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A note on $\mathcal{H}_Y$-filters and $\mathcal{H}_Y$-ideals

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Abstract

Let R be a commutative ring with identity and $Y \subseteq Spec(R)$. In this paper, we pay attention to the concept of $\mathcal{H}_Y$-filters. First, as a generalization of a prime $\mathcal{H}_Y$-filter, we define the concept of 2-absorbing $\mathcal{H}_Y$-filters. We give a correspondence between the 2-absorbing $\mathcal{H}_Y$-filters and the 2-absorbing strong $\mathcal{H}_Y$-ideals. Furthermore, we give a condition under which every $\mathcal{H}_Y$-filter containing a 2-absorbing $\mathcal{H}_Y$-filter is also a 2-absorbing $\mathcal{H}_Y$-filter. The second part of this paper deals with the case of the product rings. We define the product of $\mathcal{F}$ and $\mathcal{F}'$, denoted by $\mathcal{F} \otimes \mathcal{F}'$. We prove that all the $\mathcal{H}_{\mathcal{Y}}$-filters on $\mathcal{Y}$, where $\mathcal{Y}$ is a subset of the prime spectrum of the product rings, are all of the previous forms. As a natural result, we characterize all the prime $\mathcal{H}_{\mathcal{Y}}$-filters, $\mathcal{H}_{\mathcal{Y}}$-ultrafilters and 2-absorbing $\mathcal{H}_{\mathcal{Y}}$-filters.

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1. Introduction

In this paper, we focus on the concept of $\mathcal{H}_Y$-filter and $\mathcal{H}_Y$-ideal of a commutative ring with identity. This notion was recently introduced by Aliabad et al. (2020) as an extension of the concept of z-ideal. The notion of z-ideals was first introduced by Kohls (1957) (see [11]). These ideals play a fundamental role in the study of ideals set of the ring $C(X)$, the ring of real-valued continuous functions on a Tychonoff topological space X. An

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ideal I of the ring $C(X)$ is said to be z-ideal if $Z^{-1}[Z[I]]=I$, i.e, $Z(f)\in Z[I]$ implies that $f\in I$, where $Z(f)=\{x\in X: f(x)=0\}$ and $Z[I]=\{Z(f): f\in I\}$, (see [10]). Since the term "$Z(f)$" is only defined in $C(X)$, this definition remained exclusive to the ring $C(X)$ for many decades. Note that there exists a relationship between the maximal ideals of the ring $C(X)$ and the term "$Z(f)$". In fact, a maximal ideal M of $C(X)$ is of the form $M=M^p=\{f\in C(X): p\in cl_{\beta X}Z(f)\}$ where $p\in\beta X$ and βX is the Stone-Čech compactification of X (see [10]). Therefore, the equality between the zero-set of f and that of g is equivalent to the equality between the set of all maximal ideals of $C(X)$ containing f and that of g. This fact motivated the authors to extend this concept to any commutative ring. Thing that is done by Mason (1973) [12].

Throughout this paper, the term "ring" means a commutative ring with identity. Let R be a ring, by $Spec(R)$ (resp., $Max(R)$) we mean the set of all prime (resp., maximal) ideals of R. For every ideal I of R, we denote by $Min(I)$ the set of all minimal prime ideals over I. For any subset Y of $Spec(R)$, the set of all elements of Y containing F is denoted by $h_Y(F)$. We denote by $h(F)$ (resp., $h_M(F)$) instead of $h_{Spec(R)}(F)$ (resp., $h_{Max(R)}(F)$). Also, we usually use $h_Y(a)$ instead of $h_Y(\{a\})$. Let us now recall the Mason definition. Indeed, an ideal I of R is said to be z-ideal if whenever $h_M(a)\subseteq h_M(b)$, for $a\in I$ and $b\in R$ we have $b\in I$. Recently, in 2020, this definition was generalized by the authors (see [10]). It is not limited to $Max(R)$, as in the z-ideal definition, but it has extended to any arbitrary subset of $Spec(R)$. Let R be a ring and $Y\subseteq Spec(R)$. An ideal I of R is said to be $\mathcal{H}_Y$-ideal whenever it follows from $h_Y(a)\subseteq h_Y(b)$ and $a\in I$ that $b\in I$. It is clear that if $Y=Max(R)$, the concept of $\mathcal{H}_Y$-ideals coincide with the concept of z-ideals. Note that in the last years there are some other extensions of the concept of z-ideals (*cf.* [2, 9]).

Let S be a set, define $F(S)$ the set of all finite subsets of S. As in [1], for a ring R and $Y\subseteq Spec(R)$, we denote by $\mathcal{H}_Y$ the following set $\{h_Y(F): F\in F(R)\}=\{h_Y(I): I \textit{ is a finitely generated ideal of } R\}$. It is easy to check that, for arbitrary ideals I and J of R we have $h_Y(I)\cap h_Y(J)=h_Y(I+J)$, $h_Y(I)\cup h_Y(J)=h_Y(IJ)$, $h_Y(1)=\varnothing$ and $h_Y(0)=Y$. A filter $\mathcal{F}$ of $\mathcal{H}_Y$ is called $\mathcal{H}_Y$-filter on Y (see [1]), i.e. a nonempty subfamily of $\mathcal{H}_Y$ such that (i) if $h_Y(F_1), h_Y(F_2)\in\mathcal{F}$, then $h_Y(F_1)\cap h_Y(F_2)\in\mathcal{F}$, ($ii$) if $h_Y(F_1)\in\mathcal{F}$ and $h_Y(F_2)\in\mathcal{H}_Y$ such that $h_Y(F_1)\subseteq h_Y(F_2)$, then $h_Y(F_2)\in\mathcal{F}$. We denote the sets $\{h_Y(F): F\in F(I)\}$ and $\{a\in R: h_Y(a)\in\mathcal{F}\}$ by $\mathcal{H}_Y(I)$ and $\mathcal{H}_Y^{-1}(\mathcal{F})$, respectively.

