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# Norms over bifuzzy submodules

Rasul Rasuli
*Department of Mathematics*
*Payame Noor University (PNU)*
*Tehran*
*Iran*

**Abstract**

In this work, we introduce the concepts of bifuzzy submodules under norms and consider some of their properties and structured characteristics. Also, we investigate the intersection and direct product of them and obtaion some new results. Finally, we investigate their properties under homomorphisms of modules.

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## 1. Introduction

Fuzzy sets were introduced by Zadeh[48] and Dieter Klaua [9] in 1965 as an extension of the classical notion of set. As a generalization of a fuzzy set, the concept of a bifuzzy set was introduced by Atanassov [3, 4]. Many researches introduced and presented various types of norms ($T$ and $C$) for particular purposes [2, 7, 8, 33]. Malakooti Rad and Mahdavi [11] investigated a note on the intersection graph of submodules of a module. The author by using norms, investigated some properties of fuzzy algebraic structures [12-46]. In this work as Section 2, we define bifuzzy submodules under norms and obtain some conditions that bifuzzy submodules will be submodules. In Section 3, we introduce the concepts of intersection and direct product of bifuzzy submodules under norms and investigate some of their properties. Next, we consider some of their properties and

*E-mail:* rasulirasul@yahoo.com

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structures of bifuzzy submodules under norms by homomorphisms of *R*-submodules.

## 2. Bifuzzy submodules under norms

We have following definition from [5].

**Definition 2.1 :** Let *X* a non-empty sets. A fuzzy subset $\mu$ of *X* is a function $\mu : X \to [0,1]$. Denote by $[0,1]^X$, the set of all fuzzy subset of *X*.

We give following definition from [3, 6].

**Definition 2.2 :** Let *X* be a nonempty set. An intuitionistic fuzzy set is an object of the form: $A = \{(\mu_A(x), \nu_A(x)) : x \in X\}$ such that $\mu_A, \nu_A \in [0,1]^X$ and for all $x \in X$ we have $\mu_A(x) + \nu_A(x) \in [0,1]$. This object is also called a bifuzzy set. For the sake of simplicity, we shall use the symbol $A = (\mu_A, \nu_A)$ for the $A = \{(\mu_A(x), \nu_A(x)) : x \in X\}$.

Let *X* be a nonempty set and let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be *IFSs* in *X*. Then

(1) $A \subseteq B$ iff $\mu_A \leq \mu_B$ and $\nu_A \geq \nu_B$.

(2) $A = B$ iff $A \subseteq B$ and $B \subseteq A$.

We have following definition from [5].

**Definition 2.3 :** A *t*-norm *T* is a function $T : [0,1] \times [0,1] \to [0,1]$ having the following four properties:

(T1) $T(x,1) = x$ (neutral element),

(T2) $T(x,y) \leq T(x,z)$ if $y \leq z$ (monotonicity),

(T3) $T(x,y) = T(y,x)$ (commutativity),

(T4) $T(x,T(y,z)) = T(T(x,y),z)$ (associativity),

for all $x, y, z \in [0,1]$.

It is clear that if $x_1 \geq x_2$ and $y_1 \geq y_2$, then $T(x_1, y_1) \geq T(x_2, y_2)$.

**Example 2.4 :**

(1) Standard intersection *t*-norm $T_m(x,y) = \min\{x,y\}$.

(2) Bounded sum *t*-norm $T_b(x,y) = \max\{0, x+y-1\}$.

(3) algebraic product *t*-norm $T_p(x,y) = xy$.

(4) Drastic *t*-norm

$$T_D(x,y)=\begin{cases} y & \text{if } x=1 \\ x & \text{if } y=1 \\ 0 & \text{otherwise.} \end{cases}$$

(5) Nilpotent minimum *t*-norm

$$T_{nM}(x,y)=\begin{cases} \min\{x,y\} & \text{if } x+y>1 \\ 0 & \text{otherwise.} \end{cases}$$

(6) Hamacher product *t*-norm

$$T_{H_0}(x,y)=\begin{cases} 0 & \text{if } x=y=0 \\ \dfrac{xy}{x+y-xy} & \text{otherwise.} \end{cases}$$

The drastic *t*-norm is the pointwise smallest *t*-norm and the minimum is the pointwise largest $t$ -norm: $T_D(x,y)\le T(x,y)\le T_{\min}(x,y)$ for all $x,y\in[0,1]$.

We have following definition from [5].

**Definition 2.5 :** A *t*-conorm $C$ is a function $C:[0,1]\times[0,1]\to[0,1]$ having the following four properties:

(C1) $C(x,0)=x$,

(C2) $C(x,y)\le C(x,z)$ if $y\le z$,

(C3) $C(x,y)=C(y,x)$,

(C4) $C(x,C(y,z))=C(C(x,y),z)$,

for all $x,y,z\in[0,1]$.

(1) Standard union *t*-conorm $C_m(x,y)=\max\{x,y\}$.

