Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
Vol. 26 (2023), No. 2, pp. 521–528
DOI : 10.1080/09720529.2021.1961898

Density results for perfectly regular and perfectly edge-regular fuzzy graphs

Talal Ali Al-Hawary
Department of Mathematics
Yarmouk University
Irbid 21163
Jordan

Abstract

This paper studies the fuzzy graph properties of density and being self-complimentary in perfectly regular and perfectly edge-regular fuzzy graphs. Several structural results for these classes of fuzzy graphs are provided. These results are parameterized by the parameters which define perfectly regular and perfectly edge-regular fuzzy graphs, thus illustrating the computational benefits of using these classes of fuzzy graphs to represent and/or approximate fuzzy networks when performing analyses of fuzzy networks.

Subject Classification: *(2010) 03B52, 05C72, 05C42.*

Keywords: *Fuzzy graph, Density, Regular, Perfectly regular fuzzy graph.*

1. Introduction

Perfectly regular and perfectly edge-regular fuzzy graphs, first introduced in [13], constitute a particularly interesting case of fuzzy graphs insofar as they admit certain structural properties conducive to efficient computation. The motivation behind these two (not so unrelated) classes of fuzzy graphs lies in restricting the functions μ and σ to constant functions. These two classes of graphs may be seen as restrictions of (edge-)regular and totally (edge-) regular fuzzy graphs [2, 14, 15, 17, 18, 19, 20]. By studying a more strictly defined class of fuzzy graphs, we allow

This article has been corrected with minor changes. These changes do not impact the academic content of the article.

E-mail: `talalhawary@yahoo.com`

for greater precision when computing results. While the topic studied in this paper is density, the ability to obtain more precise results may be extended to other properties, perhaps rendering perfectly regular and perfectly edge-regular fuzzy graphs as beneficial tools for approximating fuzzy networks in general settings.

As an additional focus, we study the property of completeness, i.e. when $\mu(uv) = (\sigma(u) \wedge \sigma(v))$. The property of completeness is another assumption that can help to reduce the complexity of modelling fuzzy systems when there is a great deal of uncertainty and/or the only membership function of interest in the study is μ, e.g. when studying interactions between members of a known population. In particular we study this property in combination with at least one of the properties of being perfectly regular or perfectly edge-regular. For more on completeness, density, and related topics, the reader is referred to [1, 5, 6, 7, 8, 16].

In this paper, we prove structural properties of perfectly regular and perfectly edge-regular fuzzy graphs. In particular, we study the notions of density and being self-complimentary. We prove several structural results that are necessary to imply that a given perfectly regular (or perfectly edge-regular) fuzzy graph is either self-complimentary or balanced. Throughout this work, definitions and relevant notations are borrowed from [13] and [21]. The remainder of the paper will be structured as follows. After providing some necessary preliminaries, we will study perfectly regular fuzzy graphs and characterize necessary conditions for perfectly regular fuzzy graphs to be self-complimentary via establishing results on the density of perfectly regular fuzzy graphs in Section 2. In Section 3 we will perform the same analysis for perfectly edge-regular fuzzy graphs. We will then conclude the paper with a discussion of the results and future research directions.

Before proceeding with our study, we provide some necessary definitions and results which will be used throughout the rest of this paper.

Definition 1 : A perfectly regular fuzzy graph is a fuzzy graph that is both regular and totally regular, i.e., $d(v) = d(u)\ \forall\ u, v \in V(G)$ and $\sigma(v) = \sigma(u)$.

Definition 2 : [13] A perfectly edge-regular fuzzy graph is a fuzzy graph that is both edge-regular and totally edge-regular, i.e., $d(uv) = d(xy)$ and $\mu(uv) = \mu(xy)\ \forall\ uv, xy \in E(G)$.

Definition 3 : [13] A strongly regular fuzzy graph is a perfectly regular fuzzy graph with a regular underlying crisp graph.

Definition 4 : [13] A fuzzy graph is complete if $\mu(uv) = (\sigma(u) \wedge \sigma(v)) \forall\, u, v \in V(G)$.

Definition 5 : [5] The density of a fuzzy graph G is given by $D(G) = \frac{2\sum \mu(uv)}{\sum(\sigma(u) \wedge \sigma(v))}$ where both sums are taken over all pairs of $u, v \in V(G)$.

