Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 26 (2023), No. 2, pp. 459–478
DOI : 10.1080/09720529.2021.1959997

Kernel fuzzy ideals and *-fuzzy congruences on pseudo-complemented almost distributive lattices

Wondwosen Zemene Norahun *
Department of Mathematics
University of Gondar
P. O. Box : 196
Gondar
Ethiopia

Dawit Solomon Tadesse †
Department of Mathematics
Kotebe Metropolitan University
P. O. Box : 31248
Addis Ababa
Ethiopia

Abstract

In this paper, we introduce the concept of *-fuzzy congruence relation on a pseudo-complemented almost distributive lattice and we provide a necessary and sufficient condition for a fuzzy congruence to be a *-fuzzy congruence. We also introduce the concept of kernel fuzzy ideals in terms of *-fuzzy congruence. Moreover, we give a set of equivalent condition for a fuzzy ideal to be a kernel fuzzy ideal. We show that every fuzzy filter of a pseudo-complemented ADL is a cokernel of some *-fuzzy congruence.

Subject Classification: *(2010) 06D72, 06F15, 08A72.*

Keywords: *Almost distributive lattice (ADL), Pseudo-complemented ADL, Kernel ideal, Fuzzy ideal, Fuzzy filter, Kernel fuzzy ideal, Cokernel fuzzy filter, Fuzzy congruence relation.*

* *E-mail:* `wondie1976@gmail.com (Corresponding Author)`
† *E-mail:* `devatasol@gmail.com`

1. Introduction

The theory of pseudo-complementation was introduced and extensively studied in semi-lattices and particularly in distributive lattices by O. Frink [15] and G. Birkhoff [13]. In 1973, W. H. Cornish [14], studied congruence on pseudo-complemented distributive lattices and identified those ideals and filters that are congruence kernels and cokernels respectively. In 2012, M. S. Rao and G. C. Rao [24], introduced the concept of *-fuzzy congruences and kernel ideals in a pseudo-complemented almost distributive lattice. They provide a necessary and sufficient condition for an ideal to be a kernel ideal. Recently, G. M. Addis [1], studied on congruence relations in a pseudo-complemented almost distributive lattice. He also studied the concept of congruence kernel ideals and congruence cokernel filters in a pseudo-complemented almost distributive lattice. Furthermore, he characterized congruence kernel ideals in terms of α -ideals and he provide a set of equivalent condition for every ideal to be congruence kernel.

Many papers on fuzzy algebras have been published since Rosenfeld [26] introduced the concept of fuzzy group in 1971. In particular, fuzzy subgroups and fuzzy normal subgroups of a group (see [9, 11, 12, 16, 19, 22]), fuzzy subrings and fuzzy ideals of a ring (see [17, 18, 19, 20, 21, 23]), fuzzy sublattices and fuzzy ideals of a lattice (see [3, 4, 5, 7, 8, 10, 28, 32]). Recently, Rasul Rasuli studied the concept of fuzzy congruence on product lattices by using T-norms [25]. B. A. Alaba and W. Z. Norahun studied the concept of fuzzy ideals and fuzzy filters in a pseudo-complemented semilattice. They also studied the concept of *-fuzzy congruence relations and investigate some of its properties. Using the concept of *-fuzzy congruence, they studied the concept of kernel fuzzy ideals and cokernel fuzzy filters in a pseudo-complemented semilattice [6]. In this paper, we introduce the concept of *-fuzzy congruence on a pseudo-complemented almost distributive lattice and we provide a necessary and sufficient condition for a fuzzy congruence to be a *-fuzzy congruence. We also introduce the concept of kernel fuzzy ideals and cokernel fuzzy filters in a pseudo-complemented ADL in terms of *-fuzzy congruence and investigate some of its properties. We give a set of equivalent condition for a fuzzy ideal to be a kernel fuzzy ideal. Furthermore, for any fuzzy ideal of L, we observe that there exists the smallest kernel fuzzy ideal containing it. We show that the class of all kernel fuzzy ideals forms a complete distributive lattice. We prove that every fuzzy filter of a pseudo-complemented ADL is a cokernel of some *-fuzzy congruence. We show that there is an isomorphism between the class of all kernel fuzzy ideals $FKI(L)$ of L and the class of all fuzzy ideals of the skeletal elements $FI(S(L))$ of L.

2. Preliminaries

We refer to [27] for the elementary properties of almost distributive lattices.

Definition 2.1 : [27] An almost distributive lattice (ADL) with 0 is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying the following properties:

(1) $(x \vee y) \wedge z = (x \wedge z) \vee (y \wedge z)$,

(2) $x \wedge (y \vee z) = (x \wedge y) \vee (x \wedge z)$,

(3) $(x \vee y) \wedge y = y$,

(4) $(x \vee y) \wedge x = x$,

(5) $x \vee (x \wedge y) = x$,

(6) $0 \wedge x = 0$, for all $x, y, z \in L$.

If "$\leq$" is a binary relation on L defined by:

$$a \leq b \text{ if and only if } a \wedge b = a, \text{ or equivalently } a \vee b = b$$

then, $(L, \leq)$ is a partially ordered set (poset).

An element m in L is said to be maximal if it is maximal with respect to the partial ordering $\leq$ on L.

Theorem 2.2 : [27] *Let L be an ADL and $m \in L$. Then the following are equivalent:*

(1) *m is a maximal element*

(2) *$m \vee a = m$ for all $a \in L$*

(3) *$m \wedge a = a$ for all $a \in L$*

(4) *$a \vee m$ is maximal for all $a \in L$*

A nonempty subset I of L is called an ideal (filter) of L if $a \vee b \in I (a \wedge b \in I)$ and $a \wedge x \in I (x \vee a \in I)$ whenever $a, b \in I$ and $x \in L$.

Definition 2.3 : [29] Let L be an ADL. Then the operation $a \to a^*$ on L is called a pseudo-complementation on L if, for any $a, b \in L$ it satisfies the following conditions:

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$

(2) $a \wedge a^* = 0$

(3) $(a \vee b)^* = a^* \wedge b^*$.

In this case, $(L, \vee, \wedge, *, 0)$ is called a pseudo-complemented ADL.

Theorem 2.4 : [20] *For any two elements a, b of a pseudo-complemented ADL, we have the following:*

(1) 0^* *is a maximal element,*

(2) *If m is a maximal element, then* $m^* = 0$,

(3) $0^{**} = 0$,

(4) $0^* \wedge a = a$,

(5) $a^{**} \wedge a = a$,

(6) $a^{***} = a^*$,

(7) $a \leq b \Rightarrow b^* \leq a^*$,

(8) $a^* \wedge b^* = b^* \wedge a^*$,

(9) $(a \wedge b)^{**} = a^{**} \wedge b^{**}$,

(10) $a^* \wedge b = (a \wedge b)^* \wedge b$.

For any pseudo-complemented ADL, let us denote the set of all elements of the form x^* by *D*. Clearly *D* is a filter of *L*.

Definition 2.5 : [3] Let *X* be any non-empty set. A mapping $\mu : X \to [0,1]$ is called a fuzzy subset of *X*.

The unit interval $[0,1]$ together with the operations min and max form a complete lattice satisfying the infinite meet distributive law; i.e.,

$$\alpha \wedge (\bigvee_{\beta \in M} \beta) = \bigvee_{\beta \in M} (\alpha \wedge \beta)$$

for all $\alpha \in [0,1]$ and any $M \subseteq [0,1]$.

We often write $\wedge$ for minimum or infimum and $\vee$ for maximum or supremum. That is, for all $\alpha, \beta \in [0,1]$ we have, $\alpha \wedge \beta = min\{\alpha, \beta\}$ and $\alpha \vee \beta = max\{\alpha, \beta\}$.

Definition 2.6 : [33] Let μ be a fuzzy subset of a set X and $t \in [0,1]$. The set $\mu_t = \{x \in X : \mu(x) \geq t\}$ is called a *t*-level subset of the fuzzy subset of μ. Clearly $\mu_t \subseteq \mu_s$ whenever $t \geq s$.