(2) Bounded sum *t*-conorm $C_b(x,y)=\min\{1,x+y\}$.

(3) Algebraic sum *t*-conorm $C_p(x,y)=x+y-xy$.

(4) Drastic *T*-conorm

$$C_D(x,y)=\begin{cases} y & \text{if } x=0 \\ x & \text{if } y=0 \\ 1 & \text{otherwise,} \end{cases}$$

dual to the drastic *t*-norm.

(5) Nilpotent maximum *t*-conorm , dual to the nilpotent minimum *t*-norm:

$$C_{nM}(x,y)=\begin{cases}\max\{x,y\} & \text{if } x+y<1\\ 1 & \text{otherwise.}\end{cases}$$

(6) Einstein sum (compare the velocity-addition formula under special relativity) $C_{H_2}(x,y)=\frac{x+y}{1+xy}$ is a dual to one of the Hamacher *t*-norms. Note that all *t*-conorms are bounded by the maximum and the drastic t-conorm: $C_{\max}(x,y)\le C(x,y)\le C_D(x,y)$ for any *t*-conorm *C* and all $x,y\in[0,1]$.

Recall that *t*-norm *T* (*t*-conorm *C*) is idempotent if for all $x\in[0,1]$, $T(x,x)=x(C(x,x)=x)$.

Followind lemma is from [1].

**Lemma 2.7 :** *Let T be a t-norm and C be a t-conorm. Then*

$$T(T(x,y),T(w,z))=T(T(x,w),T(y,z))$$

*and*

$$C(C(x,y),C(w,z))=C(C(x,w),C(y,z)),$$

*for all* $x,y,w,z\in[0,1]$.
we have the following definition in [10].

**Definition 2.8 :** Let *R* be a ring. A commutative group $(M,+)$ is called a left *R*-module or a left module over *R* with respect to a mapping

$$\therefore R\times M\to M$$

if for all $r,s\in R$ and $m,n\in M$,

(1) $r.(m+n)=r.m+r.n$,

(2) $r.(s.m)=(rs).m$,

(3) $(r+s).m=r.m+s.m$.

If *R* has an identity 1 and if $1.m=m$ for all $m\in M$, then *M* is called a unitary or unital left *R*-module. A right *R*-module can be defined in a similar fashion.

Let $M$ be an $R$-module and $S$ be a nonempty subset of $M$. Then $S$ is called a submodule of M if $S$ is a subgroup of $M$ and for all $r \in R, a \in S$, we have $ra \in S$.

Now we define bifuzzy submodule of $R$-module $M$ under norms.

**Definition 2.9 :** A bifuzzy set $M^* = (\mu_M, \nu_M)$ in $R$-module $M$ is called bifuzzy submodule of $M$ with respect to a $t$-norm $T$ and a $t$-conorm $C$, if it satisfies:

(1) $M^*(x+y) \supseteq (T(\mu_M(x), \mu_M(y)), C(\nu_M(x), \nu_M(y)))$,

(2) $M^*(rx) \supseteq M^*(x)$,

(3) $M^*(0) = (1,0)$,

for all $x, y \in M$ and $r \in R$.

Denote by $BFN(M)$, the set of all bifuzzy submodules of $M$ with respect to norms ($T$ and $C$).

**Example 2.10 :** Let $R = (\mathbb{Z}, +, -)$ be a ring of integers and $M = \mathbb{Z}$ be an $R$-module. Define $\mu_{\mathbb{Z}} : \mathbb{Z} \to [0,1]$ as

$$\mu_{\mathbb{Z}}(z) = \begin{cases} 0.45 & \text{if } z \in \{\pm 1, \pm 3, \ldots\} \\ 0 & \text{if } z = 0 \\ 0.35 & \text{if } z \in \{\pm 2, \pm 4, \ldots\} \end{cases}$$

and

$\nu_{\mathbb{Z}} : \mathbb{Z} \to [0,1]$ as

$$\nu_{\mathbb{Z}}(z) = \begin{cases} 0.55 & \text{if } z \in \{\pm 1, \pm 3, \ldots\} \\ 0 & \text{if } z = 0 \\ 0.65 & \text{if } z \in \{\pm 2, \pm 4, \ldots\}. \end{cases}$$

For all $x, y \in [0,1]$ if $C(x,y) = C_p(x,y) = x + y - xy$ and $T(x,y) = T_p(x,y) = xy$, then $\mathbb{Z}^* = (\mu_{\mathbb{Z}}, \nu_{\mathbb{Z}}) \in BFN(\mathbb{Z})$.

**Proposition 2.11 :** Let $R$ a ring with identity 1, and $T$ and $C$ be idempotent. Then for any $r \in R$ and $x \in M$ the condition (2) in Definition 2.9 is equivalent to the condition $M^*(rx) = M^*(x)$.