Recall that, a $\mathcal{H}_Y$-filter on Y is said to be prime if whenever $h_Y(F_1) \cup h_Y(F_2) \in \mathcal{F}$ for $F_1, F_2 \in F(R)$, we have $h_Y(F_1) \in \mathcal{F}$ or $h_Y(F_2) \in \mathcal{F}$. All prime $\mathcal{H}_Y$-filters, $\mathcal{H}_Y$-ultrafilters and all 2-absorbing $\mathcal{H}_Y$-filters are assumed to be proper, i.e. does not contain the emptyset.

The notion of 2-absorbing ideals of a ring R was first introduced by Badawi [4]. We recall the definition, a nonzero proper ideal I of R is called a 2-absorbing ideal of R if whenever $a, b, c \in R$ and $abc \in I$, then $ab \in I$ or $ac \in I$ or $bc \in I$. It was shown that, a nonzero proper ideal I of R is a 2-absorbing ideal if and only if whenever $I_1 I_2 I_3 \subseteq I$ for some ideals I_1, I_2, I_3 of R, then $I_1 I_2 \subseteq I$ or $I_1 I_3 \subseteq I$ or $I_2 I_3 \subseteq I$, ([4], Theorem 2.13). The reader is referred to [3, 4] for more information about the notion of 2-absorbing ideals.

2. Correspondence between 2-absorbing $\mathcal{H}_Y$-filters and 2-absorbing $\mathcal{H}_Y$-ideals

Definition 2.1 : Let R be a ring, $Y \subseteq Spec(R)$ and I be an ideal of R. The ideal I is said to be a strong $\mathcal{H}_Y$-ideal of R if whenever $h_Y(F) \subseteq h_Y(a)$, for F a finite subset of I and $a \in R$, then $a \in I$.

The ideal I is said to be an $\mathcal{H}_Y$-ideal if we replace F by an element $b \in R$ in the previous definition.

For another characterization of the strong $\mathcal{H}_Y$-ideals and the $\mathcal{H}_Y$-ideals see ([1], Proposition 3.2 and Proposition 3.4).

In [6], it was shown that a z-ideal of $C(X)$ is 2-absorbing if and only if it contains a 2-absorbing ideal. In the sequel, we focus on the generalization of this result to arbitrary rings. Let is start with the following useful lemma.

Lemma 2.2 : *Let R be a ring and I is a radical ideal of R. Assume that I contains a prime ideal P such that $h(P)$ forms a chain. Then I is a prime ideal.*

Proof : Since $P \subseteq I$, and $h(P)$ forms a chain, then so is $Min(I)$. But I is a radical ideal of R, then $I = \bigcap_{Q \in Min(I)} Q$. Thus, I is a prime ideal of R. □

Theorem 2.3 : *Let R be a ring, $Y \subseteq Spec(R)$ and I be an $\mathcal{H}_Y$-ideal of R that contains a 2-absorbing ideal J of R. Suppose that for every $P \in Min(J)$, $h(P)$ forms a chain. Then I is also a 2-absorbing ideal of R.*

Proof : Since J is a 2-absorbing ideal of R, then by ([4], Theorem 2.3), $|Min(J)| = 1$ or 2.

Case 1: If $Min(J)=\{P\}$. I is an $\mathcal{H}_Y$-ideal of R, then by ([1] lemma 3.12), I is a radical ideal of R, then $P=\sqrt{J}\subseteq I$. Now, using the previous lemma, we have I is a prime ideal, hence a 2-absorbing ideal of R.

Case 2: If $Min(J)=\{P_1,P_2\}$. In this case, we have $\sqrt{J}=P_1\cap P_2\subseteq I=\bigcap_{P\in Min(I)}P$. Thus, $I=R_1\cap R_2$ where R_1 (resp., R_2) is the intersection of all minimal prime ideals over I containing P_1 (resp., P_2). Since I is an $\mathcal{H}_Y$-ideal of R, then by ([1], Theorem 3.13), for every $P\in Min(I)$, P is an $\mathcal{H}_Y$-ideal of R. Thus, R_1 is an $\mathcal{H}_Y$-ideal of R as an intersection of $\mathcal{H}_Y$-ideals of R, and hence a radical ideal. Furthermore, $P_1\subseteq R_1$ and $h(P_1)$ forms a chain, then by the previous lemma, R_1 is a prime ideal of R. In the same way, we prove that R_2 is a prime ideal of R. Therefore, $I=R_1\cap R_2$ is a 2-absorbing $\mathcal{H}_Y$-ideal of R as an intersection of two prime ideals of R. □

Remark 2.4 : In general, the concept of $\mathcal{H}_Y$-ideal and the concept of 2-absorbing ideal are different. For example, as noted in [4], a nonzero proper 2-absorbing ideal I of a principal ideal domain (PID) D, is of the form pD or p^2D or pqD where p, q are distinct prime elements of D. Now, let D be a (PID), p a prime element of D and $Y=Max(D)=Spec(D)\setminus\{0\}$. Then, the notion of $\mathcal{H}_Y$-ideal and z-ideal are the same. By ([7], Corollary 1), $I=p^2D$ is not an $\mathcal{H}_Y$-ideal of D. Furthermore, if p, q and t are three distinct prime elements of D, then $I=pqtD$ is an $\mathcal{H}_Y$-ideal of D which is not a 2-absorbing ideal of D. We have not pursued the question of the necessary and sufficient condition to prove that the $\mathcal{H}_Y$-ideals of a ring are 2-absorbing ideals or vice versa. We leave that for another occasion.

