

Computation of edge C_4 -irregularity strength of Cartesian product of graphs

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Abstract

If every edge in the graph G is also an edge of a subgraph of G isomorphic to a given graph H we say that the graph G admits an H -covering. Let G be a graph admitting an H -covering and let $H_1, H_2, \dots, H_t$ be all subgraphs in G isomorphic to H . The *edge H -irregularity strength* of a graph G is the smallest integer k for which one can find a mapping $\phi : E(G) \rightarrow \{1, 2, \dots, k\}$ such that $\sum_{e \in E(H_i)} \phi(e) \neq \sum_{e \in E(H_j)} \phi(e)$ for every $1 \leq i < j \leq t$.

In this paper, we determine the exact values of $\text{ehs}(G, C_4)$ for a grid graph and a generalized prism.

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1. Introduction

The *irregularity strength* of a simple graph G , denoted by $s(G)$, is the smallest integer k for which one can assign numbers $1, 2, \dots, k$ to the edges of G such that weighted degrees of all vertices are distinct. The weighted degree of a vertex, also known as the weight of this vertex, is defined as the sum of all labels of edges incident to the given vertex. If such k does not exist, then we set $s(G) = \infty$. It is known that $s(G)$ is finite if and only if G contains no isolated edges and has at most one isolated vertex. The corresponding labeling is called *irregular*. The concept of the irregularity strength was introduced by Chartrand *et al.* in [11] when solving the problem what is the smallest number of multiple edges we can assign to an otherwise simple graph, to ensure graph irregularity. Then the parameter $s(G)$ is equal to the smallest maximum multiplicity of an edge in such a multigraph. Some results on the irregularity strength can be found in [6, 9, 12, 13].

The *edge irregularity strength* of a graph G , denoted by $es(G)$, is the smallest integer k for which one can assign numbers $1, 2, \dots, k$ to the vertices of G such that weights of all edges are distinct. The weight of the edge is the sum of labels of its end vertices. The edge irregularity strength was defined and first studied by Ahmad *et al.* in [1]. Further results of edge irregularity strength and other types of irregularity strengths can be seen in [2, 8, 19, 3, 4, 5, 16, 14, 18].

Motivated by the concepts of the irregularity strength and the edge irregularity strength Ashraf *et al.* in [7] defined a new graph invariant called the edge H -irregularity strength. If every edge in the graph G is also an edge of a subgraph of G isomorphic to a given graph H we say that the graph G admits an H -covering. Let G be a graph admitting an H -covering and let $H_1, H_2, \dots, H_t$ be all subgraphs in G isomorphic to H . Let ϕ be a mapping that assigns integers $1, 2, \dots, k$ to the edges of G . Then the H -weight of a subgraph H_i , $i = 1, 2, \dots, t$, under the labeling ϕ is defined as the sum of all labels of edges belonging to H_i , i.e.,

$$wt_{\phi}(H_i) = \sum_{e \in E(H_i)} \phi(e).$$

If all subgraphs $H_1, H_2, \dots, H_t$ have distinct weights then the labeling ϕ is called as an *H -irregular edge k -labeling*. The smallest integer k for which the graph G has an H -irregular edge k -labeling is called the *edge H -irregularity strength* of G . In [7] some results on the edge H -irregularity

strength of some standard graphs are obtained. Moreover, the following lower bound of the edge H -irregularity strength of a graph G is established.

Theorem 1 : [7] *Let G be a graph admitting an H -covering and t is the number of all the subgraphs of G isomorphic to H . Then*

$$\text{ehs}(G, H) \geq \left\lceil 1 + \frac{t-1}{|E(H)|} \right\rceil.$$

Naeem *et al.* [15] determined exact values of the edge H -irregularity strength for prisms, antiprisms, triangular ladders, diagonal ladders, wheels and gear graphs for some H .

In this paper we describe the exact values of the edge C_4 -irregularity strength for the Cartesian product of a cycle and a path and the Cartesian product of two paths.

2. The generalized prism

The *generalized prism* can be defined as the Cartesian product $C_n \square P_m$ of a cycle C_n on n vertices with a path P_m on m vertices. Let $V(C_n \square P_m) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ be the vertex set and $E(C_n \square P_m) = \{u_i^j u_{i+1}^j : 1 \leq i \leq n, 1 \leq j \leq m\} \cup \{u_i^j u_i^{j+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$ be the edge set of the generalized prism. The indexes are taken modulo n . The graph $C_n \square P_2$ is known as the prism.

Regees *et al.* [17] determined super edge trimagic total labeling of generalized prism and topological indices of generalized prism are discussed in [10]. Naeem *et al.* [15] determined the exact value of the edge C_4 -irregularity strength of the prism $C_n \square P_2$. They proved that $\text{ehs}(C_n \square P_2, C_4) = \lceil (n+3)/4 \rceil$ for $n \geq 3, n \neq 4$. In the next theorem we extend this result for generalized prisms.