***Proof :*** As $M^*(rx) = (\mu_M(rx), \nu_M(rx))$ so from condition (2) we get that

(a) $\mu_M(rx) \geq \mu_M(x)$ and

(b) $\nu_M(rx) \leq \nu_M(x)$. Now

$$\mu_M(x) = \mu_M(rx + (1-r)x) \geq T(\mu_M(rx), \mu_M((1-r)x)) = T(\mu_M(rx), \mu_M(x - rx))$$

$$\geq T(\mu_M(rx), T(\mu_M(x), \mu_M(-rx)) \geq T(\mu_M(x), T(\mu_M(x), \mu_M(-rx))$$

$$= T(T(\mu_M(x), \mu_M(x)), \mu_M(-rx)) = T(\mu_M(x), \mu_M(-rx))$$

$$= T(\mu_M(x), \mu_M(-1rx)) \geq T(\mu_M(x), \mu_M(rx))$$

then we obtain $\mu_M(x) = T(\mu_M(x), \mu_M(x)) \geq T(\mu_M(x), \mu_M(rx))$ and so

(c) $\mu_M(x) \geq \mu_M(rx)$.

Also

$$\nu_M(x) = \nu_M(rx + (1-r)x) \leq C(\nu_M(rx), \nu_M((1-r)x)) = C(\nu_M(rx), \nu_M(x - rx))$$

$$\leq C(\nu_M(rx), C(\nu_M(x), \nu_M(-rx)) \leq C(\nu_M(x), C(\nu_M(x), \nu_M(-rx))$$

$$= C(C(\nu_M(x), \nu_M(x)), \nu_M(-rx)) = C(\nu_M(x), \nu_M(-rx))$$

$$= C(\nu_M(x), \nu_M(-1rx)) \leq C(\nu_M(x), \nu_M(rx))$$

then we get $\nu_M(x) = C(\nu_M(x), \nu_M(x)) \leq C(\nu_M(x), \nu_M(rx))$ and then

(d) $\nu_M(x) \leq \nu_M(rx)$. Then from (a) and (c) we get that $\mu_M(rx) = \mu_M(x)$ and from (b) and (d) we have $\nu_M(rx) = \nu_M(x)$. Thus $M^*(rx) = M^*(x)$.

□

**Proposition 2.12 :** Let $R$ be a field. Then the condition (2) in Definition 2.9 for all $r \in R, r \neq 0,$ will be equivalent to condition $M^*(rx) = M^*(x)$.

***Proof :*** If the the condition (2) be satisfied, then $\mu_M(rx) \geq \mu_M(x) = \mu_M(\frac{1}{r}(rx))$ $\geq \mu_M(rx)$ and so $\mu_M(rx) = \mu_M(x)$. Also $\nu_M(rx) \leq \nu_M(x) = \nu_M(\frac{1}{r}(rx)) \leq \nu_M(rx)$ and so $\nu_M(rx) = \nu_M(x)$. Therefore $M^*(rx) = M^*(x)$. □

**Proposition 2.13 :** Let there exists $x \in M$ such that $M^*(x) = (1,0)$. If $R$ a ring with identity 1, then the condition (3) in Definition 2.9 follows from conditions (1) and (2).

***Proof :*** Let conditions (1) and (2) in Definition 2.9 are hold and there exists $x \in M$ such that $M^*(x) = (\mu_M(x), \nu_M(x)) = (1,0)$. Then for such $x,$ we have

$$\mu_M(0) = \mu_M(x-x) \geq T(\mu_M(x), \mu_M(-x)) = T(1, \mu_M(-x)) = \mu_M(-x) = \mu_M(-1x) \geq \mu_M(x) = 1$$

and so $\mu_M(0) = 1$. Also

$$\nu_M(0) = \nu_M(x-x) \leq C(\nu_M(x), \nu_M(-x)) = C(0, \nu_M(-x)) = \nu_M(-x) = \nu_M(-1x) \leq \nu_M(x) = 0$$

and then $\nu_M(0) = 0$.

Thus $M^*(x) = (1,0)$ and thus condition (3) will be hold. □

The following propositions investigate the relationships between bifuzzy submodules of *R*-module *M* under norms and submodules of *R*-modules *M*.

**Proposition 2.14 :** Let $M^* = (\mu_M, \nu_M) \in BFN(M)$. Then $S = \{x \mid x \in M, M^*(x) = (1,0)\}$ is a submodule of the module *M* and $M^* \in BFN(S)$.

***Proof :*** Let $x, y \in S$ and $r \in R$. Then $\mu_M(x) = 1 = \mu_M(y)$ and $\nu_M(x) = 0 = \nu_M(y)$. Let $M^* = (\mu_M, \nu_M) \in BFN(M)$.

(1) As $\mu_M(x+y) \geq T(\mu_M(x), \mu_M(y)) = T(1,1) = 1$ so $\mu_M(x+y) = 1$.

(2) As $\nu_M(x+y) \leq C(\nu_M(x), \nu_M(y)) = C(0,0) = 0$ and $\nu_M(x+y) = 0$.

