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Codes from the incidence matrices of a zero-divisor graphs

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Abstract

In this paper, we examine the linear codes with respect to the Hamming metric from incidence matrices of the zero-divisor graphs with vertex set consisting of all non-zero zero-divisors of the ring $\mathbb{Z}_n$ and two distinct vertices being adjacent if their product is zero over $\mathbb{Z}_n$. The main parameters of the codes are obtained.

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Keywords: *Linear codes, Incidence matrix, Zero-divisor graph.*

1. Introduction

Let R be a ring. A nonzero element x in R is said to be a zero-divisor if there exists a nonzero element $y \in R$ such that $x \cdot y = 0$. A simple graph G with the set of all zero-divisors of the ring R as the vertex set is said to be a *zero-divisor graph* if two vertices x and y are adjacent if and only if

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$x \cdot y = 0$ in R. In 1988, Beck[3] introduced the zero-divisor graph. Later it was modified by Anderson and Livingston[1]. In this article, we consider the zero-divisor graph $\Gamma(R)$ as a graph with vertex set $Z^*(R)$, that is, the set of non-zero zero-divisors of the ring R. Many researchers are doing research in this area. In particular, Redmond[6] and Aamir et al. [2]. In [5], Pradeep Singh and Vijay Kumar Bhat have studied Adjacency matrix and Wiener index of zero divisor graph $\Gamma(\mathbb{Z}_n)$.

Codes from the row span of incidence matrix or adjacency matrix of various graphs are discussed in [4, 7, 8].

2. Preliminaries

Let $\mathbb{F}_q$ be a finite field with q elements. Let $x = (x_1, \ldots, x_n) \in \mathbb{F}_q^n$, then the Hamming weight $w_H(x)$ of x is defined by the number of nonzero coordinates in x. Let $x = (x_1, \ldots, x_n), y = (y_1, \ldots, y_n) \in \mathbb{F}_q^n$, the Hamming distance $d_H(x, y)$ between x and y is defined by the number of coordinates in which they differ.

A q-ary code of length n is a non-empty subset C of $\mathbb{F}_q^n$. If C is a subspace of $\mathbb{F}_q^n$, then C is called a q-ary linear code of length n. An element of C is called a *codeword*. The minimum Hamming distance of a code C is defined by

$$d_H(C) = \min\{d_H(c_1, c_2) \mid c_1 \neq c_2, c_1, c_2 \in C\}.$$

The minimum weight $w_H(C)$ of a code C is the smallest among all weights of the non-zero codewords of C. For q-ary linear code, we have $d_H(C) = w_H(C)$. For basic coding theory, we refer [10].

We denote a linear code of length n, dimension k and minimum distance d as $[n, k, d]_q$. A $k \times n$ matrix G is called a *generator matrix* of a linear code C if the rows of G form a basis of C. We denote the code generated by G as $C_q(G)$.

Let $\Gamma = (V, E)$ be a graph. We denote $[x, y]$ as an edge between the vertices x and y. An incidence matrix of Γ is a $|V| \times |E|$ matrix B with rows labelled by the vertices and columns by the edges and entries $b_{ij} = 1$ if the vertex labelled by row i is incident with the edge labelled by column j and $b_{ij} = 0$ otherwise.

Let $W, X \subseteq V$ with $W \cap X = \varnothing$ and let $E(W, X)$ be the set of edges that have one end in W and the other end in X. We denote $|E(W, X)| = q(W, X)$.

An edge-cut of a connected graph Γ is the set $S \subseteq E$ such that $\Gamma - S = (V, E - S)$ is disconnected. The edge-connectivity $\lambda(\Gamma)$ is the minimum cardinality of an edge-cut. In fact,

$$\lambda(\Gamma) = \min_{\emptyset \neq W \subseteq V} q(W, V - W).$$

For any connected graph Γ, we have $\lambda(\Gamma) \leq \delta(\Gamma)$ where $\delta(\Gamma)$ is minimum degree of the graph Γ.

Theorem 2.1 : [9] *The edge connectivity of* $\Gamma(\mathbb{Z}_n)$ *for* $n = p_1^{\alpha_1} p_2^{\alpha_2} \cdots p_k^{\alpha_k}$ *is* $r = \min_{1 \leq i \leq k}\{p_i - 1\}$.

