

On the partition dimension of directed circulant graphs

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Abstract

The directed circulant graph $C_n(1, 2, \dots, t)$ consists of the vertices $v_0, v_1, \dots, v_{n-1}$ and edges directed from v_i to v_{i+j} for every $i = 0, 1, \dots, n-1$, and $j = 1, 2, \dots, t$, where $1 \leq t \leq n-1$, the indices are taken modulo n . We present lower and upper bounds on the partition dimension of very general graphs $C_n(1, 2, \dots, t)$. Our bounds yield exact values of the partition dimension of the graphs $C_n(1, 2, \dots, t)$ for $t = 2, 3, 4$ and $n \geq 10$.

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1. Introduction

The partition dimension is defined more generally than the well-known metric dimension of graphs. It has extensive applications for example in robot navigation, pattern recognition, image processing and chemistry. The structure of a chemical compound can be represented by a graph with vertex and edge labels specifying the atom and bond types, respectively.

Let G be a directed graph having vertex set $V(G)$. The distance $d(u, v)$ from a vertex $u \in V(G)$ to a vertex $v \in V(G)$ is the length of a shortest directed path from u to v . For a vertex v and a subset S of $V(G)$ the distance from v to S is defined as

$$d(v, S) = \min\{d(v, u) \mid u \in S\}.$$

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For an ordered partition $\Pi = \{S_1, S_2, \dots, S_k\}$ of $V(G)$, the partition representation of v in terms of Π is

$$r(v | \Pi) = (d(v, S_1), d(v, S_2), \dots, d(v, S_k)).$$

Π is a resolving partition iff for each two different vertices $u, v \in V(G)$, we obtain $r(u | \Pi) \neq r(v | \Pi)$. The partition dimension $pd(G)$ of G is the cardinality of a smallest resolving partition of $V(G)$.

Let n, t and $a_1, a_2, \dots, a_t$ be positive integers, where $1 \leq a_1 < a_2 < \dots < a_t \leq n - 1$. The directed circulant graph $C_n(a_1, a_2, \dots, a_t)$ consists of the vertices $v_0, v_1, \dots, v_{n-1}$ and directed edges $v_i v_{i+a_j}$ for every $i = 0, 1, \dots, n-1$, and $j = 1, 2, \dots, t$; we take the indices modulo n . Circulant graphs are important, because they form a special class of Cayley graphs. They can be defined with the help of cyclic groups. Thus it is easier to check various properties of circulant graphs.

Graph theory can be used to study important mathematical problems. For example, domination in graphs was studied in [12] and [13], graph indices in [9] and [19], algebraic problems in [1] and [15], and some other interesting problems in [4], [6] and [21].

Chartrand, Salehi and Zhang [2] introduced the partition dimension of a graph in 2000. Since then, it has been extensively studied. For example, the partition dimension for nanotubes was studied in [18] and trees were considered in [3] and [16]. Values of the partition dimension for undirected circulant graphs $C_n(\pm 1, \pm 2)$ were given in [8] and [17]. In some cases $pd(C_n(\pm 1, \pm 2)) = 4$ and in the other cases $pd(C_n(\pm 1, \pm 2)) = 3$. Values of $pd(C_n(\pm 1, \pm 3))$ and $pd(C_n(\pm 1, \pm 4))$ were presented in [10]. Some bounds and values of $pd(C_n(\pm 1, \pm 2, \dots, \pm t))$ for $t \geq 3$ were obtained in [7] and [14]. Interesting methods were used also in [6] and [21].

We study a very general problem, the partition dimension for directed graphs $C_n(1, 2, \dots, t)$ for every $t \geq 1$. These graphs are strongly connected, because they have a directed cycle which contains all the n vertices. It is easy to see that the partition dimension of $C_n(1)$ is 2 for every $n \geq 2$. Clearly, the vertices of $V(C_n(1))$ are resolved by $\Pi = \{S_1, S_2\}$ for $S_1 = \{v_0, v_1, \dots, v_{n-2}\}$ and $S_2 = \{v_{n-1}\}$.

2. Results

The circulant graph $C_n(1, 2, \dots, t)$ is isomorphic to $C_n(-1, -2, \dots, -t)$ for any $2 \leq t \leq n - 1$. We present Theorem 1 for the graph $C_n(-1, -2, \dots, -t)$,

because it is easier to express distances from vertices in a circulant graph to vertices in chosen sets S_i when considering $C_n(-1, -2, \dots, -t)$.

