

On the category of $m\theta$ sets

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Abstract

In this text, our purpose is to define the category of modal θ -valent sets denoted ENS_{θ} and see whether it is a topos.

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1. Introduction

The modal θ -valent sets denoted $m\theta$ sets for short are defined by F.AYISSI ETEME in [1]. The subset of modal θ -valent invariants of a $m\theta$ set is a classical set. Thus a $m\theta$ set appears as an modal θ -valent chrysipian extension of a classical set, so an enrichment of the classical set in the logic view. From finite $m\theta$ sets F.AYISSI ETEME and J.A.TSIMI defined the notions of $m\theta$ codes, of $m\theta$ linear codes and of $m\theta$ cyclic codes in [2, 3, 4]. The chrysipian modal θ -valent codes design is defined in [5]. Thus, $m\theta$ sets can be realized as the objects of some category ENS_{θ} which will be a basic object of our study. In this paper we intend to present explicitly the category of $m\theta$ sets and investigate if it is a topos. In the section 2, the definition of a topos is given. The section 3 presents and

defines the notion of $m\theta$ sets. In the section 4, we present the chrysipian completion of a θ -valued Lukasiewicz algebra as an example of $m\theta$ sets. In the section 5, we define the category of $m\theta$ sets $ENS\theta$ and investigate on how properties of toposes can be satisfied.

2. The definition of a topos

Definition 2.1 : A topos is a category denoted $\mathfrak{C}$ that satisfies the five following properties.

1. There exists a final object in $\mathfrak{C}$ i.e an object F such that for every object A there exists one and only one morphism from A into F (morphism denoted 1^A).
2. In $\mathfrak{C}$, one can define the product of two objets $A \times B$, with the usual morphisms projections.
3. If f and g are two morphisms from an objet A into an object B , then there there exists an equaliser i.e an objet C and a morphism u from C into A such that $f \circ u = g \circ u$.

If a morphism $u' : C' \rightarrow A$ is such that $f \circ u' = g \circ u'$, then there exists a unique morphism $\bar{u}' : C' \rightarrow C$ such that $u' = u \circ \bar{u}'$.

$$\begin{array}{ccc} C & \xrightarrow{u} & A \xrightleftharpoons[g]{f} B \\ \bar{u}' \uparrow & \nearrow u' & \\ C' & & \end{array}$$

4. $\mathfrak{C}$ has exponential i.e: if A and B are two objects, there exists an object denoted B^A and a morphism

$e : A \times B^A \rightarrow B$, called evaluation such that for every object X and every morphism $f : A \times X \rightarrow B$, there exists a unique morphism $\bar{f} : X \rightarrow B^A$ satisfying $f = e \circ (1_A \times \bar{f})$.

$$\begin{array}{ccc} A \times B^A & \xrightarrow{e} & B \\ 1_A \times \bar{f} \uparrow & \nearrow f & \\ A \times X & & \end{array}$$

5. $\mathfrak{C}$ has a subobjects classifier i.e an object Ω and a morphism $V : F \rightarrow \Omega$ such that for every monomorphism $m : A' \rightarrow A$; there exists a unique morphism $\Phi_m : A \rightarrow \Omega$ such that the following diagram is a pullback i.e

- (a) the following diagram is commutative

$$\begin{array}{ccc} A' & \xrightarrow{1^{A'}} & F \\ m \downarrow & & \downarrow \vee \\ A & \xrightarrow{\Phi_m} & \Omega \end{array}$$

- (b) For every object B and every morphism $n : B \rightarrow A$ such that the following diagram is commutative, there exists a unique morphism $\bar{n} : B \rightarrow A'$ such that $n = m \circ \bar{n}$

$$\begin{array}{ccc} B & \xrightarrow{1^B} & F \\ n \downarrow & & \downarrow \vee \\ A & \xrightarrow{\Phi_m} & \Omega \end{array}$$

$$\begin{array}{ccccc} B & & & & \\ & \searrow \bar{n} & & \searrow & \\ & & A' & \xrightarrow{\quad} & F \\ & \searrow n & \downarrow m & & \downarrow \vee \\ & & A & \xrightarrow{\Phi_m} & \Omega \end{array}$$

Proposition 2.1 : The category of classical sets denoted $\mathbf{ENS}$ is a topos. The final object of $\mathbf{ENS}$ is every set of one element denoted $F = \{1\}$.

