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On the degree locating indices of graphs

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Abstract

Graph theory has supported chemists with topological indices which predict different physicochemical properties such as boiling point, entropy, acentric factor, etc. of chemical compounds. In this article, we introduce two new topological indices called degree locating indices, based on the degree and location of the vertices. We present exact expressions for some families of standard graphs and we get the exact values of these indices for any graph of diameter two. Finally, we compute these indices for join graphs and book graphs.

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1. Introduction

Topological indices are important in chemistry, medicine, and other fields (see [5-9, 12, 18, 19]). Many of the topological indices that are currently of interest in mathematical chemistry are defined in terms of vertex degrees of molecular graphs. The first and second Zagreb indices, established by Gutman and Trinajstic in [10], are two of the most well-known topological indices of graphs. They are defined as: $M_1(G) = \sum_{u \in E(G)} (d(u)^2)$ and $M_2(G) = \sum_{uv \in E(G)} d(u)d(v)$, respectively. The Zagreb indices have been intensively explored due to their multiple applications in place of existing chemical procedures, which take longer and are more expensive. For similar reasons, many new reformulated and expanded

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versions of the Zagreb indices have been introduced (cf. [1, 2, 4, 11, 13, 15, 16, 21-23]).

By defining the locating matrix Lo (G) over G, one of the current authors, Saleh [17], has recently presented a novel matrix representation for a graph G. This representation will be redefined in the following ways.

Definition 1 : Let $G=(V,E)$ be a connected graph with vertex set $V=\{v_1, v_2, ..., v_n\}$. A degree locating function of G denoted by $\mathbf{Ld}(G)$ is a function $\mathbf{Ld}(G): V(G) \to (\mathbb{Z}^+ \cup \{0\})^{d+1}$ where d is the diameter of the graph G such that $\mathbf{Ld}(v_j) = \overrightarrow{v_j} = (\Gamma_0(v_j), \Gamma_1(v_j), ..., \Gamma_d(v_j))$, where $\Gamma_i(v_j)$ is the number of vertices of distance i from the vertex v_j in G. The vector $\overrightarrow{v_j}$ is called the degree locating vector corresponding to the vertex v_j. Then we define first and second degree locating indices of G as

$$\mathcal{L}_1(G) = \sum_{v_i \in V(G)} (\overrightarrow{v_i})^2 \quad \text{and} \quad \mathcal{L}_2(G) = \sum_{v_i v_j \in E(G)} \overrightarrow{v_i} \cdot \overrightarrow{v_j},$$

respectively, where $\overrightarrow{v_i} \cdot \overrightarrow{v_j}$ is the dot product of the two vectors $\overrightarrow{v_i}$ and $\overrightarrow{v_j}$.

Based on the degree locating vectors, we introduced two new topological indices, the *first* and *second degree locating indices*.

Unless otherwise mentioned, all graphs in this work shall be assumed to be simple, undirected, and connected. We direct the reader to [3] for graph theoretical terminologies.

2. Some exact values of locating indices

In this section,we determine the first and second degree locating indices for some standard graphs like K_n, C_n, $K_{n,m}$, W_n, P_n,

Theorem 2 : *Let $G \cong K_n$ be the complete graph with a vertex set $V(G)=\{v_1, v_2, \cdots, v_n\}$, where $n \geq 2$. Then*

i. $\mathcal{L}_1(G) = n^3 - 2n^2 + 2n$.

ii. $\mathcal{L}_2(G) = \frac{n(n-1)(n^2-2n+2)}{2}$.

Proof : Let $G \cong K_n$ be the complete graph with vertex set $V(G)=\{v_1, v_2, ..., v_n\}$, $n \geq 2$ and obviously, for any vertex $v_i \in V(G)$ the corresponding degree locating vector $\overrightarrow{v_i} = (1, n-1)$ so $\overrightarrow{v_i}^2 = 1 + (n-1)^2$.