The distance from the vertex v_j to the vertex v_i in $C_n(-1, -2, \dots, -t)$ for $i, j \in \{0, 1, \dots, n-1\}$ is

$$d(v_j, v_i) = \begin{cases} \left\lceil \frac{j-i}{t} \right\rceil, & \text{if } j \geq i, \\ \left\lceil \frac{n+j-i}{t} \right\rceil, & \text{if } j < i. \end{cases} \quad (1)$$

Theorem 1 : Let $t, a, \frac{t}{a}, n$ be integers, such that $t \geq 2, 1 \leq a \leq t$ and $n \geq 2t$. Then $pd(C_n(-1, -2, \dots, -t)) \leq \frac{t}{a} + a$.

Proof : Let $t, a, \frac{t}{a}, n$ be integers, such that $t \geq 2, 1 \leq a \leq t$ and $n \geq 2t$. Let

$$S_0 = \{v_{t+j} \mid j = 0, a, 2a, \dots, \text{where } t+j < n\}.$$

For $i = 1, 2, \dots, a-1$, let

$$S_i = \{v_{t+j} \mid j = 0, a, 2a, \dots, \text{where } t+j < n\}.$$

For $i = 0, 1, \dots, \frac{t}{a}-1$, we define $S_{a+i} = \{v_{ia}\}$.

It is easy to check that $S_0 \cup S_1 \cup \dots \cup S_{\frac{t}{a}+a-1} = V(C_n(-1, -2, \dots, -t))$ and

$$S_i \cap S_{i'} = \emptyset \text{ for } i, i' \in \{0, 1, \dots, \frac{t}{a}+a-1\}, \text{ where } i \neq i'.$$

We show that all vertices of $V(C_n(-1, -2, \dots, -t))$ are resolved by $\Pi = \{S_0, S_1, \dots, S_{\frac{t}{a}+a-1}\}$. For $i = 0, 1, \dots, \frac{t}{a}-1$, we have $|S_{a+i}| = 1$, thus the representations of the vertices in S_{a+i} are unique in terms of Π .

For $v_0 \in S_a, i = 1, 2, \dots, a$ and $c = 1, 2, \dots, d$, with d being the diameter of $C_n(-1, -2, \dots, -t)$, by (1),

$$\begin{aligned} d(v_{i+(c-1)t}, v_0) &= c, \\ d(v_{i+a+(c-1)t}, v_0) &= c, \\ d(v_{i+2a+(c-1)t}, v_0) &= c, \\ &\vdots \\ d(v_{i+\left(\frac{t}{a}-1\right)a+(c-1)t}, v_0) &= c, \end{aligned}$$

therefore for $i = 1, 2, \dots, a$, the only vertices in S_i with the same representation in terms of S_a are $v_{i+(c-1)t}, v_{i+a+(c-1)t}, \dots, v_{i+\left(\frac{t}{a}-1\right)a+(c-1)t}$, where $c = 1, 2, \dots, d$, (if $c = d$, we consider only vertices with indices smaller than n).

For $i = 1, 2, \dots, a$ and $c = 1, 2, \dots, d$, let

$$V_{i,c} = \{v_{i+(c-1)t}, v_{i+a+(c-1)t}, \dots, v_{i+t-a+(c-1)t}\}.$$

We still need to resolve the vertices in $V_{i,c}$. If $i = a$, it is not necessary to study the case $c = 1$, because $v_a, v_{2a}, \dots, v_t$ belong to different sets (so they are resolved). Let us investigate two cases.

Case 1 : $c = 1$.

The representations of the vertices in $V_{i,1}$ for $i = 1, 2, \dots, a-1$ with respect to $S_a = \{v_0\}, S_{a+1} = \{v_a\}, \dots, S_{\frac{t}{a}+a-1} = \{v_{t-a}\}$ are given in Table 1.

For $v_{i+ka} \in V_{i,1}$ and $v_{la} \in S_{a+l}$, from (1) and (2),

$$d(v_{i+ka}, v_{la}) = \begin{cases} \left\lfloor \frac{i+ka-la}{t} \right\rfloor = 1 \text{ since } i+ka < t, & \text{if } k \geq l, \\ \left\lfloor \frac{n+i+ka-la}{t} \right\rfloor > 1 \text{ since } n \geq 2t \text{ and } la \leq t-a, & \text{if } k < l. \end{cases}$$

Table 1 shows that the vertices in $V_{i,1}$ are resolved.