Definition 2.5 : Let R be a ring, $Y\subseteq Spec(R)$ and $\mathcal{F}$ be a proper $\mathcal{H}_Y$-filter on Y. $\mathcal{F}$ is said to be 2-absorbing $\mathcal{H}_Y$-filter on Y if whenever $F_1,F_2,F_3\in F(A)$ and $h_Y(F_1)\cup h_Y(F_2)\cup h_Y(F_3)\in\mathcal{F}$, then there exist $i\neq j\in\{1,2,3\}$ such that $h_Y(F_i)\cup h_Y(F_j)\in\mathcal{F}$.

It is easy to see that a prime $\mathcal{H}_Y$-filter on Y is 2-absorbing.

Proposition 2.6 : Let R be a ring, $Y\subseteq Spec(R)$, I a strong $\mathcal{H}_Y$-ideal of R and $\mathcal{F}$ be a $\mathcal{H}_Y$-filter on Y. Then the following statements hold.

(1) $\mathcal{F}$ is a 2-absorbing $\mathcal{H}_Y$-filter on Y if and only if $\mathcal{H}_Y^{-1}(\mathcal{F})$ is a 2-absorbing $\mathcal{H}_Y$-ideal of R.

(2) I is a 2-absorbing ideal of R if and only if $\mathcal{H}_Y(I)$ is a 2-absorbing $\mathcal{H}_Y$-filter on Y.

Proof : (1) "⇒" Obvious.

"$\Leftarrow$" Let $F_1, F_2, F_3 \in F(A)$ such that $h_Y(F_1) \cup h_Y(F_2) \cup h_Y(F_3) \in \mathcal{F}$. Then, $h_Y((F_1)(F_2)(F_3)) \in \mathcal{F}$. Thus by ([1], Lemma 3.1), one has $(F_1)(F_2)(F_3) \subseteq \mathcal{H}_Y^{-1}(\mathcal{F})$. The result follows using ([4], Theorem 2.13).

(2) "$\Rightarrow$" I is a strong $\mathcal{H}_Y$-ideal of R, so by ([1], Proposition 4.1), $\mathcal{H}_Y^{-1}(\mathcal{H}_Y(I)) = I$. Then, $\mathcal{H}_Y^{-1}(\mathcal{H}_Y(I))$ is a 2-absorbing $\mathcal{H}_Y$-ideal of R. Using (1) we have, $\mathcal{H}_Y(I)$ is a 2-absorbing $\mathcal{H}_Y$-filter on Y.

"$\Leftarrow$" Since I is a strong $\mathcal{H}_Y$-ideal of R and $\mathcal{H}_Y(I)$ is a 2-absorbing $\mathcal{H}_Y$-filter on Y, then by ([1] , Proposition 4.1) and (1), we have $I = \mathcal{H}_Y^{-1}(\mathcal{H}_Y(I))$ is a 2-absorbing ideal of R. □

In the following, we give an example of a non-prime 2-absorbing $\mathcal{H}_Y$-filter.

Example 2.7 : Let R be a ring what contains at least two incomparable prime ideals P_1 and P_2 and let Y be a subset of $Spec(R)$ containing P_1 and P_2. Then, $\mathcal{H}_Y(P_1 \cap P_2)$ is a non-prime 2-absorbing $\mathcal{H}_Y$-filter. Indeed, Since $P_1, P_2 \in Y$, then P_1 and P_2 are two $\mathcal{H}_Y$-ideals of R. Moreover, P_1 and P_2 are not in chain, so $P_1 \cap P_2$ is a non-prime ideal of R. Using ([1], Propositions 4.1, 4.3), $\mathcal{H}_Y(P_1 \cap P_2)$ is a non-prime $\mathcal{H}_Y$-filter on Y. Also, since $P_1 \cap P_2$ is a 2-absorbing ideal of R, then, by the previous proposition, $\mathcal{H}_Y(P_1 \cap P_2)$ is a 2-absorbing $\mathcal{H}_Y$-filter on Y.

In Proposition 2.6, the condition "I is a strong $\mathcal{H}_Y$-ideal of R" can be replaced by another condition. This is the purpose of the ensuing proposition.

Proposition 2.8 : Let R be a ring, $Y \subseteq Spec(R)$ and I be a 2-absorbing ideal of R. Suppose that for every $P \in Min(I)$, $h(P)$ forms a chain then $\mathcal{H}_Y(I)$ is a 2-absorbing $\mathcal{H}_Y$-filter on Y (if it is proper).

Proof : By ([1], Proposition 4.1), $I \subseteq \mathcal{H}_Y^{-1}(\mathcal{H}_Y(I))$. Since I is a 2-absorbing ideal of R, then by the previous theorem, $\mathcal{H}_Y^{-1}(\mathcal{H}_Y(I))$ is a 2-absorbing $\mathcal{H}_Y$-ideal of R. Now, using the previous proposition, $\mathcal{H}_Y(I)$ is a 2-absorbing $\mathcal{H}_Y$-filter on Y. □

It is well-known that, if an ideal of a ring is contained in a union of finitely many prime ideals P_i's, then it is contained in P_i for some i. (Prime avoidance lemma). The following proposition exhibits an analogous result.