We denote the code generated by the rows of the incidence matrix G of the graph Γ is by $C_p(G)$ over the finite field $\mathbb{F}_p$.

Theorem 2.2 : [4]

1. *Let* $\Gamma = (V, E)$ *be a connected graph and let* G *be a* $|V| \times |E|$ *incidence matrix for* Γ. *Then the main parameters of the code* $C_2(G)$ *is* $[|E|, |V| - 1, \lambda(\Gamma)]_2$.
2. *Let* $\Gamma = (V, E)$ *be a connected bipartite graph and let* G *be a* $|V| \times |E|$ *incidence matrix for* Γ. *Then the incidence matrix generates* $[|E|, |V| - 1, \lambda(\Gamma)]_p$ *code for odd prime* p.

In this paper, we study a code from the incidence matrix of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ over $\mathbb{F}_p$ and we determined the parameters of the code. In Section 3, we discussed a code from incidence matrix of a zero-divisor graph $\Gamma(\mathbb{Z}_{p_1 p_2})$ over the field $\mathbb{F}_p$. In Section 4, we discussed a code from incidence matrix of a zero-divisor graph $\Gamma(\mathbb{Z}_{p_1 p_2 p_3})$ over $\mathbb{F}_2$. Finally, we studied the code from incidence matrix of the zero-divisor $\Gamma(\mathbb{Z}_{p_1 p_2 \cdots p_r})$ over $\mathbb{F}_2$ in Section 5. All the zero-divisor graphs considered in this article are simple and undirected.

3. Code from Incidence Matrix of a Zero-divisor Graph $\Gamma(\mathbb{Z}_{p_1 p_2})$

In this section, we study the codes obtained from the incidence matrix of the zero-divisor graph $\Gamma(\mathbb{Z}_{p_1 p_2})$ over $\mathbb{F}_p$ and we find the parameters of the code.

Let $n = p_1 p_2$ where p_1 and p_2 are distinct primes. Let $\mathbb{Z}_n$ be the ring of integers modulo n. Then the zero-divisors of $\mathbb{Z}_n$ is $Z^*(\mathbb{Z}_n) = \{p_1, 2p_1, \cdots, (p_2 - 1)p_1, p_2, 2p_2, \cdots, (p_1 - 1)p_2\}$.

Let $\Gamma(\mathbb{Z}_n)$ be the graph with vertex set $V = Z^*(\mathbb{Z}_n)$ and two distinct vertices a and b are adjacent if and only if $ab = 0$.

Let $A = \{xp_1 \mid x = 1, 2, \cdots, p_2 - 1\}$ and let $B = \{yp_2 \mid y = 1, 2, \cdots, p_1 - 1\}$ be the two disjoint subsets of the vertex set V. Then $|A| = p_2 - 1$ and $|B| = p_1 - 1$. The zero-divisor graph $\Gamma(\mathbb{Z}_n)$ can be viewed as follows:

That is, every vertex in A is adjacent with all vertices in B and vice versa. Clearly, $\Gamma(\mathbb{Z}_n)$ is a connected and complete bipartite graph with bipartition A and B and $|A|\,|B| = (p_1 - 1)(p_2 - 1)$ edges. Then the incidence matrix of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is

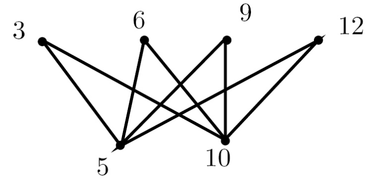
$$G = \left[\begin{array}{c|c|c|c} I_{p_2-1} & I_{p_2-1} & \cdots & I_{p_2-1} \\ \hline 1 & 0 & \cdots & 0 \\ 0 & 1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & \cdots & 1 \end{array}\right]_{(p_1+p_2-2)\times((p_1-1)(p_2-1))}$$

where I_{p_2-1} is the $(p_2-1)\times(p_2-1)$ identity matrix, $1 = [11\cdots1]_{1\times(p_2-1)}$ and $0 = [00\cdots0]_{1\times(p_2-1)}$.