Case 2 : $c = 2, 3, \dots, d$.

The representations of the vertices in $V_{i,c}$ for $i = 1, 2, \dots, a$ and $c = 2, 3, \dots, d$ with respect to $S_a = \{v_0\}, S_{a+1} = \{v_a\}, \dots, S_{\frac{t}{a}+a-1} = \{v_{t-a}\}$ are given in Table 2. For $v_{i+ka+(c-1)t} \in V_{i,c}$ and $v_{la} \in S_{a+l}$, from (1),

Table 1
The vertices in $V_{i,1}$ for $i = 1, 2, \dots, a-1$ and their representations
with respect to $S_a, S_{a+1}, \dots, S_{\frac{t}{a}+a-1}$.

Distances	v_0	v_a	v_{2a}	$\dots$	v_{t-a}
v_i	1	> 1	> 1	$\dots$	> 1
v_{i+a}	1	1	> 1	$\dots$	> 1
v_{i+2a}	1	1	1	$\dots$	> 1
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
v_{i+t-2a}	1	1	1	$\dots$	> 1
v_{i+t-a}	1	1	1	$\dots$	1

Table 2

The vertices in $V_{i,c}$ for $i = 1, 2, \dots, a$ and $c = 2, 3, \dots, d$, and their representations with respect to $S_a, S_{a+1}, \dots, S_{t_{a+1}-1}$.

Distances	v_0	v_a	v_{2a}	$\dots$	v_{t-a}
$v_{i+(c-1)t}$	c	$c-1$	$c-1$	$\dots$	$c-1$
$v_{i+a+(c-1)t}$	c	c	$c-1$	$\dots$	$c-1$
$v_{i+2a+(c-1)t}$	c	c	c	$\dots$	$c-1$
$\vdots$	$\vdots$	$\vdots$	$\vdots$		$\vdots$
$v_{i+t-2a+(c-1)t}$	c	c	c	$\dots$	$c-1$
$v_{i+t-a+(c-1)t}$	c	c	c	$\dots$	c

$$d(v_{i+ka+(c-1)t}, v_{la}) = \left\lceil \frac{(c-1)t + i + ka - la}{t} \right\rceil = \begin{cases} c, & \text{if } k \geq l, \\ c-1, & \text{if } k < l. \end{cases}$$

Table 2 shows that the vertices in $V_{i,c}$ are resolved. □

Since the graphs $C_n(-1, -2, \dots, -t)$ and $C_n(1, 2, \dots, t)$ are isomorphic, we get the following result.

Corollary 1 : Let $t, a, \frac{t}{a}, n$ be integers such that $t \geq 2, 1 \leq a \leq t$ and $n \geq 2t$. Then $pd(C_n(1, 2, \dots, t)) \leq \frac{t}{a} + a$.

Note that the function $f(a) = \frac{t}{a} + a$ is minimized for $a^2 = t$ (since $f'(a) = \frac{-t}{a^2} + 1$). Thus we obtain best upper bounds on $pd(C_n(1, 2, \dots, t))$ when a is a closest integer to $\sqrt{t}$ such that $\frac{t}{a}$ is an integer.

Example 1 : If $t = 24$, then from Corollary 1 the partition dimension $pd(C_n(1, 2, \dots, 24)) \leq \frac{24}{a} + a$, where $a, \frac{t}{a}$ are integers such that $1 \leq a \leq 24$. We choose $a = 4$ to obtain $pd(C_n(1, 2, \dots, 24)) \leq 10$. For $t = 36$ we choose $a =$

Table 3

Upper bounds for the partition dimension of $C_n(1, 2, \dots, t)$ for $2 \leq t \leq 10$ and $n \geq 2t$.

t	2	3	4	5	6	7	8	9	10
$pd(C_n(1, 2, \dots, t)) \leq$	3	4	4	6	5	8	6	6	7

6 to obtain $pd(C_n(1, 2, \dots, 36)) \leq 12$. If $t = 180$, then $a = 12$ yields the bound $pd(C_n(1, 2, \dots, 36)) \leq 27$.