Proposition 2.9 : Let $\mathcal{F}$ be $\mathcal{H}_Y$-filter on Y and $\mathcal{P}_1, \cdots, \mathcal{P}_n$ a finitely many prime $\mathcal{H}_Y$-filters on Y. Assume that $\mathcal{F} \subseteq \bigcup_{i=1}^{n} \mathcal{P}_i$, then there exists $i_0 \in \{1, \cdots, n\}$ such that $\mathcal{F} \subseteq \mathcal{P}_{i_0}$.

Proof : By ([1], Proposition 4.3) and the prime avoidance lemma.

We recall the following definition (see [1]). □

Definition 2.10 : An $\mathcal{H}_Y$-filter $\mathcal{P}$ is said to be a minimal prime $\mathcal{H}_Y$-filter over a $\mathcal{H}_Y$-filter $\mathcal{F}$, if there is no prime $\mathcal{H}_Y$-filter strictly contained in $\mathcal{P}$ that contains $\mathcal{F}$.

We denote by $Min(\mathcal{F})$ the set of all minimal prime $\mathcal{H}_Y$-filters over $\mathcal{F}$.

Theorem 2.11 : *Let R be a ring, $Y \subseteq Spec(R)$ and $\mathcal{F}$ be a 2-absorbing $\mathcal{H}_Y$-filter on Y. Then there are at most two prime $\mathcal{H}_Y$-filters on Y that are minimal over $\mathcal{F}$.*

Proof : By Proposition 2.6, $\mathcal{H}_Y^{-1}(\mathcal{F})$ is a 2-absorbing $\mathcal{H}_Y$-ideal of R, and then $|Min(\mathcal{H}_Y^{-1}(\mathcal{F}))| = 1$ or 2. Now, suppose that there exist more than two distinct prime $\mathcal{H}_Y$-filters on Y that are minimal over $\mathcal{F}$. Let $\mathcal{P}_1$, $\mathcal{P}_2$ and $\mathcal{P}_3 \in Min(\mathcal{F})$ among them. By ([1], Proposition 4.9), $\mathcal{H}_Y^{-1}(\mathcal{P}_i) \in Min(\mathcal{H}_Y^{-1}(\mathcal{F}))$, $i = 1, 2, 3$. Then, ([1], Corollary 4.2) implies that the $\mathcal{H}_Y^{-1}(\mathcal{P}_i)'s$ are distinct. Which is a contradiction, and so the result holds. □

We need the following straightforward lemmas to prove our main result of this section.

Lemma 2.12 : *Let R be a ring, $Y \subseteq Spec(R)$, $(\mathcal{F}_i)_{i \in I}$ a family of $\mathcal{H}_Y$-filters on Y and $(I_\lambda)_{\lambda \in \Lambda}$ be a family of strong $\mathcal{H}_Y$-ideals of R. Then the following statements hold.*

(1) $\mathcal{H}_Y^{-1}(\bigcap_{i \in I} \mathcal{F}_i) = \bigcap_{i \in I} \mathcal{H}_Y^{-1}(\mathcal{F}_i)$.

(2) $\mathcal{H}_Y(\bigcap_{\lambda \in \Lambda} I_\lambda) = \bigcap_{\lambda \in \Lambda} \mathcal{H}_Y(I_\lambda)$.

Lemma 2.13 : *The intersection of two prime $\mathcal{H}_Y$-filters on Y is a 2-absorbing $\mathcal{H}_Y$-filter.*

Lemma 2.14 : *Let R be a ring, $Y \subseteq Spec(R)$, $\mathcal{F}_1$ and $\mathcal{F}_2$ two $\mathcal{H}_Y$-filters on Y and $\mathcal{P}$ is a prime $\mathcal{H}_Y$-filter on Y such that $\mathcal{F}_1 \cap \mathcal{F}_2 \subseteq \mathcal{P}$, then either $\mathcal{F}_1 \subseteq \mathcal{P}$ or $\mathcal{F}_2 \subseteq \mathcal{P}$.*

Theorem 2.15 : *Let R be a ring, $Y \subseteq Spec(R)$, $\mathcal{F}$ a proper $\mathcal{H}_Y$-filter on Y containing a 2-absorbing $\mathcal{H}_Y$-filter $\mathcal{C}$ on Y. Suppose that for every $\mathcal{P} \in Min(\mathcal{C})$, the set of all prime $\mathcal{H}_Y$-filters on Y containing $\mathcal{P}$, $\mathcal{V}(\mathcal{P})$ forms a chain. Then $\mathcal{F}$ is a 2-absorbing $\mathcal{H}_Y$-filter on Y.*

Proof : Since $\mathcal{C}$ is a 2-absorbing $\mathcal{H}_Y$-filter, then by Theorem 2.11, $|Min(\mathcal{C})| = 1$ or 2.

Case 1 : If $Min(\mathcal{C}) = \{\mathcal{P}\}$. According to ([1], Corollary 4.6), we have $\mathcal{P} = \mathcal{C} \subseteq \mathcal{F} = \bigcap_{\mathcal{R} \in Min(\mathcal{F})} \mathcal{R}$. Now, it is easy to see that $\mathcal{F}$ is a prime $\mathcal{H}_Y$-filter on Y. Therefore, $\mathcal{F}$ is a 2-absorbing $\mathcal{H}_Y$-filter on Y.