Definition 2.7 : [30] A fuzzy subset μ of an ADL L is called a fuzzy ideal of L if; for all $x, y \in L$ the following conditions are satisfied:

(1) $\mu(0) = 1$,

(2) $\mu(x \vee y) \geq \mu(x) \wedge \mu(y)$,

(3) $\mu(x \wedge y) \geq \mu(x) \vee \mu(y)$.

Definition 2.8 : [31] Let L be an ADL L with a maximal element m. A fuzzy subset μ of L is called a fuzzy filter of L if; for all $x, y \in L$ the following conditions are satisfied:

(1) $\mu(m) = 1$,

(2) $\mu(x \vee y) \geq \mu(x) \vee \mu(y)$,

(3) $\mu(x \wedge y) \geq \mu(x) \wedge \mu(y)$.

Theorem 2.9 : [30] *Let L be an almost distributive lattice,* $x \in L$ *and* $\alpha \in [0,1]$. *Define a fuzzy subset* α_x *of L as:*

$$\alpha_x(y) = \begin{cases} 1, \text{ if } \; y \in (x], \\ a, \text{ if } \; y \notin (x] \end{cases}$$

is a fuzzy ideal of L. α_x *is called the* α *-level fuzzy ideal corresponding to x.*

The binary operations "+" and "$\cdot$" on the set of all fuzzy subsets of a distributive lattice L defined as:

$$(\mu + \theta)(x) = Sup\{\mu(y) \wedge \theta(z) : y, z \in L, y \vee z = x\} \text{ and}$$

$$(\mu \cdot \theta)(x) = Sup\{\mu(y) \wedge \theta(z) : y, z \in L, y \wedge z = x\}.$$

If μ and θ are fuzzy ideals of L, then $\mu \cdot \theta = \mu \wedge \theta = \mu \cap \theta$ and $\mu + \theta = \mu \vee \theta$.

If μ and θ are fuzzy filters of L, then $\mu + \theta = \mu \wedge \theta$ and $\mu \cdot \theta = \mu \vee \theta$.

Definition 2.10 : [2] A fuzzy subset θ of $L \times L$ is said to be a fuzzy congruence relation on an ADL L if and only if, for any $x, y, z, w \in L$, the following hold:

(1) $\theta(x, x) = 1$,

(2) $\theta(x, y) = \theta(y, x)$,

(3) $\theta(x, y) \wedge \theta(y, z) \leq \theta(x, z)$,

(4) $\theta(x, y) \wedge \theta(z, w) \leq \theta(x \vee z, y \vee w) \wedge \theta(x \wedge z, y \wedge w)$.

Note: Throughout the rest of this paper L stands for a pseudo-complemented almost distributive lattice $(L, \vee, \wedge, *, 0, m)$.

3. *-fuzzy congruences on ADLs

In this section, we introduce a *-fuzzy congruence relation on a pseudo-complemented ADL. We give a necessary and sufficient condition for a fuzzy congruence to be a *-fuzzy congruence.

If μ is a fuzzy filter in an ADL, then the relation defined by:

$$\Psi_{\mu}(x, y) = Sup\{\mu(a) : x \wedge a = y \wedge a, a \in L\}$$

is a fuzzy congruence on L. It is a special case of the fuzzy congruence Ψ^{η} generally defined for multiplicatively closed fuzzy subset η in [2].

Lemma 3.1 : *For any two fuzzy filters η_1 and η_2 of L,*

$$\Psi_{\eta_1} \circ \Psi_{\eta_2} = \Psi_{\eta_2} \circ \Psi_{\eta_1}.$$

In this case, $\Psi_{\eta_1} \vee \Psi_{\eta_2} = \Psi_{\eta_1} \circ \Psi_{\eta_2}$.

Proof : To prove our claim, it is enough to prove that $\Psi_{\eta_1} \circ \Psi_{\eta_2} = \Psi_{\eta_1 \vee \eta_2}$. For any $x, y \in L$,

$$\begin{aligned}
(\Psi_{\eta_1} \circ \Psi_{\eta_2})(x, y) &= Sup\{\Psi_{\eta_1}(z, y) \wedge \Psi_{\eta_2}(x, z) : z \in L\} \\
&= Sup\{Sup\{\eta_1(a) : z \wedge a = y \wedge a\} \wedge Sup\{\eta_2(b) : x \wedge b = z \wedge b\} : \\
&\quad z \in L\} \\
&= Sup\{Sup\{\eta_1(a) \wedge \eta_2(b) : z \wedge a = y \wedge a, x \wedge b = z \wedge b\} : z \in L\}
\end{aligned}$$

If $z \wedge a = y \wedge a$ and $x \wedge b = z \wedge b$, then $x \wedge a \wedge b = y \wedge a \wedge b$. Thus

$$\begin{aligned}
(\Psi_{\eta_1} \circ \Psi_{\eta_2})(x, y) &\leq Sup\{Sup\{\eta_1(a) \wedge \eta_2(b) : x \wedge a \wedge b = y \wedge a \wedge b\}\} \\
&\leq Sup\{Sup\{\eta_1(r_1) \wedge \eta_2(r_2) : r_1 \wedge r_2 = a \wedge b\} : x \wedge a \wedge b = y \wedge a \wedge b\} \\
&= Sup\{(\eta_1 \vee \eta_2)(a \wedge b) : x \wedge a \wedge b = y \wedge a \wedge b\} \\
&\leq Sup\{(\eta_1 \vee \eta_2)(r) : x \wedge r = y \wedge r, r \in L\} \\
&= \psi_{\eta_1 \vee \eta_2}(x, y)
\end{aligned}$$

This shows that $\Psi_{\eta_1} \circ \Psi_{\eta_2} \subseteq \Psi_{\eta_1 \vee \eta_2}$.

On the other hand,

$$\begin{aligned}
\Psi_{\eta_1 \vee \eta_2}(x, y) &= Sup\{(\eta_1 \vee \eta_2)(r) : x \wedge r = y \wedge r, r \in L\} \\
&= Sup\{Sup\{\eta_1(a) \wedge \eta_2(b) : a \wedge b = r\} : x \wedge a \wedge b = y \wedge a \wedge b\}
\end{aligned}$$

Now we need to find some $z \in L$ such that $z \wedge a = y \wedge a$ and $x \wedge b = z \wedge b$.

If $x \wedge a \wedge b = y \wedge a \wedge b$, then $x \wedge b \wedge a = y \wedge b \wedge a$. Put $z = (y \wedge a) \vee (x \wedge b)$. Then $z \wedge a = y \wedge a$ and $z \wedge b = x \wedge b$. Based on this we have

$$\begin{aligned}\Psi_{\eta_1 \vee \eta_2}(x, y) &\leq Sup\{Sup\{\eta_1(a) : z \wedge a = y \wedge a\} \wedge Sup\{\eta_2(b) : z \wedge b = x \wedge b\} : r \in L\} \\ &= Sup\{\Psi_{\eta_1}(z, y) \wedge \Psi_{\eta_2}(x, z) : z \in L\} \\ &= (\Psi_{\eta_1} \circ \Psi_{\eta_2})(x, y)\end{aligned}$$

This shows that $\Psi_{\eta_1 \vee \eta_2} \subseteq \Psi_{\eta_1} \circ \Psi_{\eta_2}$. Hence $\Psi_{\eta_1 \vee \eta_2} = \Psi_{\eta_1} \circ \Psi_{\eta_2}$. In the similar procedure we can easily show that $\Psi_{\eta_2 \vee \eta_1} = \Psi_{\eta_2} \circ \Psi_{\eta_1}$. Since $\eta_1 \vee \eta_2 = \eta_2 \vee \eta_1$, we get that $\Psi_{\eta_1 \vee \eta_2} = \Psi_{\eta_2 \vee \eta_1}$. Therefore, $\Psi_{\eta_1} \circ \Psi_{\eta_2} = \Psi_{\eta_2} \circ \Psi_{\eta_1}$. □

Now we introduce the concept of *-fuzzy congruence relation.

