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Binary codes from graphs related to the Johnson graph $\Gamma(n, k, k-1)$

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Abstract

We show that the binary codes generated by the row span of adjacency matrices of the uniform subset graph $\Gamma(2k,k,1)$ and the Johnson graph $\Gamma(2k,k,k-1)$ coincide despite the graphs being non-isomorphic. We extend our results to the binary codes of $\Gamma(2k,k,i)$ and $\Gamma(2k,k,k-i)$ where $k>i$, $i \neq 0, \frac{k}{2}$ (k even), by showing that their adjacency matrices are equivalent. Further, we discuss the binary codes from the generalised uniform subset graph $\Gamma(2k,k,I)$ for $I=\{1,k-1\}$, and show that they are self-orthogonal for $k \geq 3$.

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1. Introduction

The codes generated by the row span of an adjacency matrix of graphs have been studied by various authors (see [8], [9], [10], [11], [19],

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[20] and [22]). From such studies it is evident that the codes do not have any uniform properties as regards their dimension, minimum weight and minimum words. In contrast, studies on codes from incidence matrices of some classes of graphs, (see for example [6], [7], [12], [13], [14], [16] and [26] for further details) have shown that these codes have some useful common properties. It is shown that the dimension is either the size of the vertex set or one less than it. It is also shown that the minimum weight coincides with the valency of the graph and the minimum words are the non-zero scalar multiples of the rows of the incidence matrix.

One would expect that non-isomorphic graphs would generate non-isomorphic codes, considering that an adjacency matrix of a graph completely describes the graph up to isomorphism. It is surprising that this is not the case. In this paper, we show that the binary codes generated by adjacency matrices of $\Gamma(2k,k,1)$ and the Johnson graph $\Gamma(2k,k,k-1)$ coincide. We further prove that adjacency matrices of $\Gamma(2k,k,i)$ and $\Gamma(2k,k,k-i)$ for $i \neq 0, \frac{k}{2}$ are equivalent and hence their binary codes also coincide.

Chen and Lih in [5] introduced uniform subset graphs as follows: let n, k be positive integers such that $n \geq 2k$, and i a non-negative integer with $i < k$. Let Ω be a set of size n and $\Omega^{\{k\}}$ the set of all k-subsets of Ω. The **uniform subset graph** $\Gamma(n,k,i)$ is defined by

$$V(\Gamma(n,k,i)) = \Omega^{\{k\}};$$
$$\{u,v\} \in E(\Gamma(n,k,i)) \Leftarrow |u \cap v| = i,\ u,v \in \Omega^{\{k\}}.$$

In another context, $\Gamma(2k,k,1)$ can be described as a graph obtained by the method of Key and Moori [25, Proposition 1]. In that context, the group is S_{2k}, and G_α is the maximal subgroup $S_k \times S_k$. By exploring the intrinsic properties of $\Gamma(2k,k,1)$, it becomes apparent that it is not isomorphic to $\Gamma(2k,k,k-1)$. The same applies to $\Gamma(2k,k,i)$, and $\Gamma(2k,k,k-i)$ for $i \neq 0, \frac{k}{2}$.

In [15] Fish, Key and Mwambene considered binary codes from an adjacency matrix of the combined graph formed from the odd graph $\Gamma(2k+1,k,0)$ and the Johnson graph $\Gamma(2k+1,k,k-1)$. Here we take a similar approach to the construction of binary codes from an adjacency matrix A_I of the generalised uniform subset graph $\Gamma(2k,k,I)$ where $I = \{1,k-1\}$. Note that for $k \geq 3$, the non-zero entries of adjacency matrices A_1 and A_{k-1} of $\Gamma(2k,k,1)$ and $\Gamma(2k,k,k-1)$ respectively do not coincide, and hence $A_I = A_1 + A_{k-1}$.

This paper is organised as follows: in Section 2 we discuss the preliminary concepts relating to graphs, designs and codes. In Section

3 we give some properties of the graphs $\Gamma(2k,k,i)$, $\Gamma(2k,k,k-i)$ and $\Gamma(2k,k,I)$ for $I=\{1,k-1\}$. In Section 4 we show that the codes from $\Gamma(2k,k,1)$ and $\Gamma(2k,k,k-1)$ coincide. We extend this result to the general case. Further, we discuss the parameters of the binary code of $\Gamma(2k,k,I)$ where $I=\{1,k-1\}$. Our main result is as follows:

Theorem 1 : *Let C_I denote the binary code from the row span of an adjacency matrix of the generalised uniform subset graph $\Gamma(2k,k,I)$ where $I=\{1,k-1\}$, and $C_I^{\perp}$ its dual. Then the following hold:*

(a) for $k\geq 3$ odd, C_I is a self-dual $[\binom{2k}{k},\binom{2k-1}{k-1},2]_2$-code;

(b) for $k\geq 4$ even, C_I is a $[\binom{2k}{k},\binom{2k-3}{k-2}-2^{k-2},2k^2]_2$-code, and $C_I^{\perp}$ a $[\binom{2k}{k},\binom{2k}{k}-\binom{2k-3}{k-2}+2^{k-2},d^{\perp}]_2$-code where $d^{\perp}\leq k+1$. Furthermore, C_I is self-orthogonal.