Corollary 2 : *Let $\sqrt{t} \geq 2$ be an integer and let $n \geq 2t$. Then we have $pd(C_n(1, 2, \dots, t)) \leq 2\sqrt{t}$.*

Upper bounds for $pd(C_n(1, 2, \dots, t))$, where $2 \leq t \leq 10$, are given in Table 3. Let us study lower bounds for the partition dimension of directed circulant graphs. Note that the distance from the vertex v_j to the vertex v_i in $C_n(1, 2, \dots, t)$, where $i, j \in \{0, 1, \dots, n-1\}$, is

$$d(v_j, v_i) = \begin{cases} \left\lceil \frac{i-j}{t} \right\rceil, & \text{if } i \geq j, \\ \left\lceil \frac{n+i-j}{t} \right\rceil, & \text{if } i < j. \end{cases} \quad (3)$$

(4)

Lemma 1 : *For $t \geq 2$ and any n , we have $pd(C_n(1, 2, \dots, t)) \geq 3$.*

Proof : Suppose to the contrary that $pd(C_n(1, 2, \dots, t)) = 2$. Then there exists some $\Pi = \{S_1, S_2\}$ that resolves $V(C_n(1, 2, \dots, t))$. Since $t \geq 2$, we have $n \geq 3$, thus one of the sets S_1, S_2 has two or more vertices. Suppose that $|S_1| \geq 2$. Clearly, S_1 contains a vertex with the partition representation $(0, 1)$ and a vertex having the representation $(0, 2)$. We can suppose that v_0 is the vertex with $r(v_0 | \Pi) = (0, 2)$. Let j be the smallest natural number, where $v_j \in S_2$. Since $d(v_0 | S_2) = 2$, we have $j \geq t+1$. We get $r(v_{j-1} | \Pi) = r(v_{j-2} | \Pi) = (0, 1)$, a contradiction. $\square$

Lemma 1 and Table 3 yield Corollary 3.

Corollary 3 : *Let $n \geq 3$. Then $pd(C_n(1, 2)) = 3$.*

We improve the bound given in Lemma 1 for every $t \geq 3$. Clearly, for any $t+1$ consecutive vertices $v_j, v_{j+1}, \dots, v_{j+t} \notin S_i$, we have

$$x = d(v_j, S_i) \geq d(v_{j+1}, S_i) \geq \dots \geq d(v_{j+t}, S_i) = x-1 \quad (5)$$

for some natural number x . We use these inequalities in the proof of Theorem 2.

Theorem 2 : *Let $t \geq 3$ and $n \geq 10$. Then $pd(C_n(1, 2, \dots, t)) \geq 4$.*

Proof : By Lemma 1, we have $pd(C_n(1, 2, \dots, t)) \geq 3$. Let us prove this result by contradiction. Suppose that $pd(C_n(1, 2, \dots, t)) = 3$. Thus there exists a resolving partition $\Pi = \{S_1, S_2, S_3\}$ of $V(C_n(1, 2, \dots, t))$. Since $n \geq 10$, one of the sets S_1, S_2, S_3 has four or more vertices. Suppose that $|S_1| \geq 4$. We investigate the following cases:

Case 1 : S_1 contains a vertex having the partition representation $(0, 2, 2)$.

We can assume that for v_0 we have $r(v_0 | \Pi) = (0, 2, 2)$. Let j be the smallest natural number, where $v_j \notin S_1$. Since $d(v_0 | S_2) = d(v_0 | S_3) = 2$, we have $j \geq t + 1$. Without loss of generality, suppose that $v_j \in S_3$. Note that $v_{j-1}, v_{j-2}, v_{j-3} \in S_1$. Their partition representations are $r(v_{j-1} | \Pi) = (0, a_1, 1)$, $r(v_{j-2} | \Pi) = (0, a_2, 1)$, $r(v_{j-3} | \Pi) = (0, a_3, 1)$. From (5) we know that $a_1 \leq a_2 \leq a_3$ and the difference between a_3 and a_1 is at most one, so the representations of two vertices from the set $\{v_{j-3}, v_{j-2}, v_{j-1}\}$ are equal in terms of Π , which is a contradiction.

Case 2 : S_1 does not contain a vertex having the partition representation $(0, 2, 2)$.

Clearly, S_1 contains a vertex with the partition representation $(0, 1, 1)$. S_1 contains also vertices having the representations $(0, 1, 2)$ or $(0, 2, 1)$, or both, $(0, 1, 2)$ and $(0, 2, 1)$. Since $|S_1| \geq 4$, S_1 must contain also the vertex having the representation $(0, 1, 3)$ or $(0, 3, 1)$. We can suppose that $v_0 \in S_1$ has the representation $r(v_0 | \Pi) = (0, 1, 3)$. Let j be the smallest natural number, where $v_j \in S_3$. Since $d(v_0 | S_3) = 3$, we have $j \geq 2t + 1$.