Definition 3.2 : A fuzzy congruence relation ψ on L is called a *-fuzzy congruence if $\psi(x, y) \leq \psi(x^*, y^*)$ for all $x, y \in L$.

Theorem 3.3 : *A fuzzy congruence ψ on L is a *-fuzzy congruence if and only if each level subset of ψ is a *-congruence on L.*

Corollary 3.4 : *Let ψ be a congruence relation on L. Then ψ is a *-congruence on L if and only if its characteristic function χ_ψ is a *-fuzzy congruence on L.*

Theorem 3.5 : *A fuzzy congruence ψ on L is a *-fuzzy congruence if and only if*

$$\psi(x^*, m) \geq \psi(x, 0) \textit{ for all } x \in L.$$

Proof : Suppose ψ is a *-fuzzy congruence on L. Then by Definition 3.2 $\psi(x^*, 0^*) \geq \psi(x, 0)$. Suppose conversely that the condition holds. Let $a, b \in L$. Since ψ is a fuzzy congruence on L, $\psi(a, b) \leq \psi(a \wedge a^*, b \wedge a^*) = \psi(0, b \wedge a^*)$. By the assumption, we have $\psi(0, b \wedge a^*) \leq \psi(0^*, (b \wedge a^*)^*)$. Thus $\psi(a, b) \leq \psi(0^*, (b \wedge a^*)^*) \leq \psi(a^*, (b \wedge a^*)^* \wedge a^*) = \psi(a^*, a^* \wedge b^*)$.

Similarly, $\psi(a, b) \leq \psi(a^* \wedge b^*, b^*)$. Since ψ is a fuzzy congruence, $\psi(a^*, b^*) \geq \psi(a^*, a^* \wedge b^*) \wedge \psi(a^* \wedge b^*, b^*)$. Thus $\psi(a^*, b^*) \geq \psi(a, b)$. So ψ is a *-fuzzy congruence on L. □

Theorem 3.6 : *Let μ be a fuzzy filter of L. Define a fuzzy relation ψ^μ on L by:*

$$\psi^\mu(x, y) = Sup\{\mu(z) : x \wedge z^{**} = y \wedge z^{**}, z \in L\}.$$

Then ψ^{μ} is a *-fuzzy congruence.

Proof : Clearly ψ^{μ} is reflexive and symmetric.Now

(1) For any $a,b,c \in L$,

$$\begin{aligned}\psi^{\mu}(a,b)\wedge\psi^{\mu}(b,c) &= Sup\{\mu(f_1): a\wedge f_1^{**} = b\wedge f_1^{**}\}\\ &\quad \wedge Sup\{\mu(f_2): b\wedge f_2^{**} = c\wedge f_2^{**}\}\\ &= Sup\{\mu(f_1\wedge f_2): a\wedge f_1^{**} = b\wedge f_1^{**},\\ &\quad b\wedge f_2^{**} = c\wedge f_2^{**}\}\end{aligned}$$

If $a\wedge f_1^{**} = b\wedge f_1^{**}$ and $b\wedge f_2^{**} = c\wedge f_2^{**}$, then $a\wedge(f_1\wedge f_2)^{**} = c\wedge$ $\wedge(f_1\wedge f_2)^{**}$. Using this, we can easily show that $\psi^{\mu}(a,b)\wedge\psi^{\mu}(b,c)\leq\psi^{\mu}(a,c)$.

(2) For any $a,b,c,d \in L$,

$$\begin{aligned}\psi^{\mu}(a,b)\wedge\psi^{\mu}(c,d) &= Sup\{\mu(f_1): a\wedge f_1^{**} = b\wedge f_1^{**}\}\wedge\\ &\quad Sup\{\mu(f_2): c\wedge f_2^{**} = d\wedge f_2^{**}\}\\ &= Sup\{\mu(f_1\wedge f_2): a\wedge f_1^{**} = b\wedge f_1^{**},\\ &\quad c\wedge f_2^{**} = d\wedge f_2^{**}\}\end{aligned}$$

If $a\wedge f_1^{**} = b\wedge f_1^{**}$ and $c\wedge f_2^{**} = d\wedge f_2^{**}$, then $(a\wedge c)\wedge(f_1\wedge f_2)^{**} =$ $(b\wedge d)\wedge(f_1\wedge f_2)^{**}$ and $(a\vee c)\wedge(f_1\wedge f_2)^{**} = (b\vee d)\wedge(f_1\wedge f_2)^{**}$. Using this fact, we can show that $\psi^{\mu}(a,b)\wedge\psi^{\mu}(c,d)\leq\psi^{\mu}(a\wedge c,b\wedge d)$ and $\psi^{\mu}(a\wedge c,b\wedge d)\leq\psi^{\mu}(a\vee c,b\vee d)$. Thus ψ^{μ} is a fuzzy congruence on L.

To show ψ^{μ} is a *-fuzzy fuzzy congruence, let $\psi^{\mu}(a,0) = Sup\{\mu(f): a\wedge f^{**} = 0, f\in L\}$. Since $a\wedge f^{**} = 0$, we get that $a^{*}\wedge f^{**} = 0^{*}\wedge f^{**}$. This implies that $\psi^{\mu}(a,0)\leq\psi^{\mu}(a^{*},0^{*})$. Therefore ψ^{μ} is a *-fuzzy congruence. □

Theorem 3.7 : *Let μ be a fuzzy ideal of L. Define a fuzzy relation $\psi(\mu)$ on L by:*

$$\psi(\mu)(x,y) = Sup\{\mu(z): x\wedge z^{*} = y\wedge z^{*}, z\in L\}.$$

*Then $\psi(\mu)$ is a *-fuzzy congruence on L.*

Proof : Clearly $\psi(\mu)$ is both reflexive and symmetric.

(1) For any $a,b,c \in L$,

$$\begin{aligned}\psi(\mu)(a,b)\wedge\psi(\mu)(b,c) &= Sup\{\mu(x): a\wedge x^{*}=b\wedge x^{*}\}\\ &\quad\wedge Sup\{\mu(y): b\wedge y^{*}=c\wedge y^{*}\}\\ &= Sup\{\mu(x\vee y): a\wedge x^{*}=b\wedge x^{*},\\ &\quad b\wedge y^{*}=c\wedge y^{*}\}\end{aligned}$$

If $a\wedge x^{*}=b\wedge x^{*}$ and $b\wedge y^{*}=c\wedge y^{*}$, then $a\wedge(x\vee y)^{*}=c\wedge(x\vee y)^{*}$. Using this, we can easily show that $\psi(\mu)(a,b)\leq\psi(\mu)(a,c)$.

(2) For any $a,b,c,d\in L$,

$$\begin{aligned}\psi(\mu)(a,b)\wedge\psi(\mu)(c,d) &= Sup\{\mu(i_1): a\wedge i_1^{*}=b\wedge i_1^{*}\}\\ &\quad\wedge Sup\{\mu(i_2): c\wedge i_2^{*}=d\wedge i_2^{*}\}\\ &= Sup\{\mu(i_1\vee i_2^{*}): a\wedge i_1^{*}=b\wedge i_1^{*},\\ &\quad c\wedge i_2^{*}=d\wedge i_2^{*}\}\end{aligned}$$

If $a\wedge i_1^{*}=b\wedge i_1^{*}$ and $c\wedge i_2^{*}=d\wedge i_2^{*}$, we can easily verified that $(a\wedge c)\wedge(i_1\vee i_2)^{*}=(b\wedge d)\wedge(i_1\vee i_2)^{*}$ and $(a\vee c)\wedge(i_1\vee i_2)^{*}=(b\vee d)\wedge(i_1\vee i_2)^{*}$. Using this fact we can show that $\psi(\mu)(a,b)\wedge\psi(\mu)(c,d)\leq\psi(\mu)(a\wedge c,b\wedge d)$ and $\psi(\mu)(a,b)\wedge\psi(\mu)(c,d)\leq\psi(\mu)(a\vee c,b\vee d)$. Thus $\psi(\mu)$ is a fuzzy congruence on L.