2. Preliminaries

We introduce background terminology and notation on graphs, designs and codes, mostly as in [16] and [17].

In this paper, all the graphs discussed are assumed to be finite and simple. For a graph Γ, $V(\Gamma),E(\Gamma)$ and $\{u,v\}$ denote the vertex set, the edge set and an edge of Γ respectively. If $\{u,v\}\in E(\Gamma)$, it is said that u and v are **adjacent**. The **open neighbourhood** of v, denoted $N(v)$, is the set of all vertices adjacent to v; the set $N[v]=N(v)\cup\{v\}$ is the **closed neighbourhood** of v. $|N(v)|$ is the **degree** of v. If each $v\in V(\Gamma)$ has equal degree, then Γ is **regular** and has **valency** $|N(v)|$. Γ is **strongly regular** of type (n,k,λ,μ) if it has n vertices, valency k, any two adjacent vertices are commonly adjacent to λ vertices and any two non-adjacent vertices are commonly adjacent to μ vertices.

An adjacency matrix may be used to represent a given graph. An **adjacency matrix** $A=(a_{ij})$ of a graph Γ is a square matrix indexed by the vertices such that $a_{ij}=1$ if vertices v_i and v_j are adjacent, and $a_{ij}=0$ otherwise.

In a connected graph, the **distance** between any two vertices u,v, denoted $\delta(u,v)$, is the number of edges in a shortest path connecting them. The **eccentricity** of a vertex u, denoted $\varepsilon(u)$, is defined by $\varepsilon(u):=\max_{v\in V(\Gamma)}\delta(u,v)$. The **diameter** of a graph Γ, denoted $\mathrm{diam}(\Gamma)$, is defined by $\operatorname{diam}(\Gamma) := \max_{v \in V(\Gamma)} \varepsilon(v)$. Γ is **distance-regular** if for any two vertices u and v such that $\delta(u,v) = i$, the number of vertices w such that $\delta(u,w) = j$ and $\delta(w,v) = k$, is independent of u and v.

Graphs Γ and Γ^* are said to be **isomorphic** if there exists a bijection $\alpha : V(\Gamma) \to V(\Gamma^*)$ such that for all $u,v \in V(\Gamma)$, $\{u,v\} \in E(\Gamma)$ if and only if $\{\alpha(u), \alpha(v)\} \in E(\Gamma^*)$. We call the mapping α an isomorphism. We denote this $\Gamma \cong \Gamma^*$, otherwise we write $\Gamma \ncong \Gamma^*$. An **automorphism** α of a graph Γ is a permutation of the vertices of the graph such that $\{u, v\} \in E(\Gamma)$ if and only if $\{\alpha(u), \alpha(v)\} \in E(\Gamma)$. The set of automorphisms of Γ, denoted $\operatorname{Aut}(\Gamma)$, forms a group under composition. A graph Γ is **vertex-transitive** if given $u,v \in V(\Gamma)$, there exists an automorphism $\alpha \in \operatorname{Aut}(\Gamma)$ such that $\alpha(u) = v$. A graph Γ is **edge-transitive** if given $d,e \in E(\Gamma)$, there exists an automorphism $\alpha \in \operatorname{Aut}(\Gamma)$ such that $\alpha(d) = e$.

The notation for designs and codes is as in [16] and [17]. An incidence structure $\mathcal{D} = (\mathcal{P},\mathcal{B},\mathcal{I})$, with point set $\mathcal{P}$, block set $\mathcal{B}$ and incidence $\mathcal{I}$ is a $t-(v,k,\lambda)$ **design**, if $|\mathcal{P}| = v$, every block $B \in \mathcal{B}$ is incident with precisely k points and every t distinct points are together incident with exactly λ blocks. The design is said to be symmetric if it has the same number of points and blocks. An **incidence matrix** of a design is a $|\mathcal{B}| \times |\mathcal{P}|$ matrix A such that $a_{ij} = 1$ if a point p_j is incident with a block B_i in $\mathcal{B}$, and $a_{ij} = 0$ otherwise. The code $C_{\mathbb{F}}(\mathcal{D})$ of the design $\mathcal{D}$ over the finite field $\mathbb{F}$ is the space spanned by the incidence vectors of the blocks over $\mathbb{F}$. If P is any subset of $\mathcal{P}$, then we denote the incidence vector of P by v^P. Each row of A is an incidence vector v^B of the corresponding block B.

All the codes here are linear codes. For primes p, the notation $[n,k,d]_p$ is used for p-ary code C of length n, and C is a k-dimensional subspace of $\mathbb{F}_p^n$, the vector space of n-tuples over $\mathbb{F}_p$. The elements of C are called **codewords**. The **support** of a codeword $c \in C$, denoted $\operatorname{Supp}(c)$, is the set of coordinate positions i such that c_i is non-zero. The **Hamming distance** between two codewords is the number of coordinate positions in which they differ. The **Hamming weight** of a codeword $c \in C$, denoted $\operatorname{wt}(c)$, is defined as the number of its non-zero coordinate positions. Hence $\mathrm{w}t(c) = |\operatorname{Supp}(c)|$. The **minimum weight** of a code C, denoted $\mathrm{d}(C)$, is the least number of elements in the support of any non-zero codeword of C. A **generator matrix** for C is a $k \times n$ matrix made up of a basis for C, and the **dual** code $C^\perp$ is the orthogonal under the standard inner product $(\cdot,\cdot)$, i.e. $C^\perp = \{v \in \mathbb{F}^n \mid (v,c) = 0 \text{ for all } c \in C\}$. A code C is sid to be **self-orthogonal** if $C \subseteq C^\perp$ and **self-dual** if $C = C^\perp$. If A is an adjacency matrix for a graph Γ, then the code generated by the row span of A over $\mathbb{F}_p$ may be denoted $C_p(A)$.