Now from (1) and (2) we get that $x+y \in S$.

(3) As $\mu_M(rx) \geq \mu_M(x) = 1$ so $\mu_M(rx) = 1$.

(4) As $\nu_M(rx) \leq \nu_M(x) = 0$ so $\nu_M(rx) = 0$.

Then from (3) and (4) we obtain that $rx \in S$.

(5) Also since $M^*(0) = (1,0)$ so $0 \in S$.

Now from (1)-(5) we get that *S* will be a submodule of the module *M*.

The second part of the statement is obvios. □

**Proposition 2.15 :** Let $M^* = (\mu_M, \nu_M) \in BFN(M)$. If *T* and *C* be idempotent, then

$$S_t = \{x \mid x \in M, M^*(x) \supseteq M^*(t)\}$$

will be a submodule of the module *M* for any $t \in [0,1]$.

***Proof :*** Let $x, y \in S_t$ and $r \in R$. Then $\mu_M(x) \geq t$ and $\mu_M(y) \geq t$. Also $\nu_M(x) \leq t$ and $\nu_M(y) \leq t$. Let $A = (\mu_M, \nu_M) \in BFN(M)$.

(1) Since

$$\mu_M(x+y) \geq T(\mu_M(x), \mu_M(y)) \geq T(t,t) = t$$

and

$$\nu_M(x+y) \le C(\nu_M(x), \nu_M(y)) \le C(t,t) = t$$

so $\mu_M(x+y) \ge t$ and $\nu_M(x+y) \le t$. Then $x+y \in S_t$.

(2) As $\mu_M(rx) \ge \mu_M(x) = t$ and $\nu_M(rx) \le \nu_M(x) = t$ so $\mu_M(rx) \ge t$ and $\nu_M(rx) \le t$.

Thus $rx \in S_t$.

(3) Also since $M^*(0) = (1,0) \ge (\mu_M(t), \nu_M(t)) = M^*(t)$ for any $t \in [0,1]$ so $0 \in S_t$.

Now from (1)-(3) we get that $S_t$ will be a submodule of the module *M*. □

**Proposition 2.16 :** Let *M* be an *R*-module and *S* be a subset of *M*. Let $\mu_M : S \to \{0,1\}$ be the characteristic function of *S* as

$$\mu_M(x) = \begin{cases} 1 & \text{if } x \in S \\ 0 & \text{if } x \notin S \end{cases}$$

and $\nu_M : S \to \{0,1\}$ be the anti characteristic function of *S* as

$$\nu_M(x) = \begin{cases} 0 & \text{if } x \in S \\ 1 & \text{if } x \notin S. \end{cases}$$

Then $M^* = (\mu_M, \nu_M) \in BFN(M)$ if and only if *S* is a submodule of *M*.

***Proof :*** Let $M^* = (\mu_M, \nu_M) \in BFN(M)$ and we prove that *S* is a submodule of *M*. Let $x, y \in S \subseteq M$ and $r \in R$. Since $\mu_M(x+y) \ge T(\mu_M(x), \mu_M(y)) = T(1,1) = 1$ and $\nu_M(x+y) \le C(\nu_M(x), \nu_M(y)) = C(0,0) = 0$ so $\mu_M(x+y) = 1$ and $\nu_M(x+y) = 0$. Thus $x+y \in S$. Also $\mu_M(rx) \ge \mu_M(x) = 1$ and $\nu_M(rx) \le \nu_M(x) = 0$ and then $rx \in S$. Finally $M^*(0) = (1,0)$ means that $0 \in S$. Thus *S* is a submodule of *M*.

Conversely, let *S* is a submodule of *M* and we prove that $M^* = (\mu_M, \nu_M) \in BFN(M)$. Let $x, y \in M$.

(1) If $x, y \in S$, then $x+y \in S$. Now

$$\mu_M(x+y) = 1 \ge 1 = T(1,1) = T(\mu_M(x), \mu_M(y))$$

and

$$\nu_M(x+y) = 0 \le 0 = C(0,0) = C(\nu_M(x), \nu_M(y)).$$

(2) If $x \in S$ and $y \notin S$, then $x+y \notin S$. Then

$$\mu_M(x+y) = 0 \ge 0 = T(1,0) = T(\mu_M(x), \mu_M(y))$$

and

$$\nu_M(x+y) = 1 \le 1 = C(0,1) = C(\nu_M(x), \nu_M(y)).$$

(3) If $x, y \notin S$, then we have the following conditions.

(a) If $x+y \notin S$, then

$$\mu_M(x+y) = 0 \ge 0 = T(0,0) = T(\mu_M(x), \mu_M(y))$$

and

$$\nu_M(x+y) = 1 \le 1 = C(1,1) = C(\nu_M(x), \nu_M(y)).$$

(b) If $x+y \in S$, then

$$\mu_M(x+y) = 1 \ge 0 = T(0,0) = T(\mu_M(x), \mu_M(y))$$

and

$$\nu_M(x+y) = 0 \le 1 = C(1,1) = C(\nu_M(x), \nu_M(y)).$$

Therefore from (1)-(3) we obtain that

$$M^*(x+y) \supseteq (T(\mu_M(x), \mu_M(y)), C(\nu_M(x), \nu_M(y))).$$

Now let $x \in M$ and $r \in R$.