Case 2 : If $Min(\mathcal{C}) = \{\mathcal{P}_1, \mathcal{P}_2\}$ In this case $\mathcal{C} = \mathcal{P}_1 \cap \mathcal{P}_2$. Thus, $\mathcal{P}_1 \cap \mathcal{P}_2 \subseteq \mathcal{F} = \bigcap_{\mathcal{P} \in Min(\mathcal{F})} \mathcal{P}$. According to the previous lemma, we can write $\mathcal{F} = \mathcal{R}_1 \cap \mathcal{R}_2$ where

$$\mathcal{R}_i = \cap\{\mathcal{P} : \mathcal{P} \in Min(\mathcal{F}), \text{ and } \mathcal{P}_i \subseteq \mathcal{P}\} \quad i = 1,2.$$

Since $\mathcal{V}(\mathcal{P}_i)$, $i = 1,2$ form a chain, then as above, $\mathcal{R}_i$, $i = 1,2$ is a prime $\mathcal{H}_Y$-filter on Y. Consequently, $\mathcal{F}$ is a 2-absorbing $\mathcal{H}_Y$-filter on Y, by lemma 2.13. □

Remark 2.16 : In general, 2-absorbing $\mathcal{H}_Y$-filters (or ideals) are not necessarily prime. Therefore, one can raise the following question: When every 2-absorbing $\mathcal{H}_Y$-filters (or ideals) is prime. In the case of ideals, the class of these rings is called 2-AB and it was carefully studied in [8]. As a matter of fact, in the case of $\mathcal{H}_Y$-filters, the fact that every 2-absorbing $\mathcal{H}_Y$-filter is prime or an intersection of two prime $\mathcal{H}_Y$-filters (Theorem 2.11) makes it easy. In fact, by Lemma 2.13, we can easily show that every 2-absorbing $\mathcal{H}_Y$-filter is prime if and only if the set of prime $\mathcal{H}_Y$-filters forms a chain. Consequently, if R is 2-AB, then for each $Y \subseteq Spec(R)$, every 2-absorbing $\mathcal{H}_Y$-filter is prime.

3. Case of the product of rings

In this section, we focus on the case of the product of rings, $R_1 \times R_2$. As, we characterize its "$\mathcal{H}_Y$-filters", prime, minimal prime and 2-absorbing $\mathcal{H}_Y$-filters on Y. It is well-known that the prime ideals of the product of rings $R_1 \times R_2$ are of the form $R_1 \times Q$ or $P \times R_2$ where P and Q are prime ideals of R_1 and R_2 respectively. Let $Y \subseteq Spec(R_1)$, $Y' \subseteq Spec(R_2)$ and $\mathcal{Y} = (Y \times R_2) \cup (R_1 \times Y') \subseteq Spec(R_1 \times R_2)$, where $Y \times R_2 = \{P \times R_2 : P \in Y\}$

and $R_1 \times Y' = \{R_1 \times Q : Q \in Y'\}$. Also, it is easy to see that $F \in F(R_1 \times R_2)$ if and only if $F = F_1 \times F_2$ where $F_1 \in F(R_1)$ and $F_2 \in F(R_2)$. Therefore,

$$\mathcal{H}_{\mathcal{Y}} = \{h_{\mathcal{Y}}(F_1 \times F_2) : F_1 \in F(R_1) \text{ and } F_2 \in F(R_2)\}.$$

Now, let $F_1 \in F(R_1)$ and $F_2 \in F(R_2)$, then we have

$$\begin{aligned} h_{\mathcal{Y}}(F_1 \times F_2) &= \{\mathcal{P} \in \mathcal{Y} : F_1 \times F_2 \subseteq \mathcal{P}\} \\ &= \{P \times R_2 : P \in h_Y(F_1)\} \cup \{R_1 \times P : P \in h_{Y'}(F_2)\}. \end{aligned}$$

In order to simplify the notations, we denote $h_Y(F_1) \times R_2 := \{P \times R_2 : P \in h_Y(F_1)\}$, $R_1 \times h_{Y'}(F_2) := \{R_1 \times P : P \in h_{Y'}(F_2)\}$ and $h_Y(F_1) \otimes h_{Y'}(F_2) := (h_Y(F_1) \times R_2) \cup (R_1 \times h_{Y'}(F_2))$. Thus,

$$\mathcal{H}_{\mathcal{Y}} = \{h_Y(F_1) \otimes h_{Y'}(F_2) : F_1 \in F(R_1) \text{ and } F_2 \in F(R_2)\}.$$

The proof of the following lemma is straightforward and is omitted.

Lemma 3.1 : *Let R_1 and R_2 be two rings, $Y \subseteq Spec(R_1)$ and $Y' \subseteq Spec(R_2)$. Then the following statements hold.*

(1) For every $F_1, F_1' \in F(R_1)$ and $F_2, F_2' \in F(R_2)$,

$$h_Y(F_1) \otimes h_{Y'}(F_2) \subseteq h_Y(F_1') \otimes h_{Y'}(F_2') \Leftrightarrow \begin{cases} h_Y(F_1) & \subseteq \quad h_Y(F_1') \text{ and} \\ h_{Y'}(F_2) & \subseteq \quad h_{Y'}(F_2') \end{cases}$$

(2) For every $F_1, \cdots, F_n \in F(R_1)$ and $G_1, \cdots, G_n \in F(R_2)$ we have:

(a) $\bigcap_{i=1}^n (h_Y(F_i) \otimes h_{Y'}(G_i)) = (\bigcap_{i=1}^n h_Y(F_i)) \otimes (\bigcap_{i=1}^n h_{Y'}(G_i))$.