3. The graphs $\Gamma(2k,k,i)$, $\Gamma(2k,k,k-i)$ and $\Gamma(2k,k,I)$ for $I=\{1,k-1\}$

The uniform subset graphs $\Gamma(2k,k,i)$ and $\Gamma(2k,k,k-i)$ are regular with $\binom{2k}{k}$ vertices. In both cases the valency is $\binom{k}{i}\binom{k}{k-i}$. In general, these graphs are not strongly regular as can be seen from their diameters. One exception is $\Gamma(4,2,1)$ in which any two adjacent vertices are commonly adjacent to two vertices, and any two non-adjacent vertices are commonly adjacent to four vertices. Uniform subset graphs are in general vertex- and edge-transitive, as the symmetric group S_n, in its natural action, induces a transitive action both on its vertex- and edge-set. The uniform subset graph $\Gamma(2k,k,k-1)$ has been investigated elsewhere as a Johnson graph (see [29]) and the binary codes from such graphs are the content of [9].

We start by stating a result which identifies isomorphic uniform subset graphs and from which we can deduce that $n \geq 2k$. The proof can be found in [8].

Result 1 : [8, Lemma 4.1.1] For $n \geq k \geq i, \Gamma(n,k,i) \cong \Gamma(n,n-k,n-2k+i)$.

In $\Gamma(2k,k,i)$ and $\Gamma(2k,k,k-i)$ the size of the intersection between two vertices as k-sets determines their distance. This in turn determines the eccentricities of the vertices and the diameter of the graphs.

Result 2 : [3, Lemma 3] Let u,v be vertices of $\Gamma(2k,k,1)$ for $k \geq 5$, and $|u \cap v| = m > 1$. Then

$$\delta(u,v) = \min\left\{2\left\lceil \frac{k-m}{2} \right\rceil, 2\left\lceil \frac{m-1}{2} \right\rceil + 1\right\}. \tag{1}$$

Result 3 : [4, Lemma 2] Let k,i,s be positive integers with $k > i$. Let $\mathcal{V}$ denote the set of all vertex pairs (u,v) of $\Gamma(n,k,i))$ for $n = 2k-i+s$, and $|u \cap v| > i$. Then

$$\max_{(u,v)\in\mathcal{V}} \delta(u,v) = \begin{cases} \left\lceil \dfrac{k-i-1}{i+s} \right\rceil + 1 & \text{if } 0 < s < k-2i-1; \\ 2 & \text{otherwise.} \end{cases} \tag{2}$$

Result 4 : [4, Lemma 2] Let k, i, s be positive integers with $k > i$. Then for $n = 2k - i + s$,

$$\operatorname{diam}(\Gamma(n,k,i)) = \begin{cases} \left\lceil \dfrac{k-i-1}{i+s} \right\rceil + 1 & \text{if } k \geq s+2i+2; \\ 2 & \text{if } s \geq i \text{ and } 2i \leq k \leq i+s, \\ & \text{or } s < i \text{ and } i+s \leq k \leq 2i; \\ 3 & \text{otherwise.} \end{cases} \tag{3}$$

Result 5 : [8, Lemma 7.1.1, p. 83] Let u and v be vertices of the Johnson graph $\Gamma(2k, k, k-1)$. Then for $m \geq 0$,

$$\delta(u,v) = m \text{ if and only if } |u \cap v| = k - m. \tag{4}$$

From Result 5 the diameter of the Johnson graph $\Gamma(2k, k, k-1)$ is k, and two vertices u and v are at a distance k if and only if they are disjoint. However, from Result 4 the diameter of $\Gamma(2k, k, 1)$ is not k, and so the graphs are not isomorphic.

We generalise this as follows:

Theorem 2 : *Let k and i be positive integers with $k > i$. Then $\Gamma(2k,k,i) \ncong \Gamma(2k,k,k-i)$ for $k \geq 3i+2$, $i \neq 0, \frac{k}{2}$.*

Proof : By Result 4 the diameters of $\Gamma(2k,k,i)$ and $\Gamma(2k,k,k-i)$ are not equal, and the theorem follows. □

In general, the uniform subset graphs $\Gamma(2k,k,i)$ and $\Gamma(2k,k,k-i)$ are not isomorphic for $i \neq 0, \frac{k}{2}$. There are however cases when $k < 3i+2$ in Result 4 where the diameters of $\Gamma(2k,k,i)$ and $\Gamma(2k,k,k-i)$ are equal. We used Magma [1] to confirm this observation.