We study representations of the vertices $v_{j-t-1}, v_{j-t-2}, v_{j-t-3}$. Since $v_{j-t-1}, v_{j-t-2}, v_{j-t-3} \notin S_3$, we have $v_{j-t-1}, v_{j-t-2}, v_{j-t-3} \in S_1 \cup S_2$.

Assume that $v_{j-t-1} \in S_1$ (if $v_{j-t-1} \in S_2$, the proof would be analogous since we do not use the fact that $v_0 \in S_1$ in the rest of this proof). We distinguish 4 subcases.

Case 2a : $v_{j-t-2}, v_{j-t-3} \in S_1$.

From (3) we know that $d(v_{j-t-1} | S_3) = d(v_{j-t-2} | S_3) = d(v_{j-t-3} | S_3) = 2$, so $r(v_{j-t-1} | \Pi) = (0, a_1, 2)$, $r(v_{j-t-2} | \Pi) = (0, a_2, 2)$, $r(v_{j-t-3} | \Pi) = (0, a_3, 2)$. From (5) we know that $a_1 \leq a_2 \leq a_3$ and the difference between a_3 and a_1 is at most one, which implies that two vertices from the set $\{v_{j-3}, v_{j-2}, v_{j-1}\}$ have the same partition representations with respect to Π .

Case 2b : $v_{j-t-2}, v_{j-t-3} \in S_2$.

By (3), we have $r(v_{j-t-2} | \Pi) = r(v_{j-t-3} | \Pi) = (1, 0, 2)$.

Case 2c : $v_{j-t-2} \in S_1$ and $v_{j-t-3} \in S_2$.

We have $v_{j-2}, v_{j-3} \in S_1$ (otherwise if $v_{j-2} \in S_2$ or $v_{j-3} \in S_2$, then we would obtain $r(v_{j-t-1} | \Pi) = r(v_{j-t-2} | \Pi) = (0, 1, 2)$). Consequently, $v_{j-1} \in S_1$ (otherwise if $v_{j-1} \in S_2$, then from (3) we would obtain $r(v_{j-2} | \Pi) = r(v_{j-3} | \Pi) = (0, 1, 1)$). But then the vertices $v_{j-1}, v_{j-2}, v_{j-3}$ cannot be resolved by Π , since $r(v_{j-1} | \Pi) = (0, a_1, 1)$, $r(v_{j-2} | \Pi) = (0, a_2, 1)$, $r(v_{j-3} | \Pi) = (0, a_3, 1)$, where $a_1 = a_2$ or $a_2 = a_3$.

Case 2d : $v_{j-t-2} \in S_2$ and $v_{j-t-3} \in S_1$.

We have $r(v_{j-t-3} | \Pi) = (0, 1, 2)$ and $r(v_{j-t-1} | \Pi) = (0, a, 2)$. So a must be at least 2, which implies that none of $v_{j-3}, v_{j-2}, v_{j-1}$ can be in S_2 . Thus $v_{j-3}, v_{j-2}, v_{j-1} \in S_1$ and these vertices cannot be resolved by S_2 , which means that they cannot be resolved by Π , a contradiction. $\square$

From Theorem 2 and Table 3 we obtain exact values of the partition dimension for $C_n(1, 2, 3)$ and $C_n(1, 2, 3, 4)$.

Corollary 4 : Let $n \geq 10$. Then $pd(C_n(1, 2, 3)) = pd(C_n(1, 2, 3, 4)) = 4$.

Our Corollaries 1 and 2 give strong upper bounds for the partition dimension of the directed graphs $C_n(1, 2, \dots, t)$. Corollaries 3 and 4 yield exact values of $pd(C_n(1, 2, \dots, t))$ for $t = 2, 3$ and 4 which means that our lower bound on $pd(C_n(1, 2, \dots, t))$ (Theorem 2) is best possible for $t = 3$ and 4.