Definition 6 : [5] A fuzzy graph G is balanced if, for every subgraph $H \subset G$, we have $D(H) \leq D(G)$.

Theorem 1 : [5] *Every self-complimentary graph has density equals one.*

2. Perfectly Regular Fuzzy Graphs

In this section we study the density of perfectly regular fuzzy graphs in order to establish parameterized necessary conditions for perfectly regular fuzzy graphs to be self-complimentary. We begin, however, with a convenient structural result on the underlying crisp graph of a regular fuzzy graph with a constant function μ, and thus an important structural result on the underlying crisp graph of a strongly regular fuzzy graph.

Lemma 2 : *A regular fuzzy graph with constant function μ necessarily has a regular underlying crisp graph.*

Proof : Let G be a regular fuzzy graph with constant μ. Since G is regular, $d(v) = d(u) \;\forall\, u, v \in V(G)$. Since μ is constant, the degree of a given vertex $b \in V(G)$ is $d(v) = \sum_{u \in N(v)} \mu(uv) = \mu d_c(v)$ where $d_c(v)$ is the degree of v in the underlying crisp graph. Since $d(v) = d(u) \;\forall\, u, v \in V(G)$, we have that $\mu d_c(v) = \mu d_c(u)$ for all vertices $u, v \in V(G)$, which implies that the underlying crisp graph of G is necessarily d_c-regular. □

Theorem 3 : *The density of a perfectly regular fuzzy graph G is precisely* $D(G) = \left(\frac{2d(v)}{(n-1)\sigma(v)}\right)$.

Proof : Let G be a fuzzy graph on $|V(G)| = n$ vertices. Since G is perfectly regular, $\sigma(v)$ is constant, hence the denominator becomes $\sigma(v)\binom{n}{2}$. Since $\sum_E \mu(uv) = \frac{1}{2}\sum_V \mu(uv)$ (we may take the sum over the edge set in the numerator because edges that are not present have $\mu(uv)=0$ by definition), we have that the numerator may be expressed as $\sum_V d(v)$. Since $d(v)$ is constant in a perfectly regular fuzzy graph, we have that $\sum_V d(v) = d(v)\,|\,V(G)\,|$. Combined, this gives us that $D(G) = \frac{d(v)|V(G)}{\sigma(v)\binom{n}{2}} = \frac{2d(v)}{(n-1)\sigma(v)}$. □

Corollary 1 : *A necessary condition for a perfectly regular fuzzy graph to be self-complimentary is that* $\frac{d(v)}{\sigma(v)} = \frac{n-1}{2}$.

Perfectly regular fuzzy graphs are not necessarily balanced.

Example 1 : Let G be a perfectly regular fuzzy graph, in particular the following fuzzy system defined on K_4 where $\sigma = c$ and the values for $\mu(x,y) = \frac{1}{2}$ for all vertices x and y on the boundary edges and $\mu(x,y) = \frac{1}{3}$ for diagonal edges. Consider the subgraph H of G which is the same as G after removing the two diagonals. By computing the densities of G and H, we obtain that $D(G) = \frac{8}{9c} < \frac{1}{c} = D(H)$. Since we have that c is bounded between zero and one, it follows that $D(H) > D(G)$, hence G is not balanced.

Finding a closed form, simplified expression for the density of a perfectly edge-regular fuzzy graph would be nice, but such as expression does not seem obvious. The numerator reduces easily/conveniently, but the denominator does not.

Theorem 1 : *A self-complimentary perfectly regular fuzzy graph with constant* μ *must satisfy* $\sigma(v) = 2\mu(uv)$ *for all* $v \in V(G)$ *and* $uv \in E(G)$.