Hence,

$$\mathcal{L}_1(G) = n^3 - 2n^2 + 2n.$$

Similarly,

$$\mathcal{L}_2(G) = \frac{n(n-1)(n^2 - 2n + 2)}{2}. \qquad \square$$

Proposition 3 : For an even integer $n \geq 2$, let $G \cong C_n$. Then

$$\mathcal{L}_1(G) = \mathcal{L}_2(G) = 2n(n-1).$$

Proof : By labeling the vertices of the cycle C_n as $\{v_1, v_2, \cdots, v_n\}$ in the anticlockwise direction, we can see that for any vertex v_i the degree locating vector will be in the form

$$\overrightarrow{v_i} = \left(1, \overbrace{2, \cdots, 2}^{n/2-1}, 1\right), \text{ for all } i = 1, 2, \ldots, n.$$

So, $\overrightarrow{v_i}^2 = 2 + 4(n/2 - 1) = 2(n-1)$.

Hence,

$$\mathcal{L}_1(G) = \mathcal{L}_2(G) = 2n(n-1). \qquad \square$$

Proposition 4 : For an odd integer $n \geq 3$, let $G \cong C_n$. Then

$$\mathcal{L}_1(G) = \mathcal{L}_2(G) = n(2n-1).$$

Proof : By labeling the vertices of the cycle C_n as $\{v_1, v_2, \cdots, v_n\}$ in the anticlockwise direction, it is easy to see that for any vertex v_i the degree locating vector will be in the form

$$\overrightarrow{v_i} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n-1}{2}}\right), \text{ for all } i = 1, 2, \ldots, n.$$

So, $\overrightarrow{v_i}^2 = 2n - 1$.

Hence,

$$\mathcal{L}_1(G) = \mathcal{L}_2(G) = n(2n-1). \qquad \square$$

Theorem 5 : *Let* $G \cong K_{a,b}$, *be a complete bipartite graph. Then*

i. $\mathcal{L}_1(G) = a^3 + ab^2 - 2a^2 + 2a + a^2b + b^3 - 2b^2 + 2b.$

ii. $\mathcal{L}_2(G) = ab(2ab - a + 2 - b).$

Proof : Let $G \cong K_{a,b}$, with two partite sets A and B such that $|A| = a$ and $|B| = b$, by labeling the vertices as $V(G) = \{v_1, \ldots v_a, u_1, \ldots u_b\}$. Obviously for any vertex v_i the corresponding degree degree locating is of the form

$$\overrightarrow{v_i} = (1, b, a-1), \text{for all } i = 1, 2, \ldots, a,$$

and

$$\overrightarrow{u_i} = (1, a, b-1), \text{for all } i = 1, 2, \ldots, b.$$

Therefore

$$\overrightarrow{v_i}^2 = 1 + b^2 + (a-1)^2 = a^2 + b^2 - 2a + 2, \text{for all } i = 1, 2, \ldots, a,$$

and

$$\overrightarrow{u_i}^2 = 1 + a^2 + (b-1)^2 = a^2 + b^2 - 2b + 2 \text{ for all } i = 1, 2, \ldots, b.$$

In the same way we have for any adjacent vertices v_i, u_j,

$$\overrightarrow{v_i} . \overrightarrow{u_i} = 1 + ab + (a-1)(b-1), \text{for all } i = 1, 2, \ldots, b \text{ and } j = 1, 2, \ldots, b$$

Hence,

$$\mathcal{L}_1(G) = a^3 + ab^2 - 2a^2 + 2a + a^2b + b^3 - 2b^2 + 2b,$$

and

$$\mathcal{L}_2(G) = ab(2ab - a + 2 - b).$$

□

Since the following consequences of Theorem 5 are special cases and clear, we will omit their proofs.

Corollary 6 : *Let* $G \cong K_{a,a}$, *be complete bipartite graph. Then* $\mathcal{L}_1(G) = 2a^2(a^2 - a + 1)$, *and* $\mathcal{L}_2(G) = 4a(a^2 - a + 1)$.

Corollary 7 : *Let* $G \cong K_{1,a}$, *be a star graph with a+1 vertices. Then* $\mathcal{L}_1(G) = a^3 - a^2 + 3a + 1$, *and* $\mathcal{L}_2(G) = a(a+1)$.