The subobjects classifier is $\Omega = \{0, 1\}$ with V defines by $V(1) = 1$. If m is an injection from A' into A , the classifier of m , Φ_m is the characteristic function of $m(A')$ considered as a sub set of A .

3. The $m\theta$ sets [1]

Definition 3.1 : A modal θ -valent set is defined as a tuple $(E, (F_\alpha)_{\alpha \in I_*})$ where E is a non empty set, I a closed chain with first element 0 and last element 1, and $(F_\alpha)_{\alpha \in I_*}$ a family of applications from E into E satisfying the following pro-perties:

1. $\bigcap_{\alpha \in I_*} F_\alpha(E) \neq \emptyset; I_* = I \setminus \{0\}$.
2. $\alpha \neq \beta \Rightarrow F_\alpha \neq F_\beta$.
3. $\forall \alpha, \beta \in I_* = I \setminus \{0\}, F_\beta \circ F_\alpha = F_\alpha$.
4. $\forall x, y \in E, (F_\alpha(x) = F_\alpha(y), \forall \alpha \in I_*) \Rightarrow x = y$.

Definition 3.2 : Let (E, F_α) be a $m\theta$.

One calls the modal θ -valent determination theorem in (E, F_α) what follows:

$$\forall x, y \in E, (x = y \Leftrightarrow \forall \alpha \in I_*, F_\alpha(x) = F_\alpha(y)).$$

Proposition 3.1 : Let (E, F_α) be a $m\theta$. The following properties are equivalent.

- (i) $x \in \bigcap_{\alpha} F_\alpha(E)$.
- (2i) $\forall \alpha \in I_*, x = F_\alpha(x)$.
- (3i) $\forall \alpha, \beta \in I_*, F_\alpha(x) = F_\beta(x)$.
- (4i) $\exists \mu \in I_*, x = F_\mu(x)$.

Proof : [1]

□

Notation 3.1 : For short, we will write $F_\alpha E$ for $F_\alpha(E)$; $F_\alpha x$ for $F_\alpha(x)$ and $C(E, F_\alpha) = \bigcap_{\alpha} F_\alpha(E)$.

Definition 3.3 : Let (E, F_α) be a $m\theta$ set.

One calls:

1. α -modality of (E, F_α) , the map F_α ;
2. Modality of (E, F_α) every $F_\alpha, \forall \alpha \in I_*$;
3. α -modality of $x \in E$, the element $F_\alpha x$.

4. The sub-set of modal θ -valent invariant elements of (E, F_α) , the set $C(E, F_\alpha)$.

Definition 3.4 : Let (E, F_α) and (E', F'_α) be two $m\theta$ sets.

An application $f : E \rightarrow E'$ is called a $m\theta$ morphism from (E, F_α) into (E', F'_α) if $\forall \alpha \in I_*, F'_\alpha \circ f = f \circ F_\alpha$.

Remark 3.1 : Let (E, F_α) be a $m\theta$ set.

Let $(C(E, F_\alpha))^{I_*}$ be the set of I_* -families of elements of $C(E, F_\alpha)$.