Proof : Let G be a self-complimentary perfectly regular fuzzy graph with constant function μ. By Corollary 1 we know that $\frac{|E(G)|}{|V(G)|} = \frac{d(v)}{\sigma(v)}$. By Lemma 2 we know that the underlying crisp graph of G is regular. Since μ is assumed to be a constant function, we obtain the equality $\frac{|E(G)|}{|V(G)|} = \frac{\mu d_c}{\sigma}$. Since the underlying crisp graph of G is d_c-regular, we have that $|\,E(G)\,| = \frac{d_c}{|V(G)|}$. Substituting this into our relation, we obtain that $\frac{d_c}{2} = \frac{\mu d_c}{\sigma}$ and immediately we see that $\sigma = 2\mu$ is indeed a necessary condition for a perfectly regular fuzzy graph to be self-complimentary. □

Theorem 4 : *A self-complimentary perfectly regular fuzzy graph with constant* μ *must be* $\frac{(n-1)\sigma}{2\mu}$*-regular.*

Proof : Let G be a self-complimentary perfectly regular fuzzy graph with constant function μ. By Corollary 1 we know that $\frac{(n-1)}{2} = \frac{d(v)}{\sigma}$. By Lemma 2 we know that the underlying crisp graph of G is regular. Since μ is assumed to be a constant function, we obtain the equality $\frac{(n-1)}{2} = \frac{\mu d_C}{\sigma}$ from which the result follows. □

Theorem 5 : *A complete perfectly regular fuzzy graph* G *has density* $D(G) = \frac{2d_c}{n-1}$ *where* d_c *is the degree of the* d_c*-regular underlying crisp graph.*

Proof : By Theorem 3 we have that thee density of G is precisely $D(G) = \frac{2d(v)}{(n-1)\sigma(v)}$. Since G is complete, we have that μ is also a constant function. By definition of the degree of a vertex in a fuzzy graph, we now have that $d(v) = \mu d_c \ \forall\ v \in V(G)$. Thus we may rewrite the density of G as $D(G) = \frac{2\mu d_C}{(n-1)\sigma} = \frac{2d_C}{n-1}$. □

Corollary 2 : *A complete perfectly regular fuzzy graph with constant function* μ *must be* $\frac{n-1}{2}$*-regular in order to be self-complimentary.*

3. Perfectly Edge-regular Fuzzy Graphs

In a fashion similar to that of the previous section, we study the density of perfectly edge-regular fuzzy graphs in order to establish parameterized necessary conditions for perfectly edge-regular fuzzy graphs to be self-complimentary. We begin by bounding the density of a perfectly edge-regular fuzzy graph in terms of the graph parameters.

Theorem 6 : *The density of a perfectly edge-regular fuzzy graph G is bounded by* $\frac{2\mu|E(G)|}{|V(G)|} \leq D(G) \leq \frac{2|E(G)|}{|V(G)|}$.

Proof: Let G be a perfectly edge-regular fuzzy graph on $|V(G)| = n$ vertices. Since G is perfectly edge-regular, we have that $2\sum_E \mu(uv) = 2\mu(uv)|E(G)|$. Next, we bound the denominator as follows. Let $\hat{\sigma}$ represent the mean value of the function σ over $V(G)$, i.e., $\hat{\sigma} = \frac{1}{n}\sum_{V(G)} \sigma(v)$. Then the denominator can be written as $\hat{\sigma}\,|V(G)|$. Since we have that $\hat{\sigma}$ is bounded by $\mu \leq \hat{\sigma} \leq 1$, we have that the denominator is bounded between $\mu\,|V(G)|$ and $|V(G)|$. Combining these results yields the desired bounds on the density of a perfectly edge-regular fuzzy graph. □

Corollary 3 : *If G is a self-complimentary perfectly edge-regular fuzzy graph, then* $|E(G)| = \frac{1}{2}|V(G)|$.

This corollary illustrates the usefulness of perfectly edge-regular fuzzy graphs as models for fuzzy networks, at least from a computational perspective. In fact, we take the time here to note that, due to results such as this, further techniques designed to estimate structural and topological properties of fuzzy networks by simply adding or deleting conveniently designed structures from a perfectly edge-regular fuzzy graph would make for an excellent direction for future research. By reducing the extra complexity of fuzzy systems to what is expected in a crisp system, perfectly (edge-)regular fuzzy graphs make excellent modelling tools.

4. Conclusion and Future Directions

In this paper we studied the density of perfectly regular and perfectly edge-regular fuzzy graphs in detail. Consequentially, we showed that these two classes of fuzzy graphs live up to their reputation for being a strong candidate for representing complex fuzzy systems. Furthermore, we demonstrated that completeness is an additional assumption that can greatly reduce the complexity of problems arising in studying fuzzy systems.