(1) if $x \in S$, then $rx \in S$ and so $\mu_M(rx) = 1 \ge 1 = \mu_M(x)$ and $\nu_M(rx) = 0 \le 0 = \nu_M(x)$.

(2) if $x \notin S$, then $rx \notin S$ and so $\mu_M(rx) = 0 \ge 0 = \mu_M(x)$ and $\nu_M(rx) = 1 \le 1 = \nu_M(x)$.

Thus (1) and (2) give us that $M^*(rx) \supseteq M^*(x)$.

Finally, as $0 \in S$ so $\mu_M(0) = 1$ and $\nu_M(0) = 0$ and then $M^*(0) = (\mu_M(0), \nu_M(0)) = (1,0)$.

Hence $M^* = (\mu_M, \nu_M) \in BFN(M)$. □

## 3. Intersections, direct products and homomorphisms of bifuzzy submodules under norms

**Definition 3.1 :** (See [14]) Let $A=(\mu_A,\nu_A)$ and $B=(\mu_B,\nu_B)$ be two intuitionistic fuzzy sets of $X$. Define intersection of $A$ and $B$ as $A\cap B=(\mu_{A\cap B},\nu_{A\cap B})$ by

$$\mu_{A\cap B}(x)=T(\mu_A(x),\mu_B(x))$$

and

$$\nu_{A\cap B}(x)=C(\nu_A(x),\nu_B(x))$$

for all $x\in X$.

**Proposition 3.2 :** Let $M_1^*=(\mu_{M_1^*},\nu_{M_1^*})$ and $M_2^*=(\mu_{M_2^*},\nu_{M_2^*})$ be two intuitionistic fuzzy sets in $R$-module $M$. If $M_1^*,M_1^*\in BFN(M),$ then

$$M_1^*\cap M_2^*=(\mu_{M_1^*\cap M_2^*},\nu_{M_1^*\cap M_2^*})\in BFN(M).$$

***Proof :*** Let $x,y\in M$. Since

$$\mu_{M_1^*\cap M_2^*}(x+y)=T(\mu_{M_1^*}(x+y),\mu_{M_2^*}(x+y))\geq T(T(\mu_{M_1^*}(x),\mu_{M_1^*}(y)),T(\mu_{M_2^*}(x),\mu_{M_2^*}(y)))$$

$$=T(T(\mu_{M_1^*}(x),\mu_{M_2^*}(x)),T(\mu_{M_1^*}(y),\mu_{M_2^*}(y)))=T(\mu_{M_1^*\cap M_2^*}(x),\mu_{M_1^*\cap M_2^*}(y))$$

so

$$\mu_{M_1^*\cap M_2^*}(x+y)\geq T(\mu_{M_1^*\cap M_2^*}(x),\mu_{M_1^*\cap M_2^*}(y)).$$

Also

$$\nu_{M_1^*\cap M_2^*}(x+y)=C(\nu_{M_1^*}(x+y),\nu_{M_2^*}(x+y))\leq C(C(\nu_{M_1^*}(x),\nu_{M_1^*}(y)),C(\nu_{M_2^*}(x),\nu_{M_2^*}(y)))$$

$$=C(C(\nu_{M_1^*}(x),\nu_{M_2^*}(x)),C(\nu_{M_1^*}(y),\nu_{M_2^*}(y)))=C(\nu_{M_1^*\cap M_2^*}(x),\nu_{M_1^*\cap M_2^*}(y))$$

and then

$$\nu_{M_1^*\cap M_2^*}(x+y)\leq C(\nu_{M_1^*\cap M_2^*}(x),\nu_{M_1^*\cap M_2^*}(y)).$$

Thus

$$(M_1^*\cap M_2^*)(x+y)\supseteq(T(\mu_{M_1^*\cap M_2^*}(x),\mu_{M_1^*\cap M_2^*}(y)),C(\nu_{M_1^*\cap M_2^*}(x),\nu_{M_1^*\cap M_2^*}(y))).$$

Let $x\in M$ and $r\in R$. As

$$\mu_{M_1^* \cap M_2^*}(rx) = T(\mu_{M_1^*}(rx), \mu_{M_2^*}(rx)) \geq T(\mu_{M_1^*}(x), \mu_{M_2^*}(x)) = \mu_{M_1^* \cap M_2^*}(x)$$

and

$$\nu_{M_1^* \cap M_2^*}(rx) = C(\nu_{M_1^*}(rx), \nu_{M_2^*}(rx)) \leq C(\nu_{M_1^*}(x), \nu_{M_2^*}(x)) = \nu_{M_1^* \cap M_2^*}(x)$$

so $(M_1^* \cap M_2^*)(rx) \supseteq (M_1^* \cap M_2^*)(x)$.