(b) $\bigcup_{i=1}^n (h_Y(F_i) \otimes h_{Y'}(G_i)) = (\bigcup_{i=1}^n h_Y(F_i)) \otimes (\bigcup_{i=1}^n h_{Y'}(G_i))$.

Proposition 3.2 : Let $\mathcal{F}$ be a $\mathcal{H}_Y$-filter on Y and $\mathcal{F}'$ be a $\mathcal{H}_{Y'}$-filter on Y'. The set of all $h_Y(F) \otimes h_{Y'}(F')$ where $h_Y(F) \in \mathcal{F}$ and $h_{Y'}(F') \in \mathcal{F}'$ is an $\mathcal{H}_{\mathcal{Y}}$-filter on $\mathcal{Y}$, denoted by $\mathcal{F} \otimes \mathcal{F}'$ and it is called the product $\mathcal{H}_{\mathcal{Y}}$-filter of $\mathcal{F}$ and $\mathcal{F}'$.

Proof : It is straightforward. ☐

In the following, we characterize all the $\mathcal{H}_{\mathcal{Y}}$-filter on $\mathcal{Y}$. In fact, we prove that all the $\mathcal{H}_{\mathcal{Y}}$-filters on $\mathcal{Y}$ are of the form of the $\mathcal{H}_{\mathcal{Y}}$-filter what was defined in the proposition above.

Theorem 3.3 : *The $\mathcal{H}_{\mathcal{Y}}$-filters on $\mathcal{Y}$ are of the form $\mathcal{F} \otimes \mathcal{F}'$ where $\mathcal{F}$ is an $\mathcal{H}_Y$-filter on Y and $\mathcal{F}'$ is an $\mathcal{H}_{Y'}$-filter on Y'.*

Proof : Let $\mathcal{C}$ be a $\mathcal{H}_{\mathcal{Y}}$-filter on $\mathcal{Y}$. Putting

$$\begin{aligned}\mathcal{F} &= \{h_Y(F) \in \mathcal{H}_Y : \exists h_{Y'}(F') \in \mathcal{H}_{Y'},\ h_Y(F) \otimes h_{Y'}(F') \in \mathcal{C}\} \quad \text{and} \\ \mathcal{F}' &= \{h_{Y'}(F') \in \mathcal{H}_{Y'} : \exists h_Y(F) \in \mathcal{H}_Y,\ h_Y(F) \otimes h_{Y'}(F') \in \mathcal{C}\}.\end{aligned}$$

Next, we prove that $\mathcal{C} = \mathcal{F} \otimes \mathcal{F}'$.

Claim. $\mathcal{F}$ is a $\mathcal{H}_Y$-filter on Y. Indeed, let $h_Y(F_1), h_Y(F_2) \in \mathcal{F}$, then there exists $h_{Y'}(F_1'), h_{Y'}(F_2') \in \mathcal{H}_{Y'}$ such that $h_Y(F_1) \otimes h_{Y'}(F_1') \in \mathcal{C}$ and $h_Y(F_2) \otimes h_{Y'}(F_2') \in \mathcal{C}$. According to the previous lemma, $(h_Y(F_1) \otimes h_{Y'}(F_1')) \cap (h_Y(F_2) \otimes h_{Y'}(F_2')) = (h_Y(F_1) \cap h_Y(F_2)) \otimes (h_{Y'}(F_1') \cap h_{Y'}(F_2')) \in \mathcal{C}$. Therefore, $h_Y(F_1) \cap h_Y(F_2) \in \mathcal{F}$. Furthermore, let $h_Y(F_3) \in \mathcal{H}_Y$ such that $h_Y(F_1) \subseteq h_Y(F_3)$, then by the previous lemma, $h_Y(F_1) \otimes h_{Y'}(F_1') \subseteq h_Y(F_3) \otimes h_{Y'}(F_1')$. Which implies that $h_Y(F_3) \otimes h_{Y'}(F_1') \in \mathcal{C}$ and so, $h_Y(F_3) \in \mathcal{F}$. Consequently, $\mathcal{F}$ is a $\mathcal{H}_Y$-filter on Y, which proves the claim. In the same way, we prove that $\mathcal{F}'$ is a $\mathcal{H}_{Y'}$-filter on Y'. Now, for the equality. It is obvious that $\mathcal{C} \subseteq \mathcal{F} \otimes \mathcal{F}'$. Conversely, let $h_Y(F_1) \otimes h_{Y'}(F_2) \in \mathcal{F} \otimes \mathcal{F}'$, then there exists $h_{Y'}(F_1') \in \mathcal{H}_{Y'}$ and $h_Y(F_2') \in \mathcal{H}_Y$ such that $h_Y(F_1) \otimes h_{Y'}(F_1') \in \mathcal{C}$ and $h_Y(F_2') \otimes h_{Y'}(F_2) \in \mathcal{C}$. Note that $h_Y(F_1) \otimes h_{Y'}(F_1') \subseteq (h_Y(F_1) \otimes h_{Y'}(F_1')) \cup (h_Y(F_1) \otimes Y')$, and then $(h_Y(F_1) \otimes h_{Y'}(F_1')) \cup (h_Y(F_1) \otimes Y') \in \mathcal{C}$. Similarly, we prove that $(h_Y(F_2') \otimes h_{Y'}(F_2)) \cup (Y \otimes h_{Y'}(F_2)) \in \mathcal{C}$.