Proposition 8 : Let G be wheel graph W_n with $n+1$ vertices, where $(n \geq 4)$. Then we have

i. $\mathcal{L}_1(G) = n^3 - 5n^2 + 19n + 1.$

ii. $\mathcal{L}_2(G) = n(n^2 - 3n + 20).$

Proof : Let $G \cong W_n$ for $n \geq 4$, where W_n is the wheel graph with $n+1$ vertices. By labeling the vertices of $V(G)$ in the anticlockwise direction as $v_1, v_2, \cdots, v_n, v_{n+1}$ such that v_{n+1} is the center of the wheel, we obtain

$$\overrightarrow{v_i} = (1, 3, n-3), \text{ for } i = 1, 2, ..., n$$

and

$$\overrightarrow{v_{n+1}} = (1, n, 0).$$

So we get for $i = 1, ..., n$, $\overrightarrow{v_i}^2 = n^2 - 6n + 19$, $\overrightarrow{v_i}.\overrightarrow{v_{i+1}} = n^2 - 6n + 19$, $\overrightarrow{v_i}.\overrightarrow{v_{n+1}} = 3n + 1$ and $\overrightarrow{v_{n+1}}^2 = n^2 + 1$. Hence,

$$\mathcal{L}_1(G) = n^3 - 5n^2 + 19n + 1,$$

and

$$\mathcal{L}_2(G) = n(n^2 - 3n + 20).$$

□

Theorem 9 : *Let* $G \cong P_n$ *be path of even number of vertices* $(n \geq 4)$. *Then*

i. $\mathcal{L}_1(G) = \frac{n(3n-2)}{2}.$

ii. $\mathcal{L}_2(G) = \frac{3n(n-2)+4}{2}.$

Proof : Let G is be a path P_n with even number of vertices $(n \geq 4)$. By labeling the vertices from left to right as $v_1, v_2, \cdots, v_n$

$$\overrightarrow{v_1} = \Big(\overbrace{1, \cdots, 1}^{n}\Big), \qquad \overrightarrow{v_{n-1}} = \Big(\overbrace{1, \cdots, 1}^{n}\Big),$$

$$\overrightarrow{v_2} = \Big(1, 2, \overbrace{1, \cdots, 1}^{n-3}, 0\Big), \qquad \overrightarrow{v_{n-1}} = \Big(1, 2, \overbrace{1, \cdots, 1}^{n-3}, 0\Big),$$

$$\overrightarrow{v_3} = \Big(1,2,2,\overbrace{1,\cdots,1}^{n-5},0,0\Big), \qquad \overrightarrow{v_{n-2}} = \Big(1,2,2,\overbrace{1,\cdots,1}^{n-5},0,0\Big),$$

$$\vdots \qquad\qquad\qquad\qquad \vdots$$

$$\overrightarrow{v_{\frac{n}{2}}} = \Big(1,\overbrace{2,\cdots,2}^{\frac{n}{2}-1},1\overbrace{0,\cdots,0}^{\frac{n}{2}-1}\Big), \quad \overrightarrow{v_{\frac{n}{2}+1}} = \Big(1,\overbrace{2,\cdots,2}^{\frac{n}{2}-1},1\overbrace{0,\cdots,0}^{\frac{n}{2}-1}\Big).$$

By symmetry on components between the vectors easy to see that $\overrightarrow{v_1} = \overrightarrow{v_n}$, $\overrightarrow{v_2} = \overrightarrow{v_{n-1}}$, $\cdots$, $\overrightarrow{v_{\frac{n}{2}}} = \overrightarrow{v_{\frac{n}{2}+1}}$ That means, for $j = 1,2,\ldots,\frac{n}{2}$, we have

$$\overrightarrow{v_j} = \Big(1,\overbrace{2,\cdots,2}^{j-1},\overbrace{1,\cdots,1}^{n-2j+1},\overbrace{0,\cdots,0}^{j-1}\Big),$$

and

$$\overrightarrow{v_j}^2 = 1 + 4(j-1) + (n-2j+1) = n + 2j + 1.$$

Therefore,

$$\mathcal{L}_1(G) = 2\sum_{j=1}^{\frac{n}{2}}(n+2j-2) = n^2 + 4\frac{n/2(n/2+1)}{2} - 2n.$$