Next, we show that $\psi(\mu)$ is a *-fuzzy congruence. Let $a\in L$. Then $\psi(\mu)(a,0)=Sup\{\mu(x): a\wedge x^{*}=0, x\in L\}$. Since $a\wedge x^{*}=0$, we have $a^{*}\wedge x^{*}=0^{*}\wedge x^{*}$. Thus $\psi(\mu)(a,0)\leq\psi(\mu)(a^{*},0^{*})$. So $\psi(\mu)$ is a *-fuzzy congruence. □

4. Kernel fuzzy ideals

In this section, we introduce the concept of kernel fuzzy ideals and cokernel fuzzy filters in a pseudo-complemented ADL in terms of *-fuzzy congruence and investigate some of its properties. We give a set of equivalent condition for a fuzzy ideal to be a kernel fuzzy ideal. Furthermore, we show that the class of all kernel fuzzy ideals forms a complete distributive lattice.

If ψ is a fuzzy congruence on L and $x\in L$, then the fuzzy subset $[\psi]_x$ of L defined by

$$[\psi]_x(y)=\psi(x,y)\text{ for all }y\in L$$

is called a fuzzy congruence class of L determined by ψ and x. In [2], B. A. Alaba and G. M. Addis observed that, the fuzzy congruence class $[\psi]_0$ of

L determined by 0 is a fuzzy ideal of L. Now we define kernel fuzzy ideal of an almost distributive lattice in terms of *-fuzzy congruence.

Definition 4.1 : A fuzzy ideal μ of L is called a kernel fuzzy ideal if $\mu = [\psi]_0$, where ψ is a *-fuzzy congruence on L.

Theorem 4.2 : *For any fuzzy ideal μ of L the following conditions are equivalent:*

(1) μ *is a kernel fuzzy ideal,*

(2) *For* $x, y \in L$, $x^* = y^* \Rightarrow \mu(x) = \mu(y)$,

(3) $\mu(x) = \mu(x^{**})$,

(4) $\mu((x^* \wedge y^*)^*) \geq \mu(x) \wedge \mu(y)$ *for all* $x, y \in L$.

Proof : The proof of $(2) \Rightarrow (3)$ and $(3) \Rightarrow (4)$ are straightforward. Now we proceed to prove the following.

$(1) \Rightarrow (2)$: Let μ be a kernel fuzzy ideal. Then $\mu = [\psi]_0$ for some *-fuzzy congruence ψ. Suppose $x, y \in L$ such that $x^* = y^*$. Then $\mu(x) = \psi(x, 0) \leq \psi(x^*, 0^*) = \psi(y^*, 0^*)$. Since ψ is a fuzzy congruence, we have $\mu(x) \leq \psi(y^*, 0^*) \wedge \psi(y, y) \leq \psi(y^* \wedge y, 0^* \wedge y) = \psi(0, y) = \mu(y)$. Similarly, $\mu(y) \leq \mu(x)$. Thus $\mu(x) = \mu(y)$.

$(4) \Rightarrow (1)$: Suppose condition (4) holds. Consider the fuzzy relation $\psi(\mu)$ defined on L by: $\psi(\mu)(x, y) = Sup\{\mu(z) : x \wedge z^* = y \wedge z^*, z \in L\}$. It is shown from the proof of Theorem (3.7) that $\psi(\mu)$ is a *-fuzzy congruence on L. Now we proceed to show that $\mu = [\psi(\mu)]_0$. Let $x \in L$, $[\psi(\mu)]_0(x) = Sup\{\mu(a) : x \wedge a^* = 0\} \geq \mu(x)$. To show the other inequality, let $y \in L$ satisfying $x \wedge y^* = 0$. Then $y^{**} \wedge x = x$. Which implies $\mu(x) = \mu(y^{**} \wedge x) \geq \mu(x) \vee \mu(y^{**})$. By condition (4), we get that $\mu(x) \geq \mu(x) \vee \mu(y)$. Thus $\mu(x) \geq \mu(y)$. This shows that $\mu(x)$ is an upper bound of $\{\mu(a) : x \wedge a^* = 0\}$. Thus $\mu(x) \geq [\psi(\mu)]_0(x)$. So $\mu \subseteq [\psi(\mu)]_0$. Hence $\mu = [\psi(\mu)]_0$. □

Corollary 4.3 : *If μ is a kernel fuzzy ideal of L, then $\psi(\mu)$ is the smallest *-fuzzy congruence with kernel μ.*

Proof : It is shown from the proof of Theorem 3.7 and Theorem 4.2 that $\psi(\mu)$ is a *-fuzzy congruence with kernel μ. To show $\psi(\mu)$ is the smallest *-fuzzy congruence with kernel μ, let ϕ be a *-fuzzy congruence with kernel μ. For any $x, y \in L$,

$$\psi(\mu)(x, y) = Sup\{\mu(z) : x \wedge z^* = y \wedge z^*, z \in L\}.$$

Since ϕ is a *-fuzzy congruence, $\phi(x \wedge a^*, x) \geq \phi(x,x) \wedge \phi(a^*, 0^*) = \phi(a^*, 0^*) \geq \phi(a,0) = \mu(a)$. Similarly, $\phi(y \wedge a^*, y) \geq \mu(a)$. If $x \wedge a^* = y \wedge a^*$, then $\phi(x, x \wedge a^*) \wedge \phi(y \wedge a^*, y) \geq \mu(a)$. This implies that $\phi(x,y)$ is an upper bound of $\{\mu(z) : x \wedge z^* = y \wedge z^*, z \in L\}$. Thus $\psi(\mu) \subseteq \phi$. So $\psi(\mu)$ is the smallest *-fuzzy congruence with kernel μ. □

The set $S(L) = \{x^{**} : x \in L\}$ is called the skeleton of L. The elements of $S(L)$ are called skeletal.

Lemma 4.4 : *Let $\alpha \in [0,1]$. If x is a skeletal element, then α_x is a kernel fuzzy ideal.*

Theorem 4.5 : *A fuzzy subset μ of L is a kernel fuzzy ideal if and only if each level subset of μ is kernel ideal of L.*

Corollary 4.6 : *A non-empty subset I of L is a kernel ideal of L if and only if χ_I is a kernel fuzzy ideal.*

Definition 4.7 : A fuzzy filter μ of L is called a cokernel fuzzy filter if $\mu = [\psi]_m$, where ψ is a *-fuzzy congruence on L.

In the following theorem, we observe that every fuzzy filter of L is a cokerner fuzzy filter.

Theorem 4.8 : *If η is a fuzzy filter of L, then the fuzzy relation defined by:*

$$\Psi_\eta(x,y) = Sup\{\mu(a) : x \wedge a = y \wedge a, a \in L\}$$

*is a *-fuzzy congruence with cokernel fuzzy filter η. Moreover, Ψ_η is the smallest *-fuzzy congruence on L.*

Proof : Suppose that η is a fuzzy filter of L. Then clearly Ψ_η is a fuzzy congruence on L. Now we proceed to show that Ψ_η is a *-fuzzy congruence. Let $x \in L$. Then

$$\Psi_\eta(x,0) = Sup\{\eta(b) : x \wedge b = 0, b \in L\}.$$

If $x \wedge b = 0$, then $x^* \wedge b = 0^* \wedge b$. This shows that $\Psi_\eta(x,0) \leq \Psi_\eta(x^*, 0^*)$. Thus by Theorem (3.5), Ψ_η is a *-fuzzy congruence. To show η is a cokernel fuzzy filter, let $x \in L$. Then $\psi_\eta(x,m) = Sup\{\eta(a) : x \wedge a = m \wedge a\} \geq \eta(b)$.