In [9], they proved the edge-connectivity $\lambda(\Gamma(\mathbb{Z}_n))$ of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is $\min\{(p_1-1),(p_2-1)\}$. Therefore, by Theorem 2.2, the minimum distance of the code generated by the above incidence matrix G is $\min\{(p_1-1),(p_2-1)\}$ and dimension $p_1 + p_2 - 3$. Thus, we have

Theorem 3.1 : *Let p_1 and p_2 be two distinct primes. Then the main parameters of the p-ary linear code generated by the incidence matrix of the zero-divisor graph $\Gamma(\mathbb{Z}_{p_1p_2})$ is $[(p_1-1)(p_2-1), p_1+p_2-3, \min\{p_1-1, p_2-1\}]$ for any prime p.*

Example 3.2 : The zero-divisor graph $\Gamma(\mathbb{Z}_{15})$ is



Then the incidence matrix of the graph is

$$G = \left(\begin{array}{cccc|cccc} 1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\ \hline 1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 1 & 1 & 1 \end{array}\right)_{6\times 8}$$

Since any five rows are linearly independent and the graph is bipartite, implies the dimension of the code $C_p(G)$ over the finite field $\mathbb{F}_p$ generated by G is 5. The minimum distance of the code $C_p(G)$ is 2. Hence $C_p(G)$ is an $[8,5,2]_p$ code.

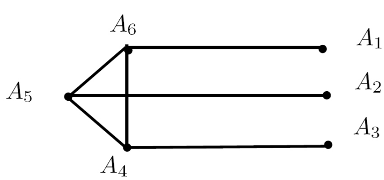
4. Code from Incidence Matrix of a zero-divisor Graph $\Gamma(\mathbb{Z}_{p_1p_2p_3})$

In this section, we study the code obtained from the incidence matrix of the zero-divisor graph $\Gamma(\mathbb{Z}_{p_1p_2p_3})$ over $\mathbb{F}_2$ and we find the main parameters of the code.

Let $n = p_1p_2p_3$ where p_1, p_2 and p_3 are distinct primes. Then the zero-divisors of $\mathbb{Z}_n$ can be partition into the following disjoints sets which is similar to that in [5]:

$$\begin{aligned} A_1 &= \{xp_1 \mid x = 1,2,\cdots,(p_2p_3-1), p_2 \nmid x, p_3 \nmid x\} \\ A_2 &= \{xp_2 \mid x = 1,2,\cdots,(p_1p_3-1), p_1 \nmid x, p_3 \nmid x\} \\ A_3 &= \{xp_3 \mid x = 1,2,\cdots,(p_1p_2-1), p_1 \nmid x, p_2 \nmid x\} \\ A_4 &= \{xp_1p_2 \mid x = 1,2,\cdots,(p_3-1)\} \\ A_5 &= \{xp_1p_3 \mid x = 1,2,\cdots,(p_2-1)\} \\ A_6 &= \{xp_2p_3 \mid x = 1,2,\cdots,(p_1-1)\} \end{aligned}$$

Then $|A_1| = (p_2-1)(p_3-1)$, $|A_2| = (p_1-1)(p_3-1)$, $|A_3| = (p_1-1)(p_2-1)$, $|A_4| = p_3-1$, $|A_5| = p_2-1$ and $|A_6| = p_1-1$. Hence $|V| = \sum_{i=1}^{6} |A_i|$. Then the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ can be viewed as follows:



where $\overset{A_i}{\bullet}\!\!\longrightarrow\!\!\!\!-\!\!\!\overset{A_j}{\bullet}$ is a complete bipartite graph. Then there are $|A_i||A_j|$ edges between the sets A_i and A_j. So the number of edges in the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is

$$|E| = |A_1||A_6| + |A_6||A_5| + |A_6||A_4| + |A_5||A_4| + |A_5||A_2| + |A_4||A_3|.$$

The set of edges between A_i and A_j is denoted by $[A_i, A_j]$. Clearly $\Gamma(\mathbb{Z}_n)$ is a connected graph. Then the incidence matrix of $\Gamma(\mathbb{Z}_n)$ is

$$G = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array} \begin{array}{c} \begin{array}{cccccc} [A_1, A_6] & [A_2, A_5] & [A_3, A_4] & [A_4, A_5] & [A_4, A_6] & [A_5, A_6] \end{array} \\ \left(\begin{array}{ccc|ccc} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ \hline 0 & 0 & 1 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 0 & 1 \\ 1 & 0 & 0 & 0 & 1 & 1 \end{array}\right)_{|V| \times |E|} \end{array}$$