Hence,

$$\mathcal{L}_1(G) = \frac{n(3n-2)}{2}.$$

Now to prove part (ii), we have, for $j = 1,2,\ldots,\frac{n}{2}-1$

$$\overrightarrow{v_i}.\overrightarrow{v_{i+1}} = \Big(1,\overbrace{2,\cdots,2}^{j-1},\overbrace{1,\cdots,1}^{n-2j+1},\overbrace{0,\cdots,0}^{j-1}\Big).\Big(1,\overbrace{2,\cdots,2}^{j-1},2,\overbrace{1,\cdots,1}^{n-2j-1},\overbrace{0,\cdots,0}^{j-1}\Big)$$

Thus,

$$\overrightarrow{v_i}.\overrightarrow{v_{i+1}} = 1 + 4(j-1) + +2 + n - 2j - 1 = 2j + n - 2.$$

But, for

$$\overrightarrow{v_{\frac{n}{2}-1}}.\overrightarrow{v_{\frac{n}{2}}} = \Big(1,\overbrace{2,\cdots,2}^{\frac{n}{2}-1},1\overbrace{0,\cdots,0}^{\frac{n}{2}-1}\Big).\Big(1,\overbrace{2,\cdots,2}^{\frac{n}{2}-1},1\overbrace{0,\cdots,0}^{\frac{n}{2}-1}\Big) = 2(n-1).$$

Then we get

$$\mathcal{L}_2(G) = 2(n-1) + 2\sum_{j=1}^{\frac{n-2}{2}}(n+2j-2) = \frac{3n(n-2)}{2} + 2$$

Hence,

$$\mathcal{L}_2(G) = \frac{3n(n-2)+4}{2}.$$

□

Theorem 10 : *Let* $G \cong P_n$ *be path of odd number of vertices* $(n \geq 5)$. *Then*

i. $\mathcal{L}_1(G) = \frac{3n^2-2n+1}{2}$.

ii. $\mathcal{L}_2(G) = \frac{3n^2-6n+3}{2}$.

Proof : Let G is be a path P_n with even number of vertices $(n \geq 5)$. By labeling the vertices from left to right as $v_1, v_2, \cdots, v_n$

$$\overrightarrow{v_1} = \left(\overbrace{1,\cdots,1}^{n}\right), \qquad \overrightarrow{v_{n-1}} = \left(\overbrace{1,\cdots,1}^{n}\right),$$

$$\overrightarrow{v_2} = \left(1,2,\overbrace{1,\cdots,1}^{n-3},0\right), \qquad \overrightarrow{v_{n-1}} = \left(1,2,\overbrace{1,\cdots,1}^{n-3},0\right),$$

$$\overrightarrow{v_3} = \left(1,2,2,\overbrace{1,\cdots,1}^{n-5},0,0\right), \qquad \overrightarrow{v_{n-2}} = \left(1,2,2,\overbrace{1,\cdots,1}^{n-5},0,0\right),$$

$$\vdots \qquad \vdots$$

$$\overrightarrow{v_{\frac{n-1}{2}}} = \left(1,\overbrace{2,\cdots,2}^{\frac{n-3}{2}},1\overbrace{0,\cdots,0}^{\frac{n-3}{2}}\right), \qquad \overrightarrow{v_{\frac{n+3}{2}}} = \left(1,\overbrace{2,\cdots,2}^{\frac{n-3}{2}},1\overbrace{0,\cdots,0}^{\frac{n-3}{2}}\right).$$