On the other hand, let $b \in L$ such that $x \wedge b = b = m \wedge b$. Since η is a fuzzy filter, $\eta(b) = \eta(x) \wedge \eta(b)$. Which implies $\eta(b) \leq \eta(x)$. This shows us $\eta(x)$ is an upper bound of $\{\eta(a) : x \wedge a = m \wedge a\}$.. Thus $\eta(x) \geq \Psi_\eta(x,m)$. So η is a cokernel of Ψ_η.

Finally, we show that Ψ_η is the smallest *-fuzzy congruence with cokernel η. Let ϕ be a *-fuzzy congruence with cokernel η. For any $x, y, a \in L$,

$$\eta(a) = \phi(m,a) = \phi(m,a) \wedge \phi(x,x) \leq \phi(m \wedge x, a \wedge x) = \phi(x, x \wedge a).$$

Similarly, $\phi(y \wedge a, y) \geq \eta(a)$.

Let $x \wedge a = y \wedge a$. Since ϕ is a fuzzy congruence, we get that $\phi(x,y) \geq \eta(a)$. It shows that $\phi(x,y)$ is an upper bound of $\{\eta(a) : x \wedge a = y \wedge a\}$. Thus $\phi(x,y) \geq \Psi_\eta(x,y)$. So Ψ_η is the smallest *-fuzzy congruence with cokernel η. □

Lemma 4.9 : *If μ is a fuzzy ideal of L, then the fuzzy subset μ_* of L defined by:*

$$\mu_*(x) = Sup\{\mu(a) : x \wedge a^* = a^*\}$$

is a fuzzy filter of L.

Theorem 4.10 : *Let μ be a nonempty fuzzy subset of L. μ is a kernel fuzzy ideal if and only if μ is a fuzzy ideal of L and $\mu(x) = \mu(x^{**})$ for all $x \in L$.*

Proof : Let μ be a kernel fuzzy ideal of L. Then $\mu = [\psi]_0$ for some *-fuzzy congruence ψ. For any $x \in L$, we have that $\mu(x^{**}) \leq \mu(x)$. Now,

$$\mu(x) = \psi(x,0) \leq \psi(x^*, 0^*) \leq \psi(x^{**}, 0^{**}) = \psi(x^{**}, 0) = \mu(x^{**}).$$

Thus $\mu(x) = \mu(x^{**})$ for all $x \in L$. Suppose conversely the condition holds. Then by the above lemma μ_* is a fuzzy filter of L and hence by Theorem 4.8, ψ_{μ_*} is a *-fuzzy congruence with cokernel μ_*. To show that $[\psi_{\mu_*}]_0 = \mu$, let $x \in L$. Then

$$[\psi_{\mu_*}]_0(x) = Sup\{\mu_*(a) : x \wedge a = 0\} = Sup\{Sup\{\mu(b) : a \wedge b^* = b^*\} : x \wedge a = 0\}.$$

If $a \wedge b^* = b^*$, then $x \wedge a \wedge b^* = 0 = x \wedge b^*$. Since L is a pseudo-complemented ADL, $b^{**} \wedge x = x$. This implies $\mu(x) \geq \mu(b^{**}) \vee \mu(x)$. By the assumption, $\mu(x) \geq \mu(b) \vee \mu(x)$. Which implies $\mu(x) \geq \mu(b)$. Thus $[\psi_{\mu_*}]_0 \subseteq \mu$. On the other hand, since $x \wedge x^* = 0$, we have $[\psi_{\mu_*}]_0(x) \geq \mu_*(x^*) = Sup\{\mu(b) : x^* \wedge b^* = b^*\} \geq \mu(x)$. Thus $\mu = [\psi_{\mu_*}]_0$. So μ is a kernel fuzzy ideal of L. □

Corollary 4.11 : *Let μ and θ be kernel fuzzy ideals of L. Then $\mu \cap \theta$ is kernel fuzzy ideal of L.*

Corollary 4.11 can be generalized as given in the following.

If $\{\mu_\alpha : \alpha \in \Delta\}$ is a nonempty family of kernel fuzzy ideals of L, then $\bigcap_{\alpha \in \Delta} \mu_\alpha$ is a kernel fuzzy ideal of L.

In the following theorem, we show that for any fuzzy ideal there exists the smallest kernel fuzzy ideal containing it.

Theorem 4.12 : *For any fuzzy ideal μ of L,*

$$\Gamma(\mu)(x) = Sup\{\mu(a) : a^{**} \wedge x = x, a \in L\}$$

is the smallest kernel fuzzy ideal containing μ.

Proof : Since $x^{**} \wedge x = x$ for all $x \in L$, we have that $\mu(x) \le \Gamma(\mu)(x)$. Thus $\mu \subseteq \Gamma(\mu)$. Now we show that $\Gamma(\mu)$ is a fuzzy ideal of L. Let $x, y \in L$. Then

$$\begin{aligned} \Gamma(\mu)(x) \wedge \Gamma(\mu)(y) &= Sup\{\mu(a) : a^{**} \wedge x = x\} \wedge Sup\{\mu(b) : b^{**} \wedge y = y\} \\ &= Sup\{\mu(a \vee b) : a^{**} \wedge x = x, b^{**} \wedge y = y\} \end{aligned}$$

Let $a^{**} \wedge x = x$ and $b^{**} \wedge y = y$. Then $(a \vee b)^{**} \wedge (x \vee y) = x \vee y$ and $(a \vee b)^{**} \wedge y = y$. Thus $(a \vee b)^{**} \wedge (x \vee y) = ((a \vee b)^{**} \wedge x) \vee ((a \vee b)^{**} \wedge y) = x \vee y$. Then we can easily verified that $\Gamma(\mu)(x) \wedge \Gamma(\mu)(y) \le \Gamma(\mu)(x \vee y)$. Moreover, we can show that $\Gamma(\mu)(x \wedge y) \ge \Gamma(\mu)(x)$ and $\Gamma(\mu)(x \wedge y) \ge \Gamma(\mu)(y)$. Thus $\Gamma(\mu)$ is a fuzzy ideal of L.

Next, we proceed to show that $\Gamma(\mu)$ is a kernel fuzzy ideal of L. To prove our claim, it is enough to show that $\Gamma(\mu(x^{**})) = \Gamma(\mu(x))$ for all $x \in L$. Since $x^{**} \wedge x = x$, we get that $\Gamma(\mu(x^{**})) \le \Gamma(\mu(x))$. On the other hand, $\Gamma(\mu)(x) = Sup\{\mu(a) : a^{**} \wedge x = x\} \le Sup\{\mu(a) : a^{**} \wedge x^{**} = x^{**}\} = \Gamma(\mu)(x^{**})$. Thus $\Gamma(\mu)(x) = \Gamma(\mu)(x^{**})$. So by Theorem 4.10, $\Gamma(\mu)$ is a kernel fuzzy ideal.

To show $\Gamma(\mu)$ is the smallest kernel fuzzy ideal, let η be a kernel fuzzy ideal containing μ. For any $x \in L$, $\Gamma(\mu)(x) = Sup\{\mu(a) : a^{**} \wedge x = x\} \le Sup\{\eta(a) : a^{**} \wedge x = x\}$. If $a^{**} \wedge x = x$, then $\eta(x) \ge \eta(a^{**})$. Since η is a kernel fuzzy ideal, then $\eta(a) = \eta(a^{**})$. This shows that $\eta(x)$ is an upper bound of $\{\mu(a) : a^{**} \wedge x = x\}$. Thus $\Gamma(\mu) \subseteq \eta$. So $\Gamma(\mu)$ is the smallest kernel fuzzy ideal containing. □

Lemma 4.13 : *For any fuzzy ideal μ of L, the map $\mu \to \Gamma(\mu)$ is a closure operator on FI(L). That is,*

(1) $\mu \subseteq \Gamma(\mu)$,

(2) $\Gamma(\Gamma(\mu)) = \Gamma(\mu)$,

(3) $\mu \subseteq \theta \Rightarrow \Gamma(\mu) \subseteq \Gamma(\theta)$, for any $\mu, \theta \in FI(L)$.