Now, obviously we have $h_Y(F_1) \otimes h_{Y'}(F_2) = [(h_Y(F_1) \otimes h_{Y'}(F_1')) \cup (h_Y(F_1)\ Y')] \cap [(h_Y(F_2') \otimes h_{Y'}(F_2)) \cup (Y \otimes h_{Y'}(F_2))] \in \mathcal{C}$, and so the equality holds. □

In the following lemma, we give a relation between the product in the classical sense and the product defined in the previous proposition.

Lemma 3.4 : *Let $\mathcal{F}$ be an $\mathcal{H}_Y$-filter on Y, $\mathcal{F}'$ an $\mathcal{H}_{Y'}$-filter on Y', I an ideal of R_1 and J be an ideal of R_2. Then the following statements hold.*

(1) $\mathcal{H}_{\mathcal{Y}}^{-1}(\mathcal{F} \otimes \mathcal{F}') = \mathcal{H}_Y^{-1}(\mathcal{F}) \times \mathcal{H}_{Y'}^{-1}(\mathcal{F}')$.

(2) $\mathcal{H}_{\mathcal{Y}}(I \times J) = \mathcal{H}_Y(I) \otimes \mathcal{H}_{Y'}(J)$.

Proof : It is straightforward. □

Proposition 3.5 : The prime $\mathcal{H}_{\mathcal{Y}}$-filters on $\mathcal{Y}$ are of the form $\mathcal{F} \otimes \mathcal{H}_{Y'}$ or $\mathcal{H}_Y \otimes \mathcal{F}'$, where $\mathcal{F}$ (resp., $\mathcal{F}'$) is a prime $\mathcal{H}_Y$-filter on $\mathcal{H}_Y$ (resp., prime $\mathcal{H}_{Y'}$-filter on Y').

Proof : It is clear that the $\mathcal{H}_{\mathcal{Y}}$-filters of the previous form are prime. Now, let $\mathcal{C}$ be a prime $\mathcal{H}_{\mathcal{Y}}$-filter on $\mathcal{Y}$. According to Theorem 3.3, there exist $\mathcal{F}$ an $\mathcal{H}_Y$-filter on Y and $\mathcal{F}'$ an $\mathcal{H}_{Y'}$-filter on Y' such that $\mathcal{C} = \mathcal{F} \otimes \mathcal{F}'$. By ([1], Theorem 4.3) and Lemma 3.4, $\mathcal{H}_{\mathcal{Y}}^{-1}(\mathcal{C}) = \mathcal{H}_{Y}^{-1}(\mathcal{F}) \times \mathcal{H}_{Y'}^{-1}(\mathcal{F}')$ is a prime ideal of $R_1 \times R_2$. Hence, $\mathcal{H}_{\mathcal{Y}}^{-1}(\mathcal{C}) = P \times R_2$ or $R_1 \times Q$ where P and Q are two prime ideals of R_1 and R_2 respectively. If $\mathcal{H}_{\mathcal{Y}}^{-1}(\mathcal{C}) = \mathcal{H}_{Y}^{-1}(\mathcal{F}) \times \mathcal{H}_{Y'}^{-1}(\mathcal{F}') = P \times R_2$, then $\mathcal{H}_{Y'}^{-1}(\mathcal{F}') = R_2$ and $\mathcal{H}_{Y}^{-1}(\mathcal{F})$ is a prime ideal of R_1. Using ([1], Proposition 4.1 and Theorem 4.3), we have $\mathcal{F}' = \mathcal{H}_{Y'}$ and that $\mathcal{F}$ is a prime $\mathcal{H}_Y$-filter on Y, therefore, $\mathcal{C} = \mathcal{F} \otimes \mathcal{H}_{Y'}$, where $\mathcal{F}$ is a prime $\mathcal{H}_Y$-filter on Y. Similarly, we show that in the second case, $\mathcal{C} = \mathcal{H}_Y \otimes \mathcal{F}'$ where $\mathcal{F}'$ is a prime $\mathcal{H}_{Y'}$-filter on Y'. □

Now, using the previous result, we can determine all the prime $\mathcal{H}_{\mathcal{Y}}$-filters on $\mathcal{Y}$ that are minimal over $\mathcal{C}$. In fact:

Corollary 3.6 : *Let $\mathcal{C} = \mathcal{F} \otimes \mathcal{F}'$ be an $\mathcal{H}_{\mathcal{Y}}$-filter on $\mathcal{Y}$. A prime $\mathcal{H}_{\mathcal{Y}}$-filter that is minimal over $\mathcal{C}$ is of the form $\mathcal{P} \otimes \mathcal{H}_{Y'}$ or $\mathcal{H}_Y \otimes \mathcal{P}'$ where $\mathcal{P} \in Min(\mathcal{F})$ and $\mathcal{P}' \in Min(\mathcal{F}')$.*

In this proposition, we give the form of the $\mathcal{H}_{\mathcal{Y}}$-ultrafilters on $\mathcal{Y}$.