The generalised uniform subset graph $\Gamma(2k,k,I)$ for $I = \{1, k-1\}$ is in essence a combination of the graphs $\Gamma(2k,k,1)$ and $\Gamma(2k,k,k-1)$ since here two vertices u and v are adjacent if and only if they are adjacent in either constituent graph. Since $k \geq 3$ in our discussions, the constituent graphs do not coincide. The number of vertices remains at $\binom{2k}{k}$ while the valency increases to $2k^2$. So with a view to finding codes with a large minimum weight relative to its length, the codes from adjacency matrices of combined graphs where the constituent graphs do not coincide could be considered.

With regard to graph isomorphisms, S_n in its natural action imbeds into the automorphism group of $\Gamma(n,k,i)$ in the following way:

Let $\alpha \in S_n$. Define a map $\sigma_\alpha : \Omega^{\{k\}} \to \Omega^{\{k\}}$ by

$$\sigma_\alpha(\{x_1, x_2, \ldots, x_k\}) = \{\alpha(x_1), \alpha(x_2), \ldots, \alpha(x_k)\}$$

to be the natural induced action of α on $\Omega^{\{k\}}$.

Further, through graph covers, the following result has been shown in [28] for $\Gamma(2k,k,I)$ for $I=\{1,k-1\}$.

Result 6 : [28, Theorem 6] For $I=\{1,k-1\}$, $\mathrm{Aut}(\Gamma(2k,k,I)) \cong S_{2k} \times S_2$.

4. Binary codes from $\Gamma(2k,k,1)$, $\Gamma(2k,k,k-1)$ and $\Gamma(2k,k,I)$ for $I=\{1,k-1\}$

For $k \geq 3$ and $\Omega = \{1,2,\ldots,2k\}$, let $\Omega^{\{k\}}$ denote the k-subsets of Ω. An adjacency matrix A_1 of $\Gamma(2k,k,1)$ is also an incidence matrix of the 1-design $\mathcal{D}_1 = (\mathcal{P}_1, \mathcal{B}_1)$ where $\mathcal{P}_1 = \Omega^{\{k\}}$, and for each $X \in \Omega^{\{k\}}$ its corresponding block $\overline{X}_1$ is defined by

$$\overline{X}_1 = \{Y \in \Omega^{\{k\}} : |Y \cap X| = 1\}.$$

The block set $\mathcal{B}_1$ is then given by

$$\mathcal{B}_1 = \{\overline{X}_1 : X \in \Omega^{\{k\}}\},$$

and the incidence vector of the block $\overline{X}_1$ by

$$v^{\overline{X}_1} = \sum_{Y \in \overline{X}_1} v^Y. \qquad (5)$$

Note that we use the notation v^Y as opposed to the more cumbersome $v^{\{Y\}}$ from this point onwards.

The incidence vector $v^{\overline{X}_1}$ corresponds to the mapping

$$v^{\overline{X}_1} : \Omega^{\{k\}} \to \mathbb{F}_2, Y \mapsto v^{\overline{X}_1}(Y) = \begin{cases} 1 & \text{if } Y \in \overline{X}_1 \\ 0 & \text{if } Y \notin \overline{X}_1 \end{cases}.$$

Hence for $Y \in \Omega^{\{k\}}$, we have $v^{\overline{X}_1}(Y) = 1$ if and only if $Y \in \overline{X}_1$.

Similarly, an adjacency matrix A_{k-1} of $\Gamma(2k,k,k-1)$ is an incidence matrix of $\mathcal{D}_{k-1} = (\mathcal{P}_{k-1}, \mathcal{B}_{k-1})$ where $\mathcal{P}_{k-1} = \mathcal{P}_1$, and each block $\overline{X}_{k-1}$ is defined by

$$\overline{X}_{k-1} = \{Y \in \Omega^{\{k\}} : |Y \cap X| = k-1\}.$$

Hence

$$v^{\overline{X}_{k-1}} = \sum_{Y \in \overline{X}_{k-1}} v^{Y}. \tag{6}$$

Note that for $X, Y \in \Omega^{\{k\}}$, we have that $Y \in \overline{X}_{k-1}$ if and only if $Y \in \overline{\complement X_1}$ where $\complement X = \Omega \setminus X$.

Hence

$$v^{\overline{X}_{k-1}} = v^{\overline{\complement X_1}}. \tag{7}$$

Now let C_1 and C_{k-1} denote the binary codes from the row span of A_1 and A_{k-1} respectively, and $C_1^{\perp}$ and $C_{k-1}^{\perp}$ their duals. Assuming that the columns of A_1 and A_{k-1} have the same ordering, we have $A_1 \sim A_{k-1}$ by equation 7. This implies that $C_1 = C_{k-1}$. The parameters of C_1 and $C_1^{\perp}$ are obtained from the parameters of the binary code C from the Johnson graph $\Gamma(n,k,k-1)$ in [9]:

Result 7 : [9, Lemma 3.1] For $k \geq 3$ odd and $n \geq 6$ even, $C = \mathbb{F}_2^{\binom{n}{k}}$ and $\mathrm{Aut}(C) = S_{\binom{n}{k}}$.