Kernel fuzzy ideals are simply the closed elements with respect to the closure operator.

Lemma 4.14 : *If* $\mu,\theta \in FKI(L),$ *the supremum of* μ *and* θ *is given by*

$$(\mu\underline{\vee}\theta)(x) = Sup\{\mu(a)\wedge\theta(b) : (a\vee b)^{**}\wedge x = x,\, a,b\in L\}.$$

Proof : To prove our claim it is enough to show that $\mu\underline{\vee}\theta = \Gamma(\mu\vee\theta).$ Let $x\in L.$

$$\begin{aligned}(\mu\underline{\vee}\theta)(x) &= Sup\{\mu(b_1)\wedge\theta(b_2) : (b_1\vee b_2)^{**}\wedge x = x,\, b_1,b_2\in L\}\\ &\leq Sup\{Sup\{\mu(b_1)\wedge\theta(b_2) : b_1\vee b_2 = b\} : b^{**}\wedge x = x\}\\ &= Sup\{(\mu\vee\theta)(b) : b^{**}\wedge x = x\}\\ &= \Gamma(\mu\vee\theta)\end{aligned}$$

Thus $\mu\underline{\vee}\theta \leq \Gamma(\mu\vee\theta).$ Since $\Gamma(\mu\vee\theta)$ is the smallest fuzzy ideal containing μ and $\theta,$ we get that $\Gamma(\mu\vee\theta)\leq\mu\underline{\vee}\theta.$ Therefore $\mu\underline{\vee}\theta$ is the smallest kernel fuzzy ideal containing μ and $\theta.$ □

The set of fuzzy ideals of $S(L)$ and the set of kernel fuzzy ideals of L are denoted by $FI(S(L))$ and $FKI(L)$ respectively.

Lemma 4.15 : *Let* $\mu,\theta\in FKI(L).$ *Then* $\Gamma(\mu\cap\theta) = \Gamma(\mu)\cap\Gamma(\theta).$

Theorem 4.16 : *The set* $FKI(L)$ *forms a complete distributive lattice with respect to inclusion ordering of fuzzy sets.*

Proof : Clearly $(FKI(L),\subseteq)$ is a partially ordered set. For $\mu,\theta\in FKI(L),$ define

$$\mu\wedge\theta = \mu\cap\theta \text{ and } \mu\underline{\vee}\theta = \Gamma(\mu\vee\theta).$$

Then $\mu\wedge\theta, \mu\underline{\vee}\theta\in FKI(L).$ Thus $(FKI(L),\underline{\vee},\cap)$ is a lattice. Now we proceed to show $FKI(L)$ is distributive. Let $\mu,\theta,\eta\in FKI(L).$ Then $(\mu\cap\theta)\underline{\vee}(\mu\cap\eta)\subseteq\mu\cap(\theta\underline{\vee}\eta).$ To show the other inclusion, $\mu\cap(\theta\underline{\vee}\eta)\subseteq\Gamma(\mu)\cap\Gamma(\theta\vee\eta) = \Gamma((\mu\cap\theta)\vee(\mu\cap\eta)) = (\mu\cap\theta)\underline{\vee}(\mu\cap\eta).$ Thus $FKI(L)$ is a distributive lattice. Finally, let $\{\mu_\alpha : \alpha\in\Delta\}$ be a subfamily of $FKI(L).$ Then by Corollary 4.11 $\bigcap_{\alpha\in\Delta}\mu_\alpha$ is a kernel fuzzy ideal of $L.$ Since $(FKI(L),\subseteq)$ is a poset with greatest element χ_L and every subfamily of $FKI(L)$ has infimum, it follows that $FKI(L)$ is complete. Therefore, $FKI(L)$ is a complete distributive lattice. □

Theorem 4.17 : *If* $\pi : FKI(L)\to FI(S(L))$ *is defined by*

$$\pi(\mu)(b) = Sup\{\mu(x) : b = x^{**}, b\in S(L),\, x\in L\}$$

for all $\mu \in FKI(L)$, then it is a lattice isomorphism between $FKI(L)$ and the lattice of fuzzy ideals of the skeleton $S(L)$.

Proof : First, we need to show $\pi(\mu)$ is a fuzzy ideal of $S(L)$. Since $0 = 0^{**}$, we have $\pi(\mu)(0) = 1$. For any $a, b \in L$,

$$\pi(\mu)(a) \wedge \pi(\mu)(b) = Sup\{\mu(x) : a = x^{**}\} \wedge Sup\{\mu(y) : b = y^{**}\}.$$

If $a = x^{**}$ and $b = y^{**}$, then $a \underline{\vee} b = x^{**} \underline{\vee} y^{**} = (x \vee y)^{**}$. It can be easily verified that $\pi(\mu)(a) \wedge \pi(\mu)(b) \leq \pi(\mu)(a \underline{\vee} b)$. Since $a, b \in S(L)$, we have $a \wedge b = (a \wedge b)^{**}$. Thus $\pi(\mu)(a) = Sup\{\mu(x) : a = x^{**}\} \leq Sup\{\mu(x \wedge b) : a \wedge b = (x \wedge b)^{**}\} = \pi(\mu)(a \wedge b)$. Similarly, $\pi(\mu)(a \wedge b) \geq \pi(\mu)(b)$. Thus $\pi(\mu)$ is a fuzzy ideal of $S(L)$.

Next, we show that π is a homomorphism. Let $\mu, \theta \in FKI(L)$. Since $\mu \cap \theta \subseteq \mu, \theta$, we have $\pi(\mu \cap \theta) \subseteq \pi(\mu) \cap \pi(\theta)$. To show the other inclusion, let $a \in L$. Then

$$\begin{aligned}
(\pi(\mu) \cap \pi(\theta))(a) &= Sup\{\mu(x) : a = x^{**}\} \wedge Sup\{\theta(y) : a = y^{**}\} \\
&\leq Sup\{\mu(x \wedge y) \wedge \theta(x \wedge y) : a = x^{**}, a = y^{**}\} \\
&\leq Sup\{(\mu \cap \theta)(x \wedge y) : a = (x \wedge y)^{**}\} \\
&\leq \pi(\mu \cap \theta)(a)
\end{aligned}$$

Thus $\pi(\mu \cap \theta) = \pi(\mu) \cap \pi(\theta)$. Since $\mu, \theta \subseteq \mu \underline{\vee} \theta$, we have $\pi(\mu) \vee \pi(\theta) \subseteq \pi(\mu \underline{\vee} \theta)$. Again,

$$\begin{aligned}
\pi(\mu \underline{\vee} \theta)(a) &= Sup\{(\mu \underline{\vee} \theta)(x) : a = x^{**}\} \\
&= Sup\{Sup\{\mu(a_1) \wedge \theta(a_2) : x = (a_1 \vee a_2)^{**} \wedge x\} : a = x^{**}\}
\end{aligned}$$

and

$$\begin{aligned}
(\pi(\mu) \vee \pi(\theta))(a) &= Sup\{\pi(\mu)(b_1) \vee \pi(\theta)(b_2) : a = b_1 \underline{\vee} b_2\} \\
&= Sup\{Sup\{\mu(x_1) : b_1 = x_1^{**}\} \wedge Sup\{\theta(x_2) : b_2 = x_2^{**}\} : \\
&\quad a = x_1^{**} \underline{\vee} x_2^{**}\} \\
&= Sup\{Sup\{\mu(x_1) \wedge \theta(x_2) : b_1 = x_1^{**}, b_2 = x_2^{**}\} : \\
&\quad a = x_1^{**} \underline{\vee} x_2^{**}\}
\end{aligned}$$

If $a = x^{**}$ and $x = (a_1 \vee a_2)^{**} \wedge x$, then $(a_1 \wedge x)^{**} \underline{\vee} (a_2 \wedge x)^{**} = a$. Put $r_1 = a_1 \wedge x$ and $r_2 = a_2 \wedge x$. Then

$$\begin{aligned}
\pi(\mu \underline{\vee} \theta)(a) &\leq Sup\{Sup\{\mu(r_1) \wedge \theta(r_2) : r_1 = (a_1 \wedge x)^{**}, r_2 = (a_2 \wedge x)^{**}\} : \\
&\quad a = r_1^{**} \underline{\vee} r_2^{**}\} \\
&= (\pi(\mu) \vee \pi(\theta))(a)
\end{aligned}$$

Thus $\pi(\mu \underline{\vee} \theta) = \pi(\mu) \vee \pi(\theta)$. So π is a homomorphism.