By symmetry on components between the vectors easy to see that $\overrightarrow{v_1} = \overrightarrow{v_n}$, $\overrightarrow{v_2} = \overrightarrow{v_{n-1}}$, $\cdots$, $\overrightarrow{v_{\frac{n-1}{2}}} = \overrightarrow{v_{\frac{n+3}{2}}}$ That means, for $j = 1,2,...,\frac{n-1}{2}$, we have

$$\overrightarrow{v_j} = \left(1,\overbrace{2,\cdots,2}^{j-1},\overbrace{1,\cdots,1}^{n-2j+1},\overbrace{0,\cdots,0}^{j-1}\right),$$

and

$$\overrightarrow{v_j}^2 = 1 + 4(j-1) + (n-2j+1) = n+2j+1.$$

For the remaining vertex $v_{\frac{n+1}{2}}$, we get

$$\overrightarrow{v_{\frac{n+1}{2}}} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n-1}{2}}, \overbrace{0, \cdots, 0}^{\frac{n-1}{2}}\right),$$

So,

$$\overrightarrow{v_{\frac{n+1}{2}}}^2 = 2n-1.$$

Therefore,

$$\mathcal{L}_1(G) = 2n-1+2\sum_{j=1}^{\frac{n-1}{2}}(n+2j-2) = \frac{3n^2-2n+1}{2}.$$

Now, to prove part (ii), we have, for $j = 1, 2, ..., \frac{n-3}{2}$

$$\overrightarrow{v_i}.\overrightarrow{v_{i+1}} = \left(1, \overbrace{2, \cdots, 2}^{j-1}, \overbrace{1, \cdots, 1}^{n-2j+1}, \overbrace{0, \cdots, 0}^{j-1}\right).\left(1, \overbrace{2, \cdots, 2}^{j-1}, 2, \overbrace{1, \cdots, 1}^{n-2j-1}, \overbrace{0, \cdots, 0}^{j-1}\right)$$

Thus,

$$\overrightarrow{v_i}.\overrightarrow{v_{i+1}} = 1+4(j-1)++2+n-2j-1 = 2j+n-2.$$

But, for $i = \frac{n-1}{2}, \frac{n+1}{2}, \frac{n+3}{2}$

$$\overrightarrow{v_{\frac{n-1}{2}}}.\overrightarrow{v_{\frac{n+1}{2}}} = \overrightarrow{v_{\frac{n+1}{2}}}.\overrightarrow{v_{\frac{n+3}{2}}} = \left(1, \overbrace{2, \cdots, 2}^{\frac{n-3}{2}}, 1 \overbrace{0, \cdots, 0}^{\frac{n-3}{2}}\right).\left(1, \overbrace{2, \cdots, 2}^{\frac{n-1}{2}}, \overbrace{0, \cdots, 0}^{\frac{n-1}{2}}\right) = 2n-3.$$

Then, we get

$$\mathcal{L}_2(G) = 2(2n-3)+2\sum_{j=1}^{\frac{n-3}{2}}(n+2j-2) = 2n+\frac{3n^2-10n+3}{2}$$

Hence,

$$\mathcal{L}_2(G) = \frac{3n^2-6n+3}{2}.$$

□

Proposition 11 : Let G be k-regular graph of $n \geq 2$ and with diameter two. Then

i. $\mathcal{L}_1(G) = n(n^2-2nk-2n+2k^2+2k+2).$

ii. $\mathcal{L}_2(G) = \frac{nk(n^2-2nk-2n+2k^2+2k+2)}{2}$.

Proof : Let G be k-regular graph of $n \geq 2$ vertices and with diameter two. Then clearly for any vertex $v \in V(G)$, we have

$$\vec{v} = (1, k, n-k-1) \text{ and } \vec{v}^{\,2} = 1 + k^2 + (n-k-1)^2, \text{ Hence,}$$

$$\mathcal{L}_1(G) = n(n^2 - 2nk - 2n + 2k^2 + 2k + 2),$$

and

$$\mathcal{L}_2(G) = \frac{nk(n^2 - 2nk - 2n + 2k^2 + 2k + 2)}{2}.$$

□

Proposition 12 : Let G be a graph of $n \geq 2$ and with diameter two. Then

i. $\mathcal{L}_1(G) = n + 2M_1(G) + (n-1)(n-2m-1)$.

ii. $\mathcal{L}_2(G) = 2M_2(G) - (n-1)M_1(G) + mn^2 + 2m - 2mn$.