Result 8 : [9, Lemma 3.4] For $k \geq 2$ and $n \geq 6$ both even, C is an $[\binom{n}{k}, \binom{n-2}{k-1}, d]_2$ code where $k+2 \leq d \leq k(n-k)$, and $C^{\perp}$ an $[\binom{n}{k}, \binom{n}{k} - \binom{n-2}{k-1}, k+1]_2$ code. If $n > 2k$, then $C^{\perp}$ does not have a basis of minimum weight vectors. $C \subseteq C^{\perp}$ and C is doubly-even. If $n > 2k$, then $\mathrm{Aut}(C) = S_n$, except when $k = 2$ and $n = 6$, in which case $\mathrm{Aut}(C) = PGL_4(2) \cong A_8$. If $n = 2k$, then $\mathrm{Aut}(C) = S_n \times Z_2$.

The observation $A_1 \sim A_{k-1}$ generalises to adjacency matrices A_i and A_{k-i} of $\Gamma(2k,k,i)$ and $\Gamma(2k,k,k-i)$ respectively. So for $k \geq 3i+2, i \neq 0, \frac{k}{2}$, the binary codes from the row span of adjacency matrices of $\Gamma(2k,k,i)$ and $\Gamma(2k,k,k-i)$ are equal, despite the graphs being non-isomorphic.

As noted before, for $k \geq 3$ A_1 and A_{k-1} have no non-zero entries which coincide. So if the rows and columns of A_1 and A_{k-1} have the same ordering, then $A_1 + A_{k-1}$ is an adjacency matrix of $\Gamma(2k,k,I)$ for $I = \{1, k-1\}$. Let A_I denote this matrix. The block $\overline{X}_I$ for $X \in \Omega^{\{k\}}$ is defined by

$$\overline{X}_I = \overline{X}_1 \dot{\cup} \overline{X}_{k-1}$$

and the incidence vector $v^{\overline{X}_I}$ by

$$v^{\overline{X}_I} = v^{\overline{X}_1} + v^{\overline{X}_{k-1}}. \tag{8}$$

By equation 7 we have

$$v^{\overline{X}_I} = v^{\overline{\complement X}_{k-1}} + v^{\overline{\complement X}_1}. \tag{9}$$

Hence

$$v^{\overline{X}_I} = v^{\overline{\complement X}_I}. \tag{10}$$

The above equations enable us to gain some insight into the relationship between any two incidence vectors.

Lemma 1 : *For* $k \geq 3, X, Y \in \Omega^{\{k\}}$:

(1) if $X = Y$ *or* $X = \complement Y$, *then* $\overline{X}_I = \overline{Y}_I$;

(2) if $|X \cap Y| = k-1$ *or* $|X \cap \complement Y| = k-1$, *then* $\overline{X}_I$ *and* $\overline{Y}_I$ *meet in* $4k-4$ *points except when* $k = 3$, *in which case they meet in 16 points;*

(3) if $|X \cap Y| = k-2$ *or* $|X \cap \complement Y| = k-2$, *then* $\overline{X}_I$ *and* $\overline{Y}_I$ *meet in 8 points except when* $k = 4$, *in which case they meet in 16 points;*

(4) if $|X \cap Y| < k-2$ *and* $|X \cap \complement Y| < k-2$, *then* $\overline{X}_I$ *and* $\overline{Y}_I$ *do not meet.*

Proof : (1) follows from the fact that $\overline{X}_I = \overline{\complement X_I}$. The rest are obtained by simple counting noting that $|X \cap \complement Y| = k - |X \cap Y|$. □

Let C_I denote the the binary code from the row span of A_I.

Lemma 2 : *For* $k \geq 3$, $C_I \subseteq C_I^{\perp}$; $j_{\binom{2k}{k}} \in C_I^{\perp}$.

Proof : By Lemma 1 the standard inner product $(v^{\overline{X}_I}, v^{\overline{Y}_I}) \equiv 0 (\mathrm{mod}\, 2)$ for all $X, Y \in \Omega^{\{k\}}$, and the first statement follows. The second statement follows from the fact that each incidence vector has weight $2k^2$. □

The parameters of C_I for $k \geq 3$ odd are discussed next.

Lemma 3 : *For* $k \geq 3$ *odd,* $S := \{v^{\overline{X}_I} : 1 \in X\}$ *is a basis for* C_I *and* $\dim(C_I) = \binom{2k-1}{k-1}$.

Proof : By equation 10 *S* spans C_I. Since *k* is odd, no set of vectors in *S* sum to 0. So *S* is a basis for C_I, and $\dim(C_I) = \binom{2k-1}{k-1}$. The vectors in *S* sum to $j_{\binom{2k}{k}}$ since each $Y \in \Omega^{\{k\}}$ is incident on k^2 blocks $\overline{X}_I$ where $1 \in X$. In this case $j_{\binom{2k}{k}} \in C_I \cap C_I^{\perp}$. □

Lemma 4 : *For* $k \geq 3$ *odd,* C_I *is self-dual.*

Proof : The result follows immediately from Lemma 2 and the fact that $\dim(C_I) = \binom{2k-1}{k-1} = \binom{2k}{k} - \binom{2k-1}{k-1} = \dim(C_I^{\perp})$. □

Lemma 5 : *For* $k \geq 3$ *odd,* C_I *has minimum weight* 2.

Proof: Let $X \in \Omega^{\{k\}}$. Consider the set of all incidence vectors $\{v^{\overline{Y}_I} : Y \in \overline{X}_{k-1}\}$, and in particular the sum $\sum_{Y \in \overline{X}_{k-1}} v^{\overline{Y}_I}$ of such vectors.