Now we proceed to show that, π is one to one and onto. To show π is one to one, let $\mu, \theta \in FKI(L)$ such that $\pi(\mu) = \pi(\theta)$. Let $x \in L$. Since μ is a kernel fuzzy ideal, we have $\mu(x) = \mu(x^{**}) \le \pi(\mu)(x^{**}) = \pi(\theta)(x^{**})$. This implies $\mu(x) \le Sup\{\theta(a) : x^{**} = a^{**}\}$. If $x^{**} = a^{**}$, then $\theta(x) = \theta(a)$ for each $a \in L$ satisfying $x^{**} = a^{**}$. This shows that $\theta(x)$ is an upper bound of $\{\theta(a) : x^{**} = a^{**}\}$. Thus $\theta(x) \ge \mu(x)$. So $\mu \subseteq \theta$. Similarly, $\theta \subseteq \mu$. Hence $\theta = \mu$. Therefore π is one to one.

To show π is onto, let θ be a fuzzy ideal of $S(L)$. Define a fuzzy subset μ of L as $\mu(x) = \theta(x^{**})$. Clearly μ is a kernel fuzzy ideal of L. Finally, we show that $\pi(\mu) = \theta$. Let $x \in L$. Then $\pi(\mu)(b) = Sup\{\mu(x) : b = x^{**}, x \in L\}$. Since $\mu(x) = \theta(x^{**})$ and $b = x^{**}$, we get that $\pi(\mu)(b) = \theta(b)$. Thus $\pi(\mu) = \theta$. So π is onto. Hence π is a lattice isomorphism. Moreover, if we define $\bar{\pi} : FI(S(L)) \to FKI(L)$ by $\bar{\pi}(x) = \mu(x^{**})$ for all $\mu \in FI(S(L))$, then π and $\bar{\pi}$ are mutually inverse lattice isomorphism between $FKI(L)$ and the lattice of fuzzy ideals of the skeleton $FI(S(L))$.

The set of dense elements of L denoted by D and it is a filter of L. For any fuzzy ideal μ of L, we define the following fuzzy relations on L.

(1) $\phi_1(x, y) = Sup\{\mu(a) : x^{**} \wedge a^{*} = y^{**} \wedge a^{*}, a \in L\}$.

(2) $\phi_2(x, y) = Sup\{\mu(a) : x \wedge b \wedge a^{*} = y \wedge b \wedge a^{*}, b \in D, a \in L\}$.

Theorem 4.18 : *For a fuzzy ideal μ of L, $\phi_1 = \phi_2$,*

Proof : Let $x, y \in L$. Then $\phi_1(x, y) = Sup\{\mu(a) : x^{**} \wedge a^{*} = y^{**} \wedge a^{*}, a \in L\}$. Since $x \vee x^{*}$ and $y \vee y^{*}$ are dense elements, then $b = (x \vee x^{*}) \wedge (y \vee y^{*}) \in D$. If $x^{**} \wedge a^{*} = y^{**} \wedge a^{*}$, then $x^{**} \wedge b \wedge a^{*} = y^{**} \wedge b \wedge a^{*}$. Since $x^{**} \wedge (x \vee x^{*}) = x$ and $y^{**} \wedge (y \vee y^{*}) = y$, we have $x \wedge b \wedge a^{*} = x^{**} \wedge b \wedge a^{*}$ and $y \wedge b \wedge a^{*} = y^{**} \wedge b \wedge a^{*}$. Based on this we have, $\phi_1(x, y) \le Sup\{\mu(a) : x \wedge b \wedge a^{*} = y \wedge b \wedge a^{*}\} = \phi_2(x, y)$. Thus $\phi_1 \subseteq \phi_2$. On the other hand, $\phi_2(x, y) = Sup\{\mu(a) : x \wedge b \wedge a^{*} = y \wedge b \wedge a^{*}, b \in D, a \in L\}$. If $x \wedge b \wedge a^{*} = y \wedge b \wedge a^{*}$, then $x^{**} \wedge b^{**} \wedge a^{*} = y^{**} \wedge b^{**} \wedge a^{*}$. Since $b \in D$, we get that $x^{**} \wedge a^{*} = y^{**} \wedge a^{*}$. Using this fact we have, $\phi_2(x, y) \le Sup\{\mu(a) : x^{**} \wedge a^{*} = y^{**} \wedge a^{*}\} = \phi_1(x, y)$ Thus $\phi_2 \subseteq \phi_1$. So $\phi_1 = \phi_2$. □

Theorem 4.19 : *For a fuzzy ideal μ of L, ϕ_2 (or ϕ_1) is a *-fuzzy congruence on L.*

***Proof* :** Clearly ϕ_2 is reflexive and symmetric. Now

(1) For $x, y, z \in L$,

$$\begin{aligned}\phi_2(x,y)\wedge\phi_2(y,z) &= Sup\{\mu(a_1) : x\wedge b_1\wedge a_1^* = y\wedge b_1\wedge a_1^*, b_1\in D, a_1\in L\}\\ &\quad\wedge Sup\{\mu(a_2) : y\wedge b_2\wedge a_2^* = z\wedge b_2\wedge a_2^*, b_2\in D, a_2\in L\}\\ &= Sup\{\mu(a_1\vee a_2) : x\wedge b_1\wedge a_1^* = y\wedge b_1\wedge a^*,\\ &\quad y\wedge b_2\wedge a_2^* = z\wedge b_2\wedge a_2^*\}\end{aligned}$$

If $x\wedge b_1\wedge a_1^* = y\wedge b_1\wedge a_1^*$ and $y\wedge b_2\wedge a_2^* = z\wedge b_2\wedge a_2^*$, then $x\wedge b_1\wedge b_2 \wedge(a_1\vee a_2)^* = y\wedge b_1\wedge b_2\wedge(a_1\vee a_2)^*$ and $y\wedge b_1\wedge b_2\wedge(a_1\vee a_2)^* = z\wedge b_1 \wedge b_2\wedge(a_1\vee a_2)^*$. Put $b = b_1\wedge b_2$ and $a = a_1\vee a_2$. Then $b\in D$. Using this fact, we can easily show that $\phi_2(x,y)\wedge\phi_2(y,z)\leq\phi_2(x,z)$. Thus ϕ_2 is transitive.

