfrak{N}$ be SGmF – ring and satisfy condition $(*)$, then $\mathfrak{N}$ is SF – ring.

Proof: Suppose that $\mathcal{M}$ be a maximal right ideal of $\mathfrak{N}$. Since $\mathfrak{N}$ to SGmF – ring, then for any $a \in \mathcal{M}$ there is $b \in \mathcal{M}$ and a fixed natural number m such that $a^m = ba^m$.

Hence $(1-b) \in \ell(a^m) \subseteq \ell(a)$ and so $a = ba$. Therefore $\mathfrak{N}$ is SF – ring.

Lemma 2.4 : [11] If $\mathfrak{N}$ is semi-prime ring, we get $r(a^n) = r(a)$ for every $a \in C(\mathfrak{N})$ and a positive integer n .

Proposition 2.5: If $\mathfrak{N}$ is a semi-prime and SGmF – ring , so the $C(\mathfrak{N})$ is regular ring.

Proof: Firstly we search about $a\mathfrak{N} + r(a) = \mathfrak{N}$ for every $a \in C(\mathfrak{N})$. If otherwise , we get a maximal right ideal $\mathcal{M}$ of $\mathfrak{N}$ so that, $a\mathfrak{N} + r(a) \subseteq \mathcal{M}$. Since $\mathfrak{N}$ is SGmF – ring, there is fixed natural number m and $b \in \mathfrak{M}$ such that, $a^m = ba^m$. Since $a \in C(\mathfrak{N})$. Then $a^m = a^m b$ and $(1-b) \in r(a^m) = r(a)$ (Lemma 2.4). So $1 \in \mathcal{M}$ a contradiction. Hence $a\mathfrak{N} + r(a) = \mathfrak{N}$ for every $a \in C(\mathfrak{N})$ and we have $axa = a$ for some $x \in \mathfrak{N}$. By ([10] Lemma 2.4), $C(\mathfrak{N})$ is regular ring.

Proposition 2.6: If $\mathfrak{N}$ is a semi-prime and SGmF – ring, so the $C(\mathfrak{N})$ is strongly regular.

Proof: Let $a^2 = 0$ for all $a \in C(\mathfrak{N})$. Then $ar = ra$, $r \in \mathfrak{N}$. Therefore $ara = ra^2 = 0$, so $a\mathfrak{N}a = 0$. Since $\mathfrak{N}$ is semi-prime ring, then $a = 0$. Hence $C(\mathfrak{N})$ is reduced and from Proposition (2.5) $C(\mathfrak{N})$ is regular. Hence $C(\mathfrak{N})$ is strongly regular.

Following [12], a ring $\mathfrak{N}$ is known as a right (left) m – regular ring, if for any $a \in \mathfrak{N}$, there is fixed natural number m and $x, y \in \mathfrak{N}$ such that $a^m = a^{m+1}x$ ($a^m = ya^{m+1}$).

Theorem 2.7: If $\mathfrak{N}$ is semi-prime and SSGmF –ring, so the $C(\mathfrak{N})$ is m – regular.

Proof: Firstly we search about $a\mathfrak{N} + r(a^m) = \mathfrak{N}$, for every $a \in C(\mathfrak{N})$ and m a fixed natural number. If not, we have a maximal right ideal $\mathcal{M}$ of $\mathfrak{N}$ so that $a\mathfrak{N} + r(a^m) \subseteq \mathcal{M}$. First observe that $\mathcal{M}$ is an essential right ideal of $\mathfrak{N}$. If not, then $\mathcal{M}$ is a direct summand of $\mathfrak{N}$. Thus $\mathcal{M} = r(g)$ for some $0 \neq g \neq g^2 \in \mathfrak{N}$. This gives $a \in \mathcal{M} = r(g)$, $0 = ga = ag$ ($a \in C(\mathfrak{N})$) and so $g \in r(a) \subseteq \mathcal{M} = r(g)$. Hence $g = 0$ and it is a contradiction. So $\mathcal{M}$ essential ideal of $\mathfrak{N}$. Therefore $\mathfrak{N}/\mathcal{M}$ is GmF so we have $b \in \mathcal{M}$ and a fixed natural number n so that $(a^m)^n = b(a^m)^n = (a^m)^n b$, $(1-b) \in r(a^m)^n = r(a^m) \subseteq \mathcal{M}$ (Lemma 2.4). Hence $(1-b) \in \mathcal{M}$ and then $1 \in \mathcal{M}$, and its contradiction. And we get $a\mathfrak{N} + r(a^m) = \mathfrak{N}$. Hence $C(\mathfrak{N})$ is right (left) m – regular ring.