By equations 8 and 7 we have

$$\sum_{Y \in \overline{X}_{k-1}} v^{\overline{Y}_I} = \sum_{Y \in \overline{\complement X}_{k-1}} v^{\overline{Y}_{k-1}} + \sum_{Y \in \overline{X}_{k-1}} v^{\overline{Y}_{k-1}}.$$

The above sum reduces to

$$k^2 v^{\complement X} + (2k-2) \sum_{\substack{y \in X \\ Y \subseteq \complement X, |Y| = k-1}} v^{\{y\} \cup Y} + 4 \sum_{\substack{\{y,y'\} \subseteq X \\ Y \subseteq \complement X, |Y| = k-2}} v^{\{y,y'\} \cup Y}$$

$$+ k^2 v^{X} + (2k-2) \sum_{\substack{y \in \complement X \\ Y \subseteq X, |Y| = k-1}} v^{\{y\} \cup Y} + 4 \sum_{\substack{\{y,y'\} \subseteq \complement X \\ Y \subseteq X, |Y| = k-2}} v^{\{y,y'\} \cup Y}.$$

Since k is odd,

$$\sum_{Y \in \overline{X}_{k-1}} v^{\overline{Y}_I} = v^{\complement X} + v^{X}.$$

Note that for $k = 3$ and $|X \cap Y| = 2$, $v^{\overline{X}_I} + v^{\overline{Y}_I} = v^{Y} + v^{\complement Y} + v^{X} + v^{\complement X}$. □

For $k \geq 4$ even, the dimension and the minimum weight of the binary code are less trivial and we will discuss them now. Firstly, we require the following definitions:

Let n, k, l be positive integers such that $n \geq k, l$ and i a non-negative integer with $i < k, l$. Let Ω be a set of size n and $\Omega^{\{k\}}, \Omega^{\{l\}}$ the k- and l-subsets of Ω respectively. The bipartite graph $\Gamma(n, k, l, i)$ is defined by

$$V(\Gamma(n,k,l,i)) = \Omega^{\{k\}} \dot{\cup} \Omega^{\{l\}};$$

$$\{u, v\} \in E(\Gamma(n,k,l,i)) \Leftarrow |u \cap v| = i,\ u \in \Omega^{\{k\}}, v \in \Omega^{\{l\}}.$$

An adjacency matrix A of a bipartite graph whose partitions have p and q vertices is given by

$$A=\begin{bmatrix}0_{p\times p} & B\\ B^T & 0_{q\times q}\end{bmatrix}.$$

The $p\times q$ submatrix B is called a **biadjacency** matrix. If B is a biadjacency matrix of $\Gamma(n,k,l,i)$, then its rows will be indexed by $\Omega^{\{k\}}$ and its columns by $\Omega^{\{l\}}$.

We also require the following result which gives the parameters of the binary code generated by the row span of an adjacency matrix of the graph $\Gamma(2k+1,k,I)$ for $I=\{0,k-1\}$ which is a combination of the odd graph $\Gamma(2k+1,k,0)$ and the Johnson graph $\Gamma(2k+1,k,k-1)$. The result and its proof can be found in [15].

Let OJ_k denote a graph formed from a combination of the odd graph $\Gamma(2k+1,k,0)$ and the Johnson graph $\Gamma(2k+1,k,k-1)$.

Result 9 : [15, Theorem 1] For $k\geq 2$, let A_k be an adjacency matrix for the odd graph $\mathcal{O}_k$ and M_k an adjacency matrix for OJ_k. Let $I=I_{\binom{2k+1}{k}}$ and $C=C_2(A_k+I)$, the row span over $\mathbb{F}_2$ of A_k+I. Then C is a

$$\left[\binom{2k+1}{k},\binom{2k}{k-1}+\binom{2k-1}{k-1}-2^{k-1},k+2\right]_2$$

code with at least $\binom{2k+2}{k}$ vectors. $C^{\perp}$ is an even-weight code with minimum weight $d^{\perp}$, where $k+4\leq d^{\perp}\leq \min\{2^k,(k+1)^2+1\}$ for k even, and $k+5\leq d^{\perp}\leq \min\{2^k,(k+1)^2\}$ for k odd. Furthermore, for $k\geq 3$,

$$\mathrm{Hull}(C)=C\cap C^{\perp}=\begin{cases}C_2(M_k) & \text{for } k \text{ odd}\\ C_2(M_k+I) & \text{for } k \text{ even}\end{cases},$$

of dimension $\binom{2k-1}{k-1}-2^{k-1}$. For k even (respectively odd), M_k (respectively M_k+I), has full 2-rank. $\mathrm{Hull}(C)$ is self-orthogonal with minimum weight d where $(k+1)^2+1\geq d\geq k+4$ for k even, $(k+1)^2\geq d\geq k+5$ for k odd.

If s_x denotes the row of A_k+I labelled by the vertex x, then the hull is spanned by the vectors $w_x=\sum_{y\sim x}s_y$, which give the corresponding rows of M_k, of weight $(k+1)^2$, for k odd, or rows of (M_k+I), of weight $(k+1)^2+1$, for k even, where $\sim$ denotes adjacency in $\mathcal{O}_k$.