Theorem 2 : *Let n, m be positive integers, $n, m \geq 3, n \neq 4$. Then*

$$\text{ehs}(C_n \square P_m, C_4) = \left\lceil \frac{n(m-1)+3}{4} \right\rceil.$$

Proof : The generalized prism $C_n \square P_m$ contains exactly $n(m-1)$ cycles isomorphic to C_4 . Let $C_4^{i,j}$, $i = 1, 2, \dots, n, j = 1, 2, \dots, m-1$ be the 4-cycle with the vertex set $V(C_4^{i,j}) = \{u_i^j, u_{i+1}^j, u_i^{j+1}, u_{i+1}^{j+1}\}$ and the edge set $E(C_4^{i,j}) = \{u_i^j u_{i+1}^j, u_i^{j+1} u_{i+1}^{j+1}, u_i^j u_i^{j+1}, u_{i+1}^j u_{i+1}^{j+1}\}$, with index i taken modulo n .

According to Theorem 1 we obtain the following lower bound for the edge C_4 -irregularity strength of $C_n \square P_m$:

$$\text{ehs}(C_n \square P_m, C_4) \geq \left\lceil \frac{n(m-1)+3}{4} \right\rceil.$$

To prove that the equality holds we construct a C_4 -irregular edge labeling $\phi : E(C_n \square P_m) \rightarrow \{1, 2, \dots, \lceil (n(m-1)+3)/4 \rceil\}$. Let $k = \lceil (n(m-1)+3)/4 \rceil$ and $s = \lfloor 2k/m \rfloor$. According to the parity of m we consider two cases.

Case 1 : Let m be an odd positive integer. In this case

$$\begin{aligned} \phi(u_i^j u_{i+1}^j) &= \frac{(m-1)(i-1)}{2} + \left\lceil \frac{j}{2} \right\rceil, & \text{for } 1 \leq i \leq s \text{ and } 1 \leq j \leq m \\ \phi(u_{s+1}^j u_{s+2}^j) &= \left\lfloor \frac{(s+1)(m-1)+3-k}{2} \right\rfloor + \frac{j-1}{2}, & \text{for } 1 \leq j \leq m \text{ and } j \text{ odd} \\ \phi(u_{s+1}^j u_{s+2}^j) &= \left\lceil \frac{(s+1)(m-1)+3-k}{2} \right\rceil + \frac{j-2}{2}, & \text{for } 2 \leq j \leq m-1 \text{ and } j \text{ even} \\ \phi(u_i^j u_{i+1}^j) &= \frac{(m-1)i}{2} + 1 - k + \left\lceil \frac{j}{2} \right\rceil, & \text{for } s+2 \leq i \leq n-1 \text{ and } 1 \leq j \leq m \\ \phi(u_n^j u_1^j) &= \left\lfloor \frac{s(m-1)+3-k}{2} \right\rfloor + \frac{j-1}{2}, & \text{for } 1 \leq j \leq m \text{ and } j \text{ odd} \\ \phi(u_n^j u_1^j) &= \left\lceil \frac{s(m-1)+3-k}{2} \right\rceil + \frac{j-2}{2}, & \text{for } 2 \leq j \leq m-1 \text{ and } j \text{ even} \\ \phi(u_i^j u_i^{j+1}) &= 1, & \text{for } 1 \leq i \leq s-1 \text{ and } 1 \leq j \leq m-1 \\ \phi(u_i^j u_i^{j+1}) &= k, & \text{for } s+2 \leq i \leq n \text{ and } 1 \leq j \leq m-1. \end{aligned}$$