Following [9], a ring $\mathfrak{N}$ is known as NJ, if $N(\mathfrak{N}) \subseteq J(\mathfrak{N})$.

Remark: Every NI rings are NJ- ring [9]

Theorem 2.8: If $\mathfrak{N}$ is NJ, and SGmF – ring, with any ideal of $\mathfrak{N}$ is principal. Then $J(\mathfrak{N}) = N(\mathfrak{N})$.

Proof: Let $0 \neq a \in N(\mathfrak{N})$, and $\mathfrak{N}$ is SGmF – ring. Then for any right maximal ideal $\mathcal{M}$ of $\mathfrak{N}$ and $a \in \mathcal{M}$ there is fixed natural number m and $b \in \mathcal{M}$ such that $a^m = ba^m = axa^m$ for some $x \in \mathcal{M}$, hence $(1 - ax)a^m = 0$. Since $a \in J(\mathfrak{N})$. Then $(1 - ax)$ is invertible and $y(1 - ax)a^m = a^m$ for some $y \in \mathfrak{N}$. So $a^m = 0$ and hence $a \in N(\mathfrak{N})$. Therefore $J(\mathfrak{N}) = N(\mathfrak{N})$ because $\mathfrak{N}$ is NJ – ring.

Lemma 2.9: [8] *If $\mathfrak{N}$ is feckly reduced ring, then all nilpotent elements of $\mathfrak{N}$ belonging to $J(\mathfrak{N})$.*

Theorem 2.10: *Let $\mathfrak{N}$ be SGmF – ring, with every ideal of $\mathfrak{N}$ is principal. Then $\mathfrak{N}$ is NJ – ring if $\mathfrak{N}$ is feckly reduced.*

Proof: Let $\mathfrak{N}$ is NJ – ring. Then by (Theorem 2.8) $J(\mathfrak{N}) = N(\mathfrak{N})$. Hence $\mathfrak{N}/J(\mathfrak{N})$ is reduced. Therefore $\mathfrak{N}$ is a feckly reduced.

Conversely: Assume $\mathfrak{N}$ is a feckly reduced, then by (Lemma 2.9) $\mathfrak{N}$ is NJ – ring.

From Theorem (2.10) and ([8], Proposition 2.6) we get:

Corollary 2.11: *Let $\mathfrak{N}$ be SGmF – ring, with any ideal of $\mathfrak{N}$ is principal. Then $\mathfrak{N}$ is NJ if $J(\mathfrak{N}) = \text{nil}(J(\mathfrak{N}))$.*

Theorem 2.12: *If $\mathfrak{N}$ satisfy condition (*) and SGmF – ring, with every ideal of $\mathfrak{N}$ is principal. Then $J(\mathfrak{N}) = 0$.*

Proof: Suppose that $a \in J(\mathfrak{N})$. Since $\mathfrak{N}$ is SGmF – ring, then for any right Maximal ideal $\mathcal{M}$ of $\mathfrak{N}$, there is a fixed natural number m and $b \in \mathcal{M}$ such that $a^m = ba^m$ ($a \in J(\mathfrak{N}) \subseteq \mathcal{M}$) and this implies that $a^m = axa^m$ for some $x \in \mathfrak{N}$. That is $(1 - ax) \in \ell(a^m) \subseteq \ell(a)$, hence $a = axa$. Since $a \in J(\mathfrak{N})$, then $(1 - ax)$ is invertible. That is $u(1 - ax) = 1$, $u \in \mathfrak{N}$ hence $a = 1 \cdot a = u(1 - ax)a = 0$. Therefore $J(\mathfrak{N}) = 0$.