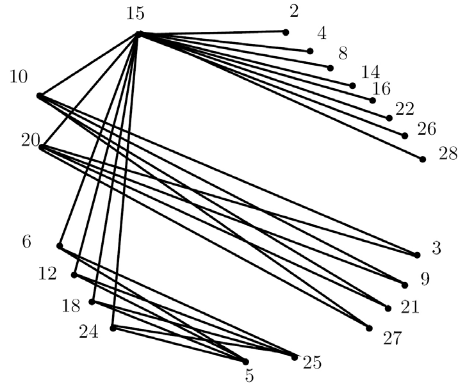
where the $(A_i, [A_k, A_j])$ entry is a $|A_i| \times |A_k||A_j|$ all one matrix for $i = k$ or $i = j$; zero matrix otherwise. Then the code generated by the incidence matrix G is $C_2(G)$ with main parameters $[|E|, |V|-1, \lambda((\Gamma(\mathbb{Z}_n)]_2$. Since the edge-connectivity $\lambda(\Gamma(\mathbb{Z}_n))$ of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$ is $\min\{(p_1-1), (p_2-1), (p_3-1)\}$ [9], thus we have

Theorem 4.1 : *Let p_1, p_2 and p_3 be two distinct primes. Then the code generated by the incidence matrix of the zero-divisor graph $\Gamma(\mathbb{Z}_{p_1p_2p_3})$ is a $C_2(G) = [n, k, d]_2$ code over the finite field $\mathbb{F}_2$ where $n = |E|, k = |V|-1$ and $d = \min\{(p_1-1), (p_2-1), (p_3-1)\}$.*

Example 4.2 : In [5], Pradeep Singh et al., gave; for $\mathbb{Z}_{30}$ with $p_1 = 2, p_2 = 3$ and $p_3 = 5$, the partitions of zero-divisors;

$$\begin{aligned} A_1 &= \{2, 4, 8, 14, 16, 22, 26, 28\} \\ A_2 &= \{3, 9, 21, 27\} \\ A_3 &= \{5, 25\} \\ A_4 &= \{6, 12, 18, 24\} \\ A_5 &= \{10, 20\} \\ A_6 &= \{15\} \end{aligned}$$

and the zero-divisor graph $\Gamma(\mathbb{Z}_{30})$ is



Then the incidence matrix of the graph is

$$G = \begin{array}{c} \\ A_1 \\ A_2 \\ A_3 \\ A_4 \\ A_5 \\ A_6 \end{array} \begin{array}{c} \begin{array}{cccccc} [A_1,A_6] & [A_2,A_5] & [A_3,A_4] & [A_4,A_5] & [A_4,A_6] & [A_5,A_6] \end{array} \\ \left(\begin{array}{ccc|ccc} \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \hline \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} & \mathbf{1} & \mathbf{0} \\ \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} & \mathbf{0} & \mathbf{1} \\ \mathbf{1} & \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{1} & \mathbf{1} \end{array}\right)_{21\times 38} \end{array}$$

where the $(A_i,[A_k,A_j])$ entry is a $|A_i|\times|A_k||A_j|$ all one matrix for $i=k$ or $i=j$; zero matrix otherwise. Since any five rows are linearly independent and the graph is bipartite, then the dimension of the code $C_2(G)$ over the finite field $\mathbb{F}_p$ generated by G is 5. The minimum distance of the code $C_2(G)$ is 2. Hence $C_2(G)$ is an $[n,k,d]_2 = [38,20,1]_2$ code.

5. Codes from incidence matrix of a zero-divisor graph $\Gamma(\mathbb{Z}_{p_1p_2\cdots p_r})$

In this section, we study the codes obtained from the incidence matrix of the zero-divisor graph $\Gamma(\mathbb{Z}_{p_1p_2\cdots p_r})$ over $\mathbb{F}_2$ and we find the parameters of the code.