(2) For $x, y, w, z \in L$,

$$\begin{aligned}\phi_2(x,y)\wedge\phi_2(w,z) &= Sup\{\mu(a_1) : x\wedge b_1\wedge a_1^* = y\wedge b_1\wedge a_1^*, b_1\in D, a_1\in L\}\\ &\quad\wedge Sup\{\mu(a_2) : w\wedge b_2\wedge a_2^* = z\wedge b_2\wedge a_2^*, b_2\in D, a_2\in L\}\\ &= Sup\{\mu(a_1\vee a_2) : x\wedge b_1\wedge a_1^* = y\wedge b_1\wedge a^*,\\ &\quad w\wedge b_2\wedge a_2^* = z\wedge b_2\wedge a_2^*\}\end{aligned}$$

If $x\wedge b_1\wedge a_1^* = y\wedge b_1\wedge a_1^*$ and $w\wedge b_2\wedge a_2^* = z\wedge b_2\wedge a_2^*$, then $x\wedge b\wedge a^* = y\wedge b\wedge a^*$ and $w\wedge b\wedge a^* = z\wedge b\wedge a^*$. This implies $(x\wedge w)\wedge b\wedge a^* = (y\wedge z)\wedge b\wedge a^*$ and $(x\vee w)\wedge b\wedge a^* = (y\vee z)\wedge b\wedge a^*$. Based on this fact we get, $\phi_2(x,y)\wedge\phi_2(w,z)\leq\phi_2(x\wedge w, y\wedge z)$ and $\phi_2(x,y)\wedge\phi_2(w,z)\leq\phi_2(x\vee w, y\vee z)$. Thus ϕ_2 is a fuzzy congruence. Finally, we show that ϕ_2 is a *-fuzzy congruence. Let $x, y\in L$. Then by the above theorem, $\phi_2(x,y) = \phi_1(x,y)$. Thus $\phi_1(x,y) = Sup\{\mu(a) : x^{**}\wedge a^* = y^{**}\wedge a^*, a\in L\}$. If $x^{**}\wedge a^* = y^{**}\wedge a^*$, then $(x^{**}\wedge a^*)^* = (y^{**}\wedge a^*)^*$. Which implies $x^*\underline{\vee} a^{**} = y^*\underline{\vee} a^{**}$ in the skeleton $(S(L))$. Since $S(L)$ is a Boolean algebra, it follows that $x^*\wedge a^* = y^*\wedge a^*$. This shows that $\phi_2(x,y)\leq\phi_2(x^*, y^*)$. Thus ϕ_2 is a *-fuzzy congruence on L. $\square$

Conclusion

In this work, we introduce the concept of kernel fuzzy ideals and *-fuzzy filters of a pseudo-complemented almost distributive lattices and investigate some of their properties. We give a necessary and sufficient conditions for a fuzzy congruence to be a *-fuzzy congruence. Furthermore, we established a set of equivalent condition for a fuzzy ideal to be a kernel fuzzy ideal. We also proved that the class of kernel fuzzy ideals form a complete distributive lattice. Finally, we prove that every fuzzy filter is the cokernel of a *-fuzzy congruence. Our future work will focus on α -fuzzy ideals in C-algebras.

References

[1] G. M. Addis, Congruence relations on pseudocomplemented almost distributive lattices, *Journal of Discrete Mathematical Sciences and Cryptography* (Accepted).

[2] B. A. Alaba and G. M. Addis, Fuzzy congruence relations on almost distributive lattices, *Ann. Fuzzy Math. Inform.* 14 (2017), No. 3, 315-330.

[3] B. A. Alaba and W. Z. Norahun, Fuzzy annihilator ideals in distributive lattices, *Ann. Fuzzy Math. Inform.* 16 (2018), No. 2, 191-200.

[4] B. A. Alaba and W. Z. Norahun, α -fuzzy ideals and space of prime α -fuzzy ideals in distributive lattices, *Ann. Fuzzy Math. Inform*. 17 (2019), No. 2, 147-163.

[5] B. A. Alaba and W. Z. Norahun, σ -fuzzy ideals of distributive p-algebras, *Ann. Fuzzy Math. Inform*. 17 (2019), No. 3, 289 - 301.

[6] B. A. Alaba and W. Z. Norahun, Fuzzy Ideals and Fuzzy Filters of Pseudo-complemented Semilattices, *Advances in Fuzzy Systems*. Volume 2019 (2019).

[7] B. A. Alaba and W. Z. Norahun, δ -fuzzy ideals in pseudo-complemented distributive lattices, *J. Appl. Math. and Informatics* Vol. 37(2019), No. 5 - 6, pp. 383 - 397.

[8] B. A. Alaba, M. A. Taye and W. Z. Norahun, d-fuzzy ideals in distributive lattices, *Ann. Fuzzy Math. Inform.* 18 (2019), No. 3, 233-243.

[9] N. Ajmal and A. S. Prajapati, Fuzzy cosets and fuzzy normal subgroups, *Inform. Sci*, 64 (1992) 237-240.

[10] N. Ajmal and K. V. Thomas, *Fuzzy lattices, Inform. Sci*, 79 (1994) 271-291.

[11] M. Akgül, Some properties of fuzzy groups, *J. Math. Anal. Appl*, 133 (1988) 93-100. 29.

[12] J. M. Anthony and H. Sherwood, Fuzzy subgroups redefined, *J. Math. Anal. Appl*, 69 (1979) 124-130.

[13] G. Birkhoff, Lattice theory, Colloquium Publication, vol. 25, Amer. Math. Soc., New York, 1948.

[14] W. H. Cornish, Congruences on distributive pseudo-complemented lattices, *Bull. Austral. Math. Sot*., 8 (1973), 161-179.

[15] O. Frink, Pseudo-complements in semi-lattices, *Duke Math. J.* 29 (1962), 505-514.

[16] I. J. Kumar, P. K. Saxtena and P. Yadav, Fuzzy normal subgroups and fuzzy quotients, *Fuzzy Sets and Systems*, 46 (1992) 121-132.

[17] R. Kumar, Fuzzy prime spectrum of a ring , *Fuzzy Sets and Systems*, 46 (1992) 147-154.

[18] R. Kumar, Certain fuzzy ideals of rings redefined, *Fuzzy Sets and Systems*, 46 (1992) 251-260.

[19] W. J. Liu, Fuzzy invariants subgroups and fuzzy ideals, F*uzzy Sets and Systems,* 8 (1982) 133-139.

[20] D. S. Malik, Fuzzy ideals of Artinian rings, *Fuzzy Sets and Systems,* 46 (1990) 111-115.

[21] D. S. Malik and J. N. Mordeson, Fuzzy primary representation of fuzzy ideals, Inform. Sci, 55 (1991) 151-165.

[22] N. P. Mukherjee and P. Bhattacharya, Fuzzy normal subgroups and fuzzy cosets, Inform. Sci, 34 (1984) 225-239.

[23] T. K. Mukherjee and M. K. Sen, On fuzzy ideals of a ring (I), *Fuzzy Sets and Systems,* 21 (1987) 99-104.

[24] M. S. Rao and G. C. Rao, Kernel ideals in almost distributive lattices, *Asian-European Journal of Mathematics,* 5 2 (2012).

[25] R. Rasuli, Fuzzy congruence product lattice under T-norms, *Journal of Information and Optimization Sciences,* 42: 2 (2021), 333 - 343.

[26] A. Rosenfeld, Fuzzy Subgroups, *J. Math. Anal. Appl.*, 35 (1971), 512-517.

[27] U. M. Swamy and G. C. Rao, Almost distributive lattice, *J. Aust. Math. Soc.* (Series A), 31(1981),77-91.

[28] U. M. Swamy and D. V. Raju, Fuzzy ideals and congruences of lattices, *Fuzzy Sets and Systems* 95 (1998) 249-253.

[29] U. M. Swamy, G. C. Rao and G. N. Rao, Pseudo-complementation on almast distributive lattices, *South Asian Bullettin of Mathematics,* 24 (2000), 95-104.

[30] U. M. Swamy, C. S. S. Raj and N. Teshale, Fuzzy ideals of almost distributive latties, *Ann. Fuzzy Math. Inform.*, 14 (2017), No. 4, 371-379.

[31] U. M. Swamy, C. S. S. Raj and N. Teshale, L-Fuzzy filters of almost distributive latties, IJMSC, 8 (2018), No. 1, 35-43.

[32] B. Yuan and W. Wu, Fuzzy ideals on a distributive lattice, *Fuzzy Sets and Systems* 35 (1990) 231-240.

[33] L. A. Zadeh, Fuzzy Sets, *Information and Control*, 8 (1965), 338-353.

Received January, 2021

Revised May, 2021