Proof : Let G be a graph of $n \geq 2$ vertices and with diameter two. Then clearly for any vertex $v \in V(G)$, we have $\vec{v} = (1, deg(v), n - deg(v) - 1)$ and $\vec{v}^{\,2} = 1 + deg(v)^2 + (n - deg(v) - 1)^2$, then

$$\mathcal{L}_1(G) = \sum_{u \in V(G)} \vec{v}^{\,2} = n + 2M_1(G) + (n-1)(n-2m-1).$$

Similarly,

$$\mathcal{L}_2(G) = \sum_{uv \in E(G)} \vec{u}.\vec{v} = \sum_{uv \in E(G)} (1 + deg(v)deg(u) + (n-1-deg(v))(n-1-deg(u))).$$

Hence,

$$\mathcal{L}_2(G) = 2M_2(G) - (n-1)M_1(G) + mn^2 + 2m - 2mn.$$

□

3. Locating indices of the join of two graphs

The join $G = G_1 + G_2$ of graphs G_1 and G_2 with disjoint vertex sets V_1 and V_2 and edge sets E_1 and E_2 is the graph union $G_1 \cup G_2$ together with all the edges joining V_1 and V_2. In the following theorem we find first and second locating indices for the join graph G.

Theorem 13 : *Let* $G \cong G_1 + G_2$ *such that* G_1 *and* G_2 *are both connected graphs, such that* G_1 *has* n_1 *vertices and* m_1 *edges while* G_2 *has* n_2 *vertices and* m_2 *edges. Then*

$$\begin{aligned}\mathcal{L}_1(G) &= 2(M_1(G_1)+M_1(G_2))+(n_2m_1+n_1m_2)+4(m_1+m_2)-4(n_1m_1+n_2m_2)\\ &\quad +(n_1^3+n_2^3)+(n_1n_2^2+n_2n_1^2)-2(n_1^2+n_2^2)+2(n_1+n_2).\end{aligned}$$

and

$$\begin{aligned}\mathcal{L}_2(G) &= (n_2-n_1+1)(M_1(G_1)+M(G_2))+2M_2(G_1)+m_1(n_1^2+n_2^2-2n_1+2)\\ &\quad +m_2(n_1^2+n_2^2-2n_2+2)+n_1n_2(m_1+m_2)\\ &\quad +2m_1m_2+n_1n_2(2n_1n_2+2-n_1-n_2).\end{aligned}$$

Proof : Let $G \cong G_1 + G_2$ such that G_1 and G_2 are both connected graphs, such that G_1 has n_1 vertices and m_1 edges while G_2 has n_2 vertices and m_2 edges. Then.by labeling the vertices of the graph G as

$$V(G)=\{v_1,v_2,\cdots,v_{n_1},u_1,u_2,\cdots,u_{n_2}\}$$

where $v_1,v_2,\cdots,v_{n_1} \in V(G_1)$ and $u_1,u_2,\cdots,v_{n_2} \in V(G_2)$. we get the degree locating vectors in G as following:

$$\overrightarrow{v_i}=(1,n_2+deg_{G_1}(v_i),n_1-deg_{G_1}(v_i)-1),$$

and

$$\overrightarrow{u_i}=(1,n_1+deg_{G_2}(u_i),n_2-deg_{G_2}(u_i)-1).$$

So,

$$\overrightarrow{v_i}^2=2deg_{G_1}(v_i)^2+2n_2deg_{G_1}(v_i)+2deg_{G_1}(v_i)-2n_1deg_{G_1}(v_i)+n_2^2+n_1^2+2-2n_1$$

and

$$\sum_{i=1}^{n_1}\overrightarrow{v_i}^2=2M_1(G_1)+4n_2m_1+4m_1-4n_1m_1+n_1^3+n_1n_2^2-2n_1^2+2n_1.$$

In the same way

$$\overrightarrow{u_i}^2 = 2deg_{G_2}(u_i)^2 + 2n_1 deg_{G_2}(u_i) + 2deg_{G_2}(u_i) - 2n_1 2deg_{G_2}(u_i) + n_2^2 + n_1^2 + 2 - 2n_2$$