Let $n = p_1p_2\cdots p_r$ where p_i's are distinct primes. Then the zero-divisors of $\mathbb{Z}_n$ can be partition into the following disjoints sets

$$A_{p_i} = \{p_i x \mid x = 1,2,\cdots,(p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_r)-1, p_j \nmid x, 1\le j\le r, i\ne j\}$$

$$A_{p_ip_j} = \{p_ip_j x \mid x = 1,2,\cdots,(p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_{j-1})-1, p_s \nmid x, 1\le s\le r, i\ne s\ne j\}$$

In general,

$$A_{p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_r} = \{x(p_1p_2\cdots p_{i-1}p_{i+1}\cdots p_r) \mid x = 1,2,\cdots,p_i - 1\}.$$

If G is the incidence matrix of the zero-divisor graph $\Gamma(\mathbb{Z}_n)$, then $C_2(G)$ is a $[|E|,|V|-1,\lambda((\Gamma(\mathbb{Z}_n)]_2$ code where $|E|$ is the length of the code, $|V|-1$ is the dimension of the code and $\lambda((\Gamma(\mathbb{Z}_n)$ is the minimum Hamming distance. Thus by Theorem 2.1, we have

Theorem 5.1 : *Let* $n = p_1p_2\cdots p_r$ *where* p_i*'s are distinct primes. Then the code generated by the incidence matrix of the zero-divisor graph* $\Gamma(\mathbb{Z}_n)$ *is a* $C_2(G) = [n,k,d]_2$ *code over the finite field* $\mathbb{F}_2$ *where* $n = |E|, k = |V| - 1$ *and* $d = \min\{(p_1 - 1),(p_2 - 1),\cdots,(p_r - 1)\}$.

Conclusion

In this paper, we studied the codes generated by the incidence matrix of the zero-divisor graphs of the different commutative ring with unity. Also we found the main parameters of the code over different finite field. We have consider only simple and undirected graphs in this article. Finding the covering radius of these codes is the further direction to work.

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References

[1] Anderson, D. F. and Livingston, P. S., The zero-divisor graph of a commutative ring, *Journal of Algebra*, 217 (1999), 434 – 447.

[2] Aamir Mukhtar, Rashid Murtaza, Shafiq U. Rehman, Saima Usman and Abdul Qudair Baig, Computing the size of zero divisor graphs, *Journal of Information and Optimization Sciences*, 41:4,(2020), 855–864.

[3] Beck I, Coloring of commutative rings, *Journal of Algebra*, 116 (1988), 208 – 226.

[4] Dankelmann P, Key J. D. and Rodrigues B. G, Codes from incidence matrices of graphs, *Design, Codes and Cryptography*, 68 (2011), 1 – 21.

[5] Pradeep Singh and Vijay Kumar Bhat, Adjacency matrix and Wiener index of zero divisor graph $\Gamma(\mathbb{Z}_n)$, *Journal of Applied Mathematics and Computing*, (2020), https://doi.org/10.1007/s12190-020-01460-2.

[6] Redmond, S. P., The zero-divisor graph of a non-commutative ring, International Journal of Commutative Ring, 1(2002), 203 – 211.

[7] Saranya R. and Durairajan C., Codes from Incidence Matrix of some Regular Graphs, *Discrete Mathematics, Algorithms and Applications*, 2020, DOI: 10.1142/S179383092150035X.

[8] Saranya R. and Durairajan C., Codes from incidence matrices of $(n,1)$-arrangement graphs and $(n,2)$-arrangement graphs, *Journal of Discrete Mathematical Sciences and Cryptography,* 2020, DOI: 10.1080/09720529.2019.1681674.

[9] Surendranath Reddy B., Rupali S. Jain and Laxmikanth N., Vertex and Edge Connectivity of the Zero Divisor Graph $\Gamma[\mathbb{Z}_n]$, *Communications in Mathematics and Applications,* 11(2)(2020), 253 – 258.

[10] San Ling, Chaoping Xing, Coding Theory: A First Course, Cambridge University Press, New York, (2004).

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