By (Theorem 2.8) and (Theorem 2.12) we get:

Corollary 2.13: *Let $\mathfrak{N}$ be NJ and satisfy condition (*), with any ideal of $\mathfrak{N}$ is principal. Then $\mathfrak{N}$ is reduced. If $\mathfrak{N}$ is SGmF – ring.*

In Cohn [13], a ring $\mathfrak{N}$ is known as ZC, if $ab = 0$ then $ba = 0$ for $a, b \in \mathfrak{N}$.

Theorem 2.14: *Let $\mathfrak{N}$ be ZC – ring, with every ideal of $\mathfrak{N}$ is principal. The equivalent statements as follows :*

1. $\mathfrak{N}$ is m – regular ring.

2. $\mathfrak{N}$ is right (left) SGmF – ring.
3. $\mathfrak{N}$ is right (left) SSGmF – ring.

Proof: Clearly (1) $\rightarrow$ (2) $\rightarrow$ (3) holds

(2) $\rightarrow$ (1) Assume that $r(\mathfrak{a}^m) + \mathfrak{a}\mathfrak{N} \neq \mathfrak{N}$, for every $\mathfrak{a} \in \mathfrak{N}$, and for some fixed natural number m . So there is a maximal ideal $\mathcal{M}$ of $\mathfrak{N}$ to the right where that $r(\mathfrak{a}^m) + \mathfrak{a}\mathfrak{N} \subseteq \mathcal{M}$. Since $\mathfrak{N}/\mathcal{M}$ is SGmF, then there is $x \in \mathcal{M}$ such that $\mathfrak{a}^m = x\mathfrak{a}^m = \mathfrak{a}y\mathfrak{a}^m$, for some $x \in \mathfrak{N}$. Hence $(1 - \mathfrak{a}y) \in \ell(\mathfrak{a}^m) = r(\mathfrak{a}^m) \subseteq \mathcal{M}$, $1 - \mathfrak{a}y + \mathfrak{a}y = 1 \in \mathcal{M}$, a contradiction. Therefore $r(\mathfrak{a}^m) + \mathfrak{a}\mathfrak{N} = 1$. In particular $b + \mathfrak{a}z = 1$ for some $z \in \mathfrak{N}$, $b \in r(\mathfrak{a}^m)$ and $\mathfrak{a}^m b + \mathfrak{a}^{m+1}z = \mathfrak{a}^m$. So $\mathfrak{N}$ is right m – regular ring. Since $\mathfrak{N}$ is ZC–ring. Therefore $\mathfrak{N}$ is m –regular ring.

(3) $\rightarrow$ (1) we'll prove that $r(\mathfrak{a}^m) + r(\mathfrak{a}) = \mathfrak{N}$ for any $\mathfrak{a} \in \mathfrak{N}$ and for some a fixed natural number m . If $r(\mathfrak{a}^m) + \mathfrak{a}\mathfrak{N} \neq \mathfrak{N}$ So there is a maximal ideal $\mathcal{M}$ of $\mathfrak{N}$ to the right where that $r(\mathfrak{a}^m) + \mathfrak{a}\mathfrak{N} \subseteq \mathcal{M}$. First look at $\mathcal{M}$ is an essential right ideal of $\mathfrak{N}$. If otherwise, we have $\mathcal{M}$ is a direct summand of $\mathfrak{N}$. Thus $\mathcal{M} = r(g)$ for some $0 \neq g = g^2 \in \mathfrak{N}$. This gives $\mathfrak{a} \in \mathfrak{N} = r(g)$ $0 = g\mathfrak{a} = \mathfrak{a}g$ ($\mathfrak{N}$ is ZC – ring) and so $g \in r(\mathfrak{a}) \subseteq \mathfrak{N} = r(g)$. Hence $g = 0$, and it is a contradiction. So $\mathcal{M}$ is essential right ideal of $\mathfrak{N}$. Since $\mathfrak{N}$ is SSGmF – ring, then the similar discussion to the proof of (2) $\rightarrow$ (1), we get $\mathfrak{N}$ is m – regular ring.

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