Finally, from

$$\mu_{M_1^* \cap M_2^*}(0) = T(\mu_{M_1^*}(0), \mu_{M_2^*}(0)) = T(1,1) = 1$$

and

$$\nu_{M_1^* \cap M_2^*}(0) = C(\nu_{M_1^*}(0), \nu_{M_2^*}(0)) = C(0,0) = 0$$

we obtain that

$$(M_1^* \cap M_2^*)(0) = (\mu_{M_1^* \cap M_2^*}(0), \nu_{M_1^* \cap M_2^*}(0)) = (1,0).$$

Hence

$$M_1^* \cap M_2^* = (\mu_{M_1^* \cap M_2^*}, \nu_{M_1^* \cap M_2^*}) \in BFN(M).$$

□

**Corollary 3.3 :** *Let* $\{M_i = (\mu_{M_i}, \nu_{M_i}) \mid i = 1,2,3,...,n\} \subseteq BFN(M)$. *Then* $\cap_{M_i} \in BFN(M)$.

**Definition 3.4 :** (See [14]) Let $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in $X$ and $Y$, respectively. The direct product of $A$ and $B$, denoted by $A \times B = (\mu_{A \times B}, \nu_{A \times B})$, is an intuitionistic fuzzy set in $X \times Y$ such that for all $x$ in $X$ and $y$ in $Y$, define

$$\mu_{A \times B}(x,y) = T(\mu_A(x), \mu_B(y))$$

and

$$\nu_{A \times B}(x,y) = C(\nu_A(x), \nu_B(y)).$$

**Proposition 3.5 :** Let $M_i$ be $R$-modules and $M_i^* = (\mu_{M_i^*}, \nu_{M_i^*}) \in BFN(M)$ for $i = 1,2$. Then

$$M_1^* \times M_2^* = (\mu_{M_1^* \times M_2^*}, \nu_{M_1^* \times M_2^*}) \in BFN(M_1 \times M_2).$$

***Proof :***

(1) Let $(x_1, y_1), (x_2, y_2) \in M_1 \times M_2$. Then

$$\mu_{M_1^* \times M_2^*}((x_1, y_1) + (x_2, y_2)) = \mu_{M_1^* \times M_2^*}(x_1 + x_2, y_1 + y_2)$$

$$= T(\mu_{M_1^*}(x_1 + x_2), \mu_{M_2^*}(y_1 + y_2)) \geq T(T(\mu_{M_1^*}(x_1), \mu_{M_1^*}(x_2)), T(\mu_{M_2^*}(y_1), \mu_{M_2^*}(y_2)))$$

$$= T(T(\mu_{M_1^*}(x_1), \mu_{M_2^*}(y_1)), T(\mu_{M_1^*}(x_2), \mu_{M_2^*}(y_2))) = T(\mu_{M_1^* \times M_2^*}(x_1, y_1), \mu_{M_1^* \times M_2^*}(x_2, y_2)).$$

Also

$$\nu_{M_1^* \times M_2^*}((x_1, y_1) + (x_2, y_2)) = \nu_{M_1^* \times M_2^*}(x_1 + x_2, y_1 + y_2)$$

$$= C(\nu_{M_1^*}(x_1 + x_2), \nu_{M_2^*}(y_1 + y_2)) \leq C(C(\nu_{M_1^*}(x_1), \nu_{M_1^*}(x_2)), C(\nu_{M_2^*}(y_1), \nu_{M_2^*}(y_2)))$$

$$= C(C(\nu_{M_1^*}(x_1), \nu_{M_2^*}(y_1)), C(\nu_{M_1^*}(x_2), \nu_{M_2^*}(y_2))) = C(\nu_{M_1^* \times M_2^*}(x_1, y_1), \nu_{M_1^* \times M_2^*}(x_2, y_2)).$$

Thus

$$(M_1^* \times M_2^*)((x_1, y_1) + (x_2, y_2))$$

$$\supseteq (T(\mu_{M_1^* \times M_2^*}(x_1, y_1), \mu_{M_1^* \times M_2^*}(x_2, y_2)), C(\nu_{M_1^* \times M_2^*}(x_1, y_1), \nu_{M_1^* \times M_2^*}(x_2, y_2))).$$

(2) Let $(x, y) \in M_1 \times M_2$ and $r \in R$. Now

$$\mu_{M_1^* \times M_2^*}(r(x, y)) = \mu_{M_1^* \times M_2^*}(rx, ry) = T(\mu_{M_1^*}(rx), \mu_{M_2^*}(ry))$$

$$\geq T(\mu_{M_1^*}(x), \mu_{M_2^*}(y)) = \mu_{M_1^* \times M_2^*}(x, y).$$

Also

$$\nu_{M_1^* \times M_2^*}(r(x, y)) = \nu_{M_1^* \times M_2^*}(rx, ry) = C(\nu_{M_1^*}(rx), \nu_{M_2^*}(ry))$$

$$\leq C(\nu_{M_1^*}(x), \nu_{M_2^*}(y)) = \nu_{M_1^* \times M_2^*}(x, y).$$

Therefore $(M_1^* \times M_2^*)(r(x, y)) \supseteq (M_1^* \times M_2^*)(x, y)$.