So,

$$\sum_{i=1}^{n_2} \overrightarrow{u_i}^2 = 2M_1(G_2) + 4n_1 m_2 + 4m_2 - 4n_2 m_2 + n_2^3 + n_2 n_1^2 + -2n_2^2 + 2n_2.$$

Clearly,

$$\mathcal{L}_1(G) = \sum_{i=1}^{n_1} \overrightarrow{v_i}^2 + \sum_{i=1}^{n_2} \overrightarrow{u_i}^2,$$

then

$$\begin{aligned}\mathcal{L}_1(G) &= 2(M_1(G_1) + M_1(G_2)) + (n_2 m_1 + n_1 m_2) + 4(m_1 + m_2) - 4(n_1 m_1 + n_2 m_2) \\ &\quad + (n_1^3 + n_2^3) + (n_1 n_2^2 + n_2 n_1^2) - 2(n_1^2 + n_2^2) + 2(n_1 + n_2)\end{aligned}$$

To prove (ii), according to different types of edges in G, we have

$$\mathcal{L}_2(G) = \sum_{v_i v_j \in E(G)} (\overrightarrow{v_i}.\overrightarrow{v_j}) + \sum_{u_i u_j \in E(G)} (\overrightarrow{u_i}.\overrightarrow{u_j}) + \sum_{v_i u_j \in E(G)} (\overrightarrow{v_i}.\overrightarrow{u_j})$$

Since,

$$\begin{aligned}\overrightarrow{v_i}.\overrightarrow{v_j} &= 1 + (n_2 + deg_{G1}(v_i))(n_2 + deg_{G_1}(v_j)) \\ &\quad + (n_1 - deg_{G1}(v_i) - 1)(n_1 - deg_{G1}(v_j) - 1),\end{aligned}$$

so,

$$\begin{aligned}\overrightarrow{v_i}.\overrightarrow{v_j} &= n_2 deg_{G1}(v_j) + n_2 deg_{G1}(v_i) + 2deg_{G1}(v_i) deg_{G1}(v_j) \\ &\quad + deg_{G1}(v_i) + deg_{G1}(v_j) - n_1 deg_{G1}(v_j) - n_1 deg_{G1}(v_i) \\ &\quad + n_2^2 + n_1^2 + 2 - 2n_1.\end{aligned}$$

Then

$$\begin{aligned}&\sum_{v_i v_j \in E(G)} (\overrightarrow{v_i}.\overrightarrow{v_j}) \\ &= \sum_{v_i v_j \in E(G)} (n_2(deg_{G_1}(v_i) + deg_{G_1}(v_j)) + 2(deg_{G_1}(v_i) deg_{G_1}(v_j)) \\ &\quad + (deg_{G_1}(v_i) + deg_{G_1}(v_j)) - n_1(deg_{G_1}(v_i) + deg_{G_1}(v_j)) + n_1^2 + n_2^2 + 2 - 2n_1) \\ &= (n_2 - n_1 + 1)M_1(G_1) + 2M_2(G_1) + m_1(n_1^2 + n_2^2 - 2n_1 + 2)\end{aligned}$$

Also, since

$$\overrightarrow{u_i}.\overrightarrow{u_j} = 1 + (n_1 + deg_{G_2}(u_i))(n_1 + deg_{G_2}(u_j)) + (n_2 - deg_{G_2}(u_i) - 1)(n_2 - deg_{G_2}(u_j) - 1)$$

In the same way we get,

$$\sum_{u_iu_j \in E(G)} (\overrightarrow{u_i}.\overrightarrow{u_j}) = (n_1 - n_2 + 1)M_1(G_2) + 2M_2(G_2) + m_2(n_1^2 + n_2^2 - 2n_2 + 2).$$

Similarly, since,

$$\overrightarrow{v_i}.\overrightarrow{u_j} = 1 + (n_2 + deg_{G_1}(v_i))(n_1 + deg_{G_2}(u_j)) + (n_1 - deg_{G_1}(v_i) - 1)(n_2 - deg_{G_2}(v_j) - 1)$$