Case 2 : Let m be an even positive integer. Now

$$\begin{aligned} \phi(u_i^j u_{i+1}^j) &= \frac{(m-1)(i-1)}{2} + \left\lceil \frac{j}{2} \right\rceil, & \text{for } 1 \leq i \leq s \text{ and } i \text{ odd and } 1 \leq j \leq m \\ \phi(u_i^j u_{i+1}^j) &= \frac{(m-1)(i-2)}{2} + \frac{m}{2} + \left\lfloor \frac{j}{2} \right\rfloor, & \text{for } 2 \leq i \leq s \text{ and } i \text{ even and } 1 \leq j \leq m \\ \phi(u_{s+1}^j u_{s+2}^j) &= \left\lfloor \frac{(s+1)(m-1)+3-k}{2} \right\rfloor + \frac{j-1}{2}, & \text{for } 1 \leq j \leq m-1 \text{ and } j \text{ odd} \\ \phi(u_{s+1}^j u_{s+2}^j) &= \left\lceil \frac{(s+1)(m-1)+3-k}{2} \right\rceil + \frac{j-2}{2}, & \text{for } 2 \leq j \leq m \text{ and } j \text{ even} \\ \phi(u_i^j u_{i+1}^j) &= \left\lfloor \frac{(m-1)i+4-2k}{2} \right\rfloor + \frac{j-1}{2}, & \text{for } s+2 \leq i \leq n-1 \text{ and } 1 \leq j \leq m-1 \text{ and } j \text{ odd} \\ \phi(u_i^j u_{i+1}^j) &= \left\lceil \frac{(m-1)i+4-2k}{2} \right\rceil + \frac{j-2}{2}, & \text{for } s+2 \leq i \leq n-1 \text{ and } 2 \leq j \leq m \text{ and } j \text{ even} \\ \phi(u_n^j u_1^j) &= \left\lfloor \frac{s(m-1)+3-k}{2} \right\rfloor + \frac{j-1}{2}, & \text{for } 1 \leq j \leq m \text{ and } j \text{ odd} \\ \phi(u_n^j u_1^j) &= \left\lceil \frac{s(m-1)+3-k}{2} \right\rceil + \frac{j-2}{2}, & \text{for } 2 \leq j \leq m-1 \text{ and } j \text{ even} \\ \phi(u_i^j u_i^{j+1}) &= 1, & \text{for } 1 \leq i \leq s+1 \text{ and } 1 \leq j \leq m-1 \\ \phi(u_i^j u_i^{j+1}) &= k, & \text{for } s+2 \leq i \leq n \text{ and } 1 \leq j \leq m-1. \end{aligned}$$

Evidently, in both cases the edge labels are from the set $\{1, 2, \dots, \lceil (n(m-1)+3)/4 \rceil\}$. Moreover, for the weight of the cycle $C_4^{i,j}$, $i=1, 2, \dots, n, j=1, 2, \dots, m-1$ we obtain

$$wt_\phi(C_4^{i,j}) = \begin{cases} (m-1)(i-1)+j+3, & \text{if } 1 \leq i \leq s \text{ and } 1 \leq j \leq m-1 \\ (m-1)i+j+3, & \text{if } s+1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m-1 \\ s(m-1)+j+3, & \text{if } i=n \text{ and } 1 \leq j \leq m-1. \end{cases}$$

Thus the weights of 4-cycles are distinct numbers from the set $\{1, 2, \dots, n(m-1)+3\}$. This concludes the proof. $\square$

3. The grid graph

The *grid graph* can be defined as the Cartesian product $P_n \square P_m$ of a path on n vertices with a path on m vertices. Let $V(P_n \square P_m) = \{u_i^j : 1 \leq i \leq n, 1 \leq j \leq m\}$ be the vertex set and $E(P_n \square P_m) = \{u_i^j u_{i+1}^j : 1 \leq i \leq n-1, 1 \leq j \leq m\} \cup \{u_i^j u_i^{j+1} : 1 \leq i \leq n, 1 \leq j \leq m-1\}$ be the edge set of $P_n \square P_m$. The graph $P_n \square P_2$ is also known as the ladder. Ashraf *et al.* [7] determined the exact value of the edge C_4 -irregularity strength of the ladder $P_n \square P_2$. They proved that $\text{ehs}(P_n \square P_2, C_4) = \lceil (n+2)/4 \rceil$ for $n \geq 2, n \neq 4$. In the next theorem we extend this result for grids $P_n \square P_m$ for $n, m \geq 3$.

Theorem 3 : Let n, m be positive integers, $n, m \geq 3$. Then

$$\text{ehs}(P_n \square P_m, C_4) = \left\lceil \frac{(n-1)(m-1)+3}{4} \right\rceil.$$

Proof : The grid graph $P_n \square P_m$ contains $(n-1)(m-1)$ subgraphs isomorphic to C_4 . Let $C_4^{i,j}$, $i=1, 2, \dots, n-1, j=1, 2, \dots, m-1$ be the 4-cycle with the vertex set $V(C_4^{i,j}) = \{u_i^j, u_{i+1}^j, u_i^{j+1}, u_{i+1}^{j+1}\}$ and the edge set $E(C_4^{i,j}) = \{u_i^j u_{i+1}^j, u_i^{j+1} u_{i+1}^{j+1}, u_i^j u_i^{j+1}, u_{i+1}^j u_{i+1}^{j+1}\}$.

Using Theorem 1 we obtain the following lower bound for the studied graph parameter

$$\text{ehs}(P_n \square P_m, C_4) \geq \left\lceil \frac{(n-1)(m-1)+3}{4} \right\rceil.$$

To prove the equality we define a C_4 -irregular edge labeling $\phi : E(P_n \square P_m) \rightarrow \{1, 2, \dots, \lceil ((n-1)(m-1)+3)/4 \rceil\}$. Let $k = \lceil ((n-1)(m-1)+3)/4 \rceil$. We again consider two cases according to the parity of m .