(3) Let $(0_{M_1}, 0_{M_2}) \in M_1 \times M_2$. Then

$$\mu_{M_1^* \times M_2^*}(0_{M_1}, 0_{M_2}) = T(\mu_{M_1^*}(0_{M_1}), \mu_{M_2^*}(0_{M_2})) = T(1, 1) = 1$$

and

$$\nu_{M_1^* \times M_2^*}(0_{M_1}, 0_{M_2}) = C(\nu_{M_1^*}(0_{M_1}), \nu_{M_2^*}(0_{M_2})) = C(0,0) = 0.$$

Then $(M_1^* \times M_2^*)(0_{M_1}, 0_{M_2}) = (1,0)$.

Hence (1)-(3) give us that $M_1^* \times M_2^* = (\mu_{M_1^* \times M_2^*}, \nu_{M_1^* \times M_2^*}) \in BFN(M_1 \times M_2)$.

□

**Corollary 3.6 :** *Let* $M_i$ *be R-modules and* $M_i^* = (\mu_{M_i^*}, \nu_{M_i^*}) \in BFN(M)$ *for* $i = 1,2,...,n$. *Then*

$$M_1^* \times M_2^* \times ... \times M_n^* \in BFN(M_1 \times M_2 \times ... \times M_n).$$

**Definition 3.7 :** (See [14]) Let Let $\varphi$ be a morphism from set $X$ into set $Y$ such that $A = (\mu_A, \nu_A)$ and $B = (\mu_B, \nu_B)$ be two intuitionistic fuzzy sets in $X$ and $Y$ respectively. For all $x \in X, y \in Y$, we define

$$\varphi(A)(y) = (\varphi(\mu_A)(y), \varphi(\nu_A)(y))$$

$$= \begin{cases} (\sup\{\mu_A(x) \mid x \in X, \varphi(x) = y\},\ \inf\{\nu_A(x) \mid x \in X, \varphi(x) = y\}) & \text{if } \varphi^{-1}(y) \neq \varnothing \\ (0,1) & \text{if } \varphi^{-1}(y) \neq \varnothing. \end{cases}$$

Also $\varphi^{-1}(B)(x) = (\varphi^{-1}(\mu_B)(x), \varphi^{-1}(\nu_B)(x)) = (\mu_B(\varphi(x)), \nu_B(\varphi(x)))$.

**Proposition 3.8 :** Let $\varphi$ be an epimorphism from $R$-module $M$ into $R$-module $S$. If $M^* = (\mu_{M^*}, \nu_{M^*}) \in BFN(M)$, then $\varphi(M^*) = (\mu_{\varphi(M^*)}, \nu_{\varphi(M^*)}) \in BFN(S)$.

***Proof :*** (1) Let $y_1, y_2 \in S$. Then

$$\mu_{\varphi(M^*)}(y_1 + y_2) = \sup\{\mu_{M^*}(x_1 + x_2) \mid x_1, x_2 \in M, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$

$$\geq \sup\{T(\mu_{M^*}(x_1), \mu_{M^*}(x_2)) \mid x_1, x_2 \in M, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$

$$= T(\sup\{\mu_{M^*}(x_1) \mid x_1 \in M, \varphi(x_1) = y_1\}, \sup\{\mu_{M^*}(x_2) \mid x_2 \in M, \varphi(x_2) = y_2\})$$

$$= T(\mu_{\varphi(M^*)}(y_1), \mu_{\varphi(M^*)}(y_2)).$$

Also

$$\nu_{\varphi(M^*)}(y_1 + y_2) = \inf\{\nu_{M^*}(x_1 + x_2) \mid x_1, x_2 \in M, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$

$$\leq \inf\{C(\nu_{M^*}(x_1), \nu_{M^*}(x_2)) \mid x_1, x_2 \in M, \varphi(x_1) = y_1, \varphi(x_2) = y_2\}$$

$$= C(\inf\{\nu_{M^*}(x_1) \mid x_1 \in M, \varphi(x_1) = y_1\}, \inf\{\nu_{M^*}(x_2) \mid x_2 \in M, \varphi(x_2) = y_2\})$$

$$= C(\nu_{\varphi(M^*)}(y_1), \nu_{\varphi(M^*)}(y_2)).$$

Therefore

$$\varphi(M^*)(y_1 + y_2) \supseteq (T(\mu_{\varphi(M^*)}(y_1), \mu_{\varphi(M^*)}(y_2)), C(\nu_{\varphi(M^*)}(y_1), \nu_{\varphi(M^*)}(y_2))).$$