We get

$$\sum_{v_iu_j \in E(G)} (\overrightarrow{v_i}.\overrightarrow{u_j}) = \sum_{i=1}^{n_1}\sum_{j=1}^{n_2} (\overrightarrow{v_i}.\overrightarrow{u_j}),$$

and

$$\begin{aligned}\sum_{i=1}^{n_1}\sum_{j=1}^{n_2} (\overrightarrow{v_i}.\overrightarrow{u_j}) &= \sum_{i=1}^{n_1} (n_2m_2 + n_1n_2deg_{G_1}(v_i) + 2m_2deg_{G_1}(v_i) + n_2deg_{G_1}(v_i) \\ &\quad + m_2 - n_2deg_{G_1}(v_i) - n_1m_2 + n_2(2n_1n_2 + 2 - n_2 - n_1)) \\ &= n_1n_2(m_1 + m_2) + 2m_1m_2 + n_1n_2(2n_1n_2 + 2 - n_1 - n_2).\end{aligned}$$

Hence,

$$\begin{aligned}\mathcal{L}_2(G) &= (n_2 - n_1 + 1)M_1(G_1) + 2M_2(G_1) + m_1(n_1^2 + n_2^2 - 2n_1 + 2) \\ &\quad (n_1 - n_2 + 1)M_1(G_2) + 2M_2(G_2) + m_2(n_1^2 + n_2^2 - 2n_2 + 2) \\ &\quad + n_1n_2(m_1 + m_2) + 2m_1m_2 + n_1n_2(2n_1n_2 + 2 - n_1 - n_2).\end{aligned}$$

□

4. Locating indices of the book graph

Theorem 14 : *Let G be a book graph* $B_t = P_2 \times S_t$ *with* $(2t+2)$ *vertices. Then,*

i. $\mathcal{L}_1(G) = 4t^3 + 16t + 4.$

ii. $\mathcal{L}_2(G) = 4t^3 + 4t^2 + 14t + 2.$

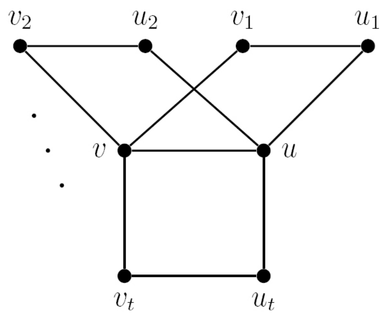

Figure 1

A Book Graph B_t.

Proof : Suppose, we have G is the book graph $B_t = P_2 \times S_t$ with $(2t+2)$ by labeling the vertices as $V(G) = \{v, u, v_1, v_2, ...v_t, u_1, u_2, ..., u_t\}$ as in Figure 1, we get,

$$\vec{v} = (1, t+1, t, 0) \quad , \quad \vec{u} = (1, t+1, t, 0)$$

And for $i = 1, 2, ..., m$, we have

$$\overrightarrow{v_i} = (1, 2, t, t-1) \quad , \quad \vec{u} = (1, 2, t, t-1),$$

Then,

$$\vec{v}^2 = \vec{u}^2 = 1 + (t+1)^2 + t^2 = 2(t^2 + t + 1),$$

and for $i = 1, 2, 3, ..., t$

$$\overrightarrow{v_i}^2 = \overrightarrow{u_i}^2 = 1 + 4 + t^2 + (t-1)^2 = 2(t^2 - t + 3).$$

Hence,

$$\mathcal{L}_1(G) = 4t^3 + 16t + 4.$$

Similarly, to prove (ii) we have, $\vec{v}.\vec{u} = 2(t^2 + t + 1)$, and for any $i = 1, 2, 3, ..., t$,

$$\overrightarrow{v_i}.\overrightarrow{u_i} = 2(t^2 - t + 3)$$

and in the same way,

$$\vec{v}.\overrightarrow{v_i} = \vec{u}.\overrightarrow{u_i} = t^2 + 2t + 3.$$

Hence,

$$\mathcal{L}_2(G) = 4t^3 + 4t^2 + 14t + 2. \qquad \square$$

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