Case 1 : Let m be an odd positive integer. Then

$$\begin{aligned}
 \phi(u_i^j u_{i+1}^{j+1}) &= 1, & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } 1 \leq j \leq m-1 \\
 \phi(u_i^j u_{i+1}^{j+1}) &= k, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n \text{ and } 1 \leq j \leq m-1 \\
 \phi(u_i^j u_{i+1}^j) &= \frac{(m-1)(i-1)}{2} + \left\lceil \frac{j}{2} \right\rceil, & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } 1 \leq j \leq m \\
 \phi(u_{\left\lfloor \frac{n}{2} \right\rfloor}^j u_{\left\lfloor \frac{n}{2} \right\rfloor+1}^j) &= \left\lfloor \frac{(i-1)(m-1)+3-k}{2} \right\rfloor + \frac{j-1}{2}, & \text{for } 1 \leq j \leq m \text{ and } j \text{ odd} \\
 \phi(u_{\left\lfloor \frac{n}{2} \right\rfloor}^j u_{\left\lfloor \frac{n}{2} \right\rfloor+1}^j) &= \left\lceil \frac{(i-1)(m-1)+3-k}{2} \right\rceil + \frac{j-2}{2}, & \text{for } 2 \leq j \leq m-1 \text{ and } j \text{ even} \\
 \phi(u_i^j u_{i+1}^j) &= \left\lfloor \frac{(m-1)(i-1)+4-2k}{2} \right\rfloor + \frac{j-1}{2}, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m \text{ and } j \text{ odd} \\
 \phi(u_i^j u_{i+1}^j) &= \left\lceil \frac{(m-1)(i-1)+4-2k}{2} \right\rceil + \frac{j-2}{2}, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-1 \text{ and } 2 \leq j \leq m-1 \text{ and } j \text{ even}
 \end{aligned}$$

Case 2 : Let m be an even positive integer. Then

$$\begin{aligned}
 \phi(u_i^j u_{i+1}^{j+1}) &= 1, & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor \text{ and } 1 \leq j \leq m-1 \\
 \phi(u_i^j u_{i+1}^{j+1}) &= k, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n \text{ and } 1 \leq j \leq m-1 \\
 \phi(u_i^j u_{i+1}^j) &= \frac{(m-1)(i-1)}{2} + \left\lceil \frac{j}{2} \right\rceil, & \text{for } 1 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } i \text{ odd and } 1 \leq j \leq m \\
 \phi(u_i^j u_{i+1}^j) &= \frac{(m-1)(i-1)}{2} + \frac{m}{2} + \left\lfloor \frac{j}{2} \right\rfloor, & \text{for } 2 \leq i \leq \left\lfloor \frac{n}{2} \right\rfloor - 1 \text{ and } j \text{ even and } 1 \leq j \leq m \\
 \phi(u_{\left\lfloor \frac{n}{2} \right\rfloor}^j u_{\left\lfloor \frac{n}{2} \right\rfloor+1}^j) &= \left\lfloor \frac{(i-1)(m-1)+3-k}{2} \right\rfloor + \frac{j-1}{2}, & \text{for } 1 \leq j \leq m-1 \text{ and } j \text{ odd} \\
 \phi(u_{\left\lfloor \frac{n}{2} \right\rfloor}^j u_{\left\lfloor \frac{n}{2} \right\rfloor+1}^j) &= \left\lceil \frac{(i-1)(m-1)+3-k}{2} \right\rceil + \frac{j-2}{2}, & \text{for } 2 \leq j \leq m \text{ and } j \text{ even} \\
 \phi(u_i^j u_{i+1}^j) &= \left\lfloor \frac{(m-1)(i-1)+4-2k}{2} \right\rfloor + \frac{j-1}{2}, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-1 \text{ and } 1 \leq j \leq m-1 \text{ and } j \text{ odd} \\
 \phi(u_i^j u_{i+1}^j) &= \left\lceil \frac{(m-1)(i-1)+4-2k}{2} \right\rceil + \frac{j-2}{2}, & \text{for } \left\lfloor \frac{n}{2} \right\rfloor + 1 \leq i \leq n-1 \text{ and } 2 \leq j \leq m \text{ and } j \text{ even.}
 \end{aligned}$$

A routine check leads to the following conclusions. All edge labels are at most k and the weights of the cycles $C_4^{i,j}$, $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, m-1$, are all distinct as

$$wt_{\phi}(C_4^{i,j}) = i + (j-1)(n-1) + 3.$$

We arrive at the desired result. $\square$

4. Conclusion

In this paper we investigated the edge C_4 -irregularity strength of Cartesian product of a cycle and a path and Cartesian product of two paths. We obtained the exact values of these parameters. An interesting

problem for further investigation is to find the edge C_4 -irregularity strength of Cartesian product of two cycles.

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