(2) Let $y \in S$ and $r \in R$. Thus

$$\mu_{\varphi(M^*)}(ry) = \sup\{\mu_{M^*}(rx) \mid x \in M, \varphi(rx) = ry\}$$

$$\geq \sup\{\mu_{M^*}(x) \mid x \in M, \varphi(x) = y\} = \mu_{\varphi(M^*)}(y)$$

and

$$\nu_{\varphi(M^*)}(ry) = \inf\{\nu_{M^*}(rx) \mid x \in M, \varphi(rx) = ry\}$$

$$\leq \inf\{\nu_{M^*}(x) \mid x \in M, \varphi(x) = y\} = \nu_{\varphi(M^*)}(y).$$

Then $\varphi(M^*)(ry) \supseteq \varphi(M^*)(y)$.

(3) As $\mu_{M^*}(0_M) = 1$ and $\nu_{M^*}(0_M) = 0$ so we get that

$$\mu_{\varphi(M^*)}(0_S) = \sup\{\mu_{M^*}(0_M) \mid 0_M \in M, \varphi(0_M) = 0_S\} = \sup\{1 \mid 0_M \in M, \varphi(0_M) = 0_S\} = 1$$

and

$$\nu_{\varphi(M^*)}(0_S) = \inf\{\nu_{M^*}(0_M) \mid 0_M \in M, \varphi(0_M) = 0_S\} = \inf\{0 \mid 0_M \in M, \varphi(0_M) = 0_S\} = 0$$

and then $\varphi(M^*)(0_S) = (1, 0)$.

Hence from (1) to (3) we get that $\varphi(M^*) = (\mu_{\varphi(M^*)}, \nu_{\varphi(M^*)}) \in BFN(S)$. □

**Proposition 3.9 :** Let Let $\varphi$ be a morphism from $R$-module $M$ into $R$-module $S$. If $S^* = (\mu_{S^*}, \nu_{S^*}) \in BFN(S)$, then $\varphi^{-1}(S^*) = (\mu_{\varphi^{-1}(S^*)}, \nu_{\varphi^{-1}(S^*)}) \in BFN(M)$.

***Proof :*** (1) Let $x_1, x_2 \in M$.

As

$$\mu_{\varphi^{-1}(S^*)}(x_1 + x_2) = \mu_{S^*}(\varphi(x_1 + x_2)) = \mu_{S^*}(\varphi(x_1) + \varphi(x_2))$$

$$\geq T(\mu_{S^*}(\varphi(x_1)), \mu_{S^*}(\varphi(x_2))) = T(\mu_{\varphi^{-1}(S^*)}(x_1), \mu_{\varphi^{-1}(S^*)}(x_2))$$

and

$$\nu_{\varphi^{-1}(S^*)}(x_1 + x_2) = \nu_{S^*}(\varphi(x_1 + x_2)) = \nu_{S^*}(\varphi(x_1) + \varphi(x_2))$$

$$\leq C(\nu_{S^*}(\varphi(x_1)), \nu_{S^*}(\varphi(x_2))) = C(\nu_{\varphi^{-1}(S^*)}(x_1), \nu_{\varphi^{-1}(S^*)}(x_2))$$

so $\varphi^{-1}(S^*)(x_1 + x_2) \supseteq (T(\mu_{\varphi^{-1}(S^*)}(x_1), \mu_{\varphi^{-1}(S^*)}(x_2)), C(\nu_{\varphi^{-1}(S^*)}(x_1), \nu_{\varphi^{-1}(S^*)}(x_2)))$.

(2) Let $x \in M$ and $r \in R$. Then

$$\mu_{\varphi^{-1}(S^*)}(rx) = \mu_{S^*}(\varphi(rx)) = \mu_{S^*}(r\varphi(x)) \geq \mu_{S^*}(\varphi(x)) = \mu_{\varphi^{-1}(S^*)}(x)$$

and

$$\nu_{\varphi^{-1}(S^*)}(rx) = \nu_{S^*}(\varphi(rx)) = \nu_{S^*}(r\varphi(x)) \leq \nu_{S^*}(\varphi(x)) = \nu_{\varphi^{-1}(S^*)}(x).$$

Then $\varphi^{-1}(S^*)(rx) \supseteq \varphi^{-1}(S^*)(x)$.

(3) Since

$$\mu_{\varphi^{-1}(S^*)}(0_M) = \mu_{S^*}(\varphi(0_M)) = \mu_{S^*}(0_S) = 1$$

and

$$\nu_{\varphi^{-1}(S^*)}(0_M) = \nu_{S^*}(\varphi(0_M)) = \nu_{S^*}(0_S) = 0$$

so $\varphi^{-1}(S^*)(0_M) = (1, 0)$.

Thus (1) to (3) will give us that $\varphi^{-1}(S^*) = (\mu_{\varphi^{-1}(S^*)}, \nu_{\varphi^{-1}(S^*)}) \in BFN(M)$.

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