For $k\geq 3$, the symmetric group S_{2k+2} acts as a primitive automorphism group of OJ_k, C and $\mathrm{Hull}(C)$, and is edge-transitive on OJ_k. For $k=4$, $\mathrm{Hull}(C)^{\perp}$ is the code of a 2-$(126,6,9)$ design on which S_{10} acts primitively on points, transitively on blocks.

Lemma 6 : *For* $k \geq 4$ *even,* $\dim(C_I) = \binom{2k-3}{k-2} - 2^{k-2}$.

Proof : An adjacency matrix A_I of $\Gamma(2k, k, I)$ for $I = \{1, k-1\}$ can be viewed as follows:

$$A_I = \begin{bmatrix} A_{\Gamma(2k-1,k-1,K)} & B_{\Gamma(2k-1,k-1,k,I)} \\ B_{\Gamma(2k-1,k,k-1,I)} & A_{\Gamma(2k-1,k,I)} \end{bmatrix},$$

where

(a) $A_{\Gamma(2k-1,k-1,K)}$ is an adjacency matrix of $\Gamma(2k-1, k-1, K)$, for $K = \{0, k-2\}$ and has order $\binom{2k-1}{k-1} \times \binom{2k-1}{k-1}$;

(b) $B_{\Gamma(2k-1,k-1,k,I)}$ is a biadjacency matrix of $\Gamma(2k-1, k-1, k, I)$, for $I = \{1, k-1\}$ and has order $\binom{2k-1}{k-1} \times \binom{2k-1}{k}$;

(c) $B_{\Gamma(2k-1,k,k-1,I)}$ is a biadjacency matrix of $\Gamma(2k-1, k, k-1, I)$, for $I = \{1, k-1\}$ and has order $\binom{2k-1}{k} \times \binom{2k-1}{k-1}$;

(d) $A_{\Gamma(2k-1,k,I)}$ is an adjacency matrix of $\Gamma(2k-1, k, I)$, for $I = \{1, k-1\}$ and has order $\binom{2k-1}{k} \times \binom{2k-1}{k}$.

By equation 10 the dimension of the row span of A_I is equal to that of $A_{\Gamma(2k-1,k-1,K)}$ for $K = \{0, k-2\}$. Letting $r = k-1$, $\Gamma(2k-1, k-1, K)$ for $K = \{0, k-2\}$ is the graph OJ_r described in Result 9 given in [15]. The binary code from OJ_r has dimension $\binom{2r-1}{r-1} - 2^{r-1}$ and hence the binary code from $\Gamma(2k-1, k-1, K)$ for $K \in \{0, k-2\}$ has dimension $\binom{2k-3}{k-2} - 2^{k-2}$. Hence $\dim(C_I) = \binom{2k-3}{k-2} - 2^{k-2}$. □

Lemma 7 : For $k \geq 4$ even, C_I has minimum weight $2k^2$.

Proof : By equation 10 we need only consider the rows of A_I indexed by $R = \{X : X \in \Omega^{\{k\}}, 1 \in X\}$. Since A_1 and A_{k-1} have no non-zero entries which coincide,

$$\sum_{\substack{X \in T \\ T \subseteq R}} v^{\overline{X}_I} = \sum_{\substack{X \in T \\ T \subseteq R}} v^{\overline{X}_1} + \sum_{\substack{X \in T \\ T \subseteq R}} v^{\overline{X}_{k-1}}.$$

By [8, Proposition 7.2.4]

$$\sum_{\substack{X \in T \\ T \subseteq R}} v^{\overline{X}_{k-1}} \geq k^2.$$

Since $C_1 = C_{k-1}$,

$$\sum_{\substack{X \in T \\ T \subseteq R}} v^{\overline{X}_I} \geq 2k^2.$$

Now each incidence vector $v^{\overline{X}_I}, X \in \Omega^{\{k\}}$, has this weight. Hence C_I has minimum weight $2k^2$. □

For a discussion on C_I for $k \geq 4$ even, we give the following definitions: for $Y \in \Omega^{\{k-1\}}$, define

$$v(Y) = \sum_{y \in \complement Y} v^{\{y\} \cup Y}, \tag{11}$$

and for $Z \in \Omega^{\{k+1\}}$, define

$$w(Z) = \sum_{\substack{Y \subseteq Z \\ |Y| = k}} v^{Y}. \tag{12}$$

Lemma 8 : *For* $Y \in \Omega^{\{k-1\}}, Z \in \Omega^{\{k+1\}}$, *we have that* $v(Y), w(Z) \in C_I^{\perp}$.

Proof : It is easy to check that $(v^{\overline{X}_I}, v(Y)) \equiv 0 (\bmod 2)$ and $(v^{\overline{X}_I}, w(Z)) \equiv 0 (\bmod 2)$ for all $Y \in \Omega^{\{k-1\}}, X \in \Omega^{\{k\}}, Z \in \Omega^{\{k+1\}}$, giving us the result. □

Lemma 9 : $C_I^{\perp}$ *has minimum weight* $d^{\perp}$ *where* $d^{\perp} \leq k+1$.

Proof : This follows from Lemma 8 and the fact that $v(Y), Y \in \Omega^{\{k-1\}}$, and $w(Z), Z \in \Omega^{\{k+1\}}$, each have weight $k+1$. □

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References

[1] W. Bosma, J. Cannon and C. Playoust, The Magma algebra system I: The user language, *J. Symbolic Comput.*, 24 (1997), 235-265.