Proposition 3.7 : *The $\mathcal{H}_{\mathcal{Y}}$-ultrafilters on $\mathcal{Y}$ are of the form $\mathcal{U} \otimes \mathcal{H}_{Y'}$ or $\mathcal{H}_Y \otimes \mathcal{U}'$ where $\mathcal{U}$ is an $\mathcal{H}_Y$-ultrafilter on Y and $\mathcal{U}'$ is an $\mathcal{H}_{Y'}$-ultrafilter on Y'.*

Proof : It is clear that the previous $\mathcal{H}_{\mathcal{Y}}$-filters are $\mathcal{H}_{\mathcal{Y}}$-ultrafilters on $\mathcal{Y}$. Now, let $\mathcal{U} \otimes \mathcal{U}'$ an $\mathcal{H}_{\mathcal{Y}}$-ultrafilter on $\mathcal{Y}$. Then either $\mathcal{U}$ or $\mathcal{U}'$ is proper. Saying $\mathcal{U}$, so there exists $\mathcal{W}$ an $\mathcal{H}_Y$-ultrafilter on Y such that $\mathcal{U} \subseteq \mathcal{W}$. Using Lemma 3.1, we have $\mathcal{U} \otimes \mathcal{U}' \subseteq \mathcal{W} \otimes \mathcal{H}_{Y'}$. Thus, $\mathcal{U} \otimes \mathcal{U}' = \mathcal{W} \otimes \mathcal{H}_{Y'}$ and the result holds. □

Corollary 3.8 : *The maximal proper strong $\mathcal{H}_{\mathcal{Y}}$-ideals of $R_1 \times R_2$ are of the form $I_1 \times R_2$ or $R_1 \times I_2$ where I_1 (resp., I_2) is a maximal proper strong $\mathcal{H}_Y$-ideal (resp., $\mathcal{H}_{Y'}$-ideal) of R_1 (resp., of R_2).*

Now, let is specify the 2-absorbing $\mathcal{H}_{\mathcal{Y}}$-filters on $\mathcal{Y}$.

Proposition 3.9 : The 2-absorbing $\mathcal{H}_{\mathcal{Y}}$-filters on $\mathcal{Y}$ are of the form $\mathcal{F} \otimes \mathcal{H}_{Y'}$ or $\mathcal{H}_Y \otimes \mathcal{F}'$, where $\mathcal{F}$ (resp., $\mathcal{F}'$) is a 2-absorbing $\mathcal{H}_Y$-filter on Y (resp., $\mathcal{H}_{Y'}$-filter on Y'), or $\mathcal{F} \otimes \mathcal{F}'$ where $\mathcal{F}$ is a prime $\mathcal{H}_Y$-filter on Y and $\mathcal{F}'$ is a prime $\mathcal{H}_{Y'}$-filter on Y'.

Proof : It is clear that the previous $\mathcal{H}_{\mathcal{Y}}$-filters are 2-absorbing $\mathcal{H}_{\mathcal{Y}}$-filters on $\mathcal{Y}$. Now, let $\mathcal{C}$ be a 2-absorbing $\mathcal{H}_{\mathcal{Y}}$-filter on $\mathcal{Y}$. Using Theorem 2.11, we have $\mathcal{C}$ is either a prime $\mathcal{H}_{\mathcal{Y}}$-filter or an intersection of two prime $\mathcal{H}_{\mathcal{Y}}$-filters.

Case 1 : If $\mathcal{C}$ is a prime $\mathcal{H}_{\mathcal{Y}}$-filter, then, by Proposition 3.5, we have $\mathcal{C} = \mathcal{P} \otimes \mathcal{H}_{Y'}$ or $\mathcal{H}_{Y} \otimes \mathcal{P}'$ where $\mathcal{P}$ (resp., $\mathcal{P}'$) is a prime $\mathcal{H}_{Y}$-filter (resp., $\mathcal{H}_{Y'}$-filter) and so a 2-absorbing on Y (resp., Y'). Therefore $\mathcal{C}$ is in the suitable form.

Case 2 : If $\mathcal{C} = \mathcal{P}_1 \cap \mathcal{P}_2$ where $\mathcal{P}_1$ and $\mathcal{P}_2$ are two prime $\mathcal{H}_{\mathcal{Y}}$-filter on $\mathcal{Y}$. According to Proposition 3.5, $\mathcal{P}_1 = \mathcal{Q}_1 \otimes \mathcal{H}_{Y'}$ or $\mathcal{H}_{Y} \otimes \mathcal{Q}_1'$ and $\mathcal{P}_2 = \mathcal{Q}_2 \otimes \mathcal{H}_{Y'}$ or $\mathcal{H}_{Y} \otimes \mathcal{Q}_2'$ where $\mathcal{Q}_1$ and $\mathcal{Q}_2$ (resp., $\mathcal{Q}_1'$ and $\mathcal{Q}_2'$) are two prime $\mathcal{H}_{Y}$-filters on Y (resp., $\mathcal{H}_{Y'}$-filters on Y'). Now, by lemma 3.1 and Lemma 2.13, it is easy to see that in all cases, $\mathcal{C}$ is in the suitable form. □

In [9], the authors have shown that the z-ideals of the product ring $R_1 \times R_2$ are of the form $I \times J$ where I is a z-ideal of R_1 and J is a z-ideal of R_2. In the following proposition, we will extend this result to the "$\mathcal{H}_{Y}$-ideal". In fact, the same result holds for the strong $\mathcal{H}_{Y}$-ideals.

Proposition 3.10 : The $\mathcal{H}_{\mathcal{Y}}$-ideals of $R_1 \times R_2$ are of the form $I \times J$ where I is an $\mathcal{H}_{Y}$-ideal of R_1 and J is an $\mathcal{H}_{Y'}$-ideal of R_2.

Proof : The proof is essentially the same as that in [9]. □

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