In order to achieve these ends, we introduced strongly regular fuzzy graphs, i.e. perfectly regular fuzzy graphs whose underlying crisp graphs are also regular in the traditional graph theoretic sense. However, their perfectly edge-regular analog was not studied in this paper; the future study of such fuzzy graphs may prove a fruitful endeavor. Additionally, a highly related property to that of density, namely balance, was not studied rigorously in this paper. Characterizing which classes of fuzzy graphs are balanced and determining what role perfectly (edge-)regular fuzzy graphs play would prove a very useful contribution as well. We end this paper by asking for a solution to the following problem:

Question

What is the density of a balanced perfectly edge-regular fuzzy graph?

References

[1] M. Akram, D. Saleem and T. Al-Hawary, Spherical Fuzzy Graphs with Application to Decision-Making, *Mathematical and Computational Applications*, 25(1) (2020), 1-32.

[2] M. Akram and W.A. Dudek, *Regular bipolar fuzzy graphs Neural Computing and Applications* 21 (1) 197-205, 2012.

[3] M. Akram and A. Habib, q-Rung picture fuzzy graphs: A creative view on regularity with applications *Journal of Applied Mathematics and Computing* 1-46, 2019.

[4] M. Akram and A. Habib and A. Koam, A novel description on edge-regular q-rung picture fuzzy graphs with application Symmetry 11 (4) 489-516, 2019.

[5] T. Al-Hawary, Complete fuzzy graphs, *International Journal of Mathematical Combinatorics* 4, 26-34, 2011.

[6] T. Al-Hawary, Certain classes of fuzzy graphs, *European Journal of Pure and Applied Mathematics* 10 (2), 552-560, 2017.

[7] T. Al-Hawary, S. Al-Shalaldeh and M. Akram, Certain Matrices and Energies of Fuzzy Graphs, *TWMS Journal of Applied and Engineering Mathematics*, 2, 2146-1147, 2021.

[8] T. Al-Hawary and L. Al-Momani, ∗-Balanced fuzzy graphs, arXiv preprint arXiv:1804.08677 2018.

[9] T. Al-Hawary and B. Hourani, On intuitionistic product fuzzy graphs, *Italian Journal of Pure and Applied Mathematics*, 113-126, 2017.

[10] T. Al-Hawary and B. Hourani, On product fuzzy graphs, *Annals of Fuzzy Mathematics and Informatics* 12 (2), 279-294, 2016.

[11] T. Al-Hawary, T. Mahmood, N. Jan, K. Ullah, and A. Hussain On intuitionistic fuzzy graphs and some operations on picture fuzzy graphs, *Italian Journal of Pure and Applied Mathematics,* 1, 1-15, 2018.

[12] M. Cary, Perfectly regular and perfectly edge-regular fuzzy graphs, *Annals of Pure and Applied Mathematics* 16 (2), 461-469, 2018.

[13] A.N. Gani and K. Radha, On regular fuzzy graphs, *Journal of Physical Sciences*, 12, 33-40, 2008.

[14] K. Radha and N. Kumaravel, On edge regular fuzzy graphs, *International Journal of Mathematical Archive* (IJMA) 5 (9), 100-112, 2014.

[15] E. Kartheek & S. Sharief Basha, The minimum dominating energy of a fuzzy graph, *Journal of Information and Optimization Sciences,* 38:3-4, 443-453, 2017.

[16] K. Radha and N. Kumaravel, The edge degree and the edge regular properties of truncations of fuzzy graphs, *Bulletin of Mathematics and Statistics Research* (BOMSR) 4 (3), 7-16, 2016.

[17] K. Radha and M. Vijaya, Totally regular property of the join of two fuzzy graphs, *International Journal of Fuzzy Mathematical Archive* 8 (1), 9-17, 2015.

[18] A.E. Samuel and C. Kayalvizhi, On k-regular chromatic fuzzy graph, *Advances in Fuzzy Sets and Systems* 19 (2), 155-167, 2015.

[19] S. Sebastian and A.M. Philip, Regular and edge regular interval valued fuzzy graphs, *Journal of Computer and Mathematical Sciences* 8 (7), 309-322, 2017.

[20] M.S. Sunitha and S. Mathew, Fuzzy graph theory: a survey, *Annals of Pure and Applied Mathematics* 4 (1), 92-110, 2013.

Received January, 2021

Revised May, 2021