[2] A. E. Brouwer and J. H. van Eijl, On the p-rank of the adjacency matrices of strongly regular graphs, *J. Algebraic Combin.*, 1 (1992), 329-346.

[3] Y. Chen and Y. Wang, On the diameter of generalised Kneser graphs, *Discrete Math.*, 308 (2008), 4276-4279.

[4] Y. Chen and W. Wang, Diameters of uniform subset graphs, *Discrete Math.*, 308 (2008), 6645-6649.

[5] B-L Chen and K-W Lih, Hamiltonian uniform subset graphs, *J. Combin. Theory Ser. B*, 42 (1987), 257-263.

[6] P. Dankelman, J. D. Key and B. G. Rodrigues, A characterisation of graphs by codes from their incidence matrices, *Electron. J. Combin.*, 20 (3) (2013), # P18.

[7] P. Dankelman, J. D. Key and B. G. Rodrigues, Codes from incidence matrices of graphs, *Des. Codes Cryptogr.*, 68 (2013), 373-393.

[8] W. Fish, Codes from uniform subset graphs and cycle products, Ph.D. Thesis, University of the Western Cape, 2007.

[9] W. Fish, Binary Codes and Partial Permutation Decoding Sets from the Johnson Graphs, *Graphs Combin.*, 31 (2015), 1381-1396.

[10] W. Fish, R. Fray and E. Mwambene, Binary codes and partial permutation decoding sets from the odd graphs, *Cent. Eur. J. Math.*, 12 (2014), 1362-1371.

[11] W. Fish, J. D. Key and E. Mwambene, Binary codes of line graphs from the n-cube. *J. Symbolic Comput.*, 45 (2010) 800-812.

[12] W. Fish, J. D. Key and E. Mwambene, Codes from the incidence matrices of graphs on 3-sets, *Discrete Math.*, 311 (2011), 1823-1840.

[13] W. Fish, J. D. Key and E. Mwambene, Codes from incidence matrices and line graphs of Hamming graphs, *Discrete Math.*, 310 (2010), 1884-1897.

[14] W. Fish, J. D. Key and E. Mwambene, Codes from the incidence matrices and line graphs of Hamming graphs $H^k(n,2)$ for $k \geq 2$, *Adv. Math. Commun.*, 5 (2011), 373-394.

[15] W. Fish, J. D. Key and E. Mwambene, Self-orthogonal binary codes from odd graphs, *Util. Math.*, 103 (2017), 73-97.

[16] W. Fish, K. Kumwenda and E. Mwambene, Codes and graphs from triangular graphs and their line graphs, *Cent. Eur. J. Math.*, 9 (2011), 1411-1423.

[17] W. Fish, N. Mumba, E. Mwambene and B. G. Rodrigues, Binary codes and partial permutation decoding sets from biadjacency matrices of bipartite graphs $\Gamma(2k+1,k,k+2,1)$; *Graphs Combin.*, 33, (2017), 357-368.

[18] D. M. Gordon, Minimal permutation sets for decoding the binary Golay codes, *IEEE Trans. Inform. Theory*, 28 (1982), 541-543.

[19] W. H. Haemers, Matrices for graphs, designs and codes, *NATO Sci. Peace Secur. Ser. D Inf. Commun. Secur.*, 29 (1981), 253-277.

[20] W. H. Haemers, R. Peeters and J. M. van Rijckevorsel, Binary codes of strongly regular graphs, *Des. Codes Cryptogr.*, 17 (1999), 187-209.

[21] J. D. Key, J. Moori, B. G. Rodrigues, Permutation decoding sets for the binary codes from triangular graphs, *European J. Combin.*, 25 (2004), 113-123.

[22] J. D. Key, J. Moori, B. G. Rodrigues, Ternary codes from graphs on triples, *Discrete Math.*, 309 (2009), 4663-4681.

[23] J. D. Key, J. Moori, B. G. Rodrigues Codes associated with triangular graphs and permutation decoding, *Int. J. Inform. and Coding Theory*, 1 (2010), 334-349.

[24] J. D. Key, Permutation decoding: An update. Available from http://www3.risc.jku.at/research/theorema/Groebner-Bases-Bibliography/gbbib_files/publication_148.pdf, 2003.

[25] J. D. Key and J. Moori, Codes, Designs and Graphs from the Janko Groups J_1 and J_2, *J. Combin. Math. Combin. Comput.*, 40, (2002), 143-160.

[26] K. Kumwenda *Codes, Graphs and Designs Related to Iterated Line Graphs of Complete Graphs*, Ph.D. dissertation, Univ. of the Western Cape, 2011.

[27] N. B. Mumba, Codes, graphs and designs from maximal subgroups of alternating groups, Ph.D. Thesis, University of the Western Cape, 2018.

[28] N. B. Mumba and E. Mwambene, Automorphism groups of graph covers and uniform subset graphs, *AKCE Int. J. Graphs Comb.*, 15, (2018), 27-30.

[29] M. Ramras and E. Donovan, The automorphism group of a Johnson graph, *SIAM J. Discrete Math.*, 25 (2011), 267-270.

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