References

- [1] P. Bhunia, S. Bag and K. Paul, Bounds for eigenvalues of the adjacency matrix of a graph, *Journal of Interdisciplinary Mathematics* 22(4) (2019) 415–431.
- [2] G. Chartrand, E. Salehi and P. Zhang, The partition dimension of a graph, *Aequationes Math.* 59 (2000) 45–54. doi.org/10.1007/PL00000127
- [3] K. Q. Fredlina and E. T. Baskoro, The partition dimension of some families of trees, *Procedia Comput. Sci.* 74 (2015) 60–66. doi.org/10.1016/j.procs.2015.12.076
- [4] X. Geng, M. Fang and D. Li, Gallai's conjecture for outerplanar graphs, *Journal of Interdisciplinary Mathematics* 18(5) (2015) 593–598.

- [5] M. Ghorbani and M. Hakimi-Nezhaad, Co-spectrality distance of graphs, *Journal of Information and Optimization Sciences* 40 (2019) 1221–1235. doi.org/10.1080/02522667.2018.1480464
- [6] M. Gionfriddo, A remarkable connection between-permutations and colourings of graphs, *Journal of Interdisciplinary Mathematics* 14(1) (2011) 95–104.
- [7] C. Grigorious, S. Stephen, R. Rajan and M. Miller, On the partition dimension of circulant graphs, *Comput. J.* 60 (2017) 180–184. doi.org/10.1093/comjnl/bxw079
- [8] C. Grigorious, S. Stephen, B. Rajan, M. Miller and A. William, On the partition dimension of a class of circulant graphs, *Inform. Process. Lett.* 114 (2014) 353–356. doi.org/10.1016/j.ipl.2014.02.005
- [9] S. M. Hosamani, P. V. Patil and S. H. Malghan, First Zagreb coindex of Hamiltonian graphs, *Journal of Information and Optimization Sciences* 38(3-4) (2017) 417–422.
- [10] I. Javaid, N. Khalid Raja, M. Salman and M. Naeem Azhar, The partition dimension of circulant graphs, *World Appl. Sci. J.* 18 (2012) 1705–1717. doi.org/10.5829/idosi.wasj.2012.18.12.1590
- [11] S. Khuller, B. Raghavachari and A. Rosenfeld, Landmarks in graphs, *Discrete Appl. Math.* 70 (1996) 217–229. doi.org/10.1016/0166-218X(95)00106-2
- [12] V. R. Kulli, B. Chaluvvaraju and M. Kumara, Graphs with equal secure total domination and inverse secure total domination numbers, *Journal of Information and Optimization Sciences* 39(2) (2018) 467–473.
- [13] V. R. Kulli and S. C. Sigarkanti, The γ -domination number of a graph, *Journal of Interdisciplinary Mathematics*, 3(2-3) (2000) 191–194.
- [14] E. Maritz and T. Vetrík, The partition dimension of circulant graphs, *Quaest. Math.* 41 (2018) 49–63. doi.org/10.2989/16073606.2017.1370031
- [15] J. Meksawang and W. Wongpinit, Isomorphism conditions for Cayley digraphs of Brandt semigroups, *Journal of Information and Optimization Sciences* 39(2) (2018) 481–493.
- [16] J. A. Rodríguez-Velázquez, I. G. Yero and M. Lema ska, On the partition dimension of trees, *Discrete Appl. Math.* 166 (2014) 204–209. doi.org/10.1016/j.dam.2013.09.026

- [17] M. Salman, I. Javaid and M. A. Chaudhry, Resolvability in circulant graphs, *Acta Math. Sin. (Engl. Ser.)* 28 (2012) 1851–1864. doi.org/10.1007/s10114-012-0417-4
- [18] H. M. A. Siddiqui and M. Imran, Computation of metric dimension and partition dimension of nanotubes, *J. Comput. Theor. Nanosci.* 12 (2015) 199–203. doi.org/10.1166/jctn.2015.3717
- [19] K. G. Sreekumar and K. Manilal, Hosoya polynomial and Harary index of SM family of graphs, *Journal of Information and Optimization Sciences* 39(2) (2018) 581–590.
- [20] X. Zhang, F. A. Shah, Y. Li, L. Yan, A. Q. Baig and M. R. Farahani, A family of fifth-order convergent methods for solving nonlinear equations using variational iteration technique, *Journal of Information and Optimization Sciences* 39(3) (2018) 673–694. doi.org/10.1080/02522667.2018.1443628
- [21] K. Zhao, Neighborhood conditions and Hamiltonian-connected graphs, *Journal of Interdisciplinary Mathematics*, 16(2-3) (2013) 137–145.

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