Let $(x_\lambda)_{\lambda \in I_*}$ in $(C(E, F_\alpha))^{I_*}$. $\forall a \in C(E, F_\alpha)$, one can identify a with a I_* -family of elements of $C(E, F_\alpha)$, $(a_\lambda)_{\lambda \in I_*}$ where $\forall \lambda \in I_*, a_\lambda = a \ \forall \alpha \in I_*$, let define $\hat{F} : (C(E, F_\alpha))^{I_*} \rightarrow (C(E, F_\alpha))^{I_*}$ such that $\hat{F}_\alpha((x_\lambda)_{\lambda \in I_*}) = x_\alpha \in C(E, F_\alpha)$.

Then $((C(E, \hat{F}_\alpha))^{I_*}, \hat{F}_\alpha)$ is a $m\theta$ set and $x \mapsto (F_\alpha x)_\lambda$ is an injection from E into $(C(E, F_\alpha))^{I_*}$.

We also have $\forall \alpha \in I_*, \hat{F}_\alpha \circ (x \mapsto (F_\alpha x)_\lambda) = (x \mapsto (F_\alpha x)_\lambda) \circ F_\alpha$. Therefore $x \mapsto (F_\alpha x)_\lambda$ is an $m\theta$ injective morphism.

In what follows, we will denote the $m\theta$ set $(C(E, F_\alpha))^{I_*}, \hat{F}_\alpha$ for short by $B^\theta(E, F_\alpha)$; so, one can identify the $m\theta$ set (E, F_α) by its $m\theta$ injective image by the application $(x \mapsto (F_\alpha x)_\lambda)$; then one can consider the $m\theta$ set (E, F_α) as a $m\theta$ sub-set of $B^\theta(E, F_\alpha)$.

Definition 3.5 : One calls $m\theta$ completion of the $m\theta$ set (E, F_α) any $m\theta$ set (E', F'_α) in $m\theta$ isomorphism with $B^\theta(E, F_\alpha)$.

4. Chrysippian completion of a θ -valued Lukasiewicz algebra [2]

Definition 4.1 : Let I be a θ -valued closed chain.

Let $I_* = I \setminus \{0\}$.

A θ -valued Lukasiewicz algebra is a structure $(L, \vee, \wedge, 1, 0, (\Phi_\alpha)_{\alpha \in I_*})$ where:

1. $(L, \vee, \wedge, 1, 0)$ is a closed distributive lattice.
2. $\forall \alpha \in I_*, \Phi_\alpha$ is an endomorphism that conservs 0 and 1.
3. $\forall \alpha, \beta \in I_*, \Phi_\alpha \circ \Phi_\beta = \Phi_\beta$.
4. $\forall \alpha, \beta \in I_*, (\alpha \leq \beta \Rightarrow \Phi_\beta \leq \Phi_\alpha)$.
5. $(\forall \alpha \in I_*, \Phi_\alpha(x) = \Phi_\alpha(y)) \Rightarrow x = y$.

6. $\forall \alpha \in I_*, \Phi_\alpha$ is an chrysipian operator (i.e $\forall x \in L, \Phi_\alpha(x)$ is chrysipian i.e $\neg(\Phi_\alpha(x))$ exists in L).

Proposition 4.1 : Let $(L, \vee, \wedge, 1, 0, (\Phi_\alpha)_{\alpha \in I_*})$ be a θ -valued Lukasiewicz algebra.

$(L, (\Phi_\alpha)_{\alpha \in I_*})$ is a $m\theta$ set.

Remark 4.1 : Let (L, Φ_α) be a θ -valued Lukasiewicz algebra.

Let set $B = C(L) = \{\Phi_\alpha(x); \alpha \in I_*, x \in L\}$ the boolean ring of chrysipian elements of L .

For any $x \in L$, let set $x_\Phi = (\Phi_\alpha(x))_{\alpha \in I_*}$ the I_* -family of decreasing elements of B defined by x .

One can observ that $x_\Phi \in B^{I_*}$, and according to the modal θ -valent determination theorem in (L, Φ_α) , the maps $x \mapsto x_\Phi$ is an injection from L into B^{I_*} .

Definition 4.2 : x_Φ is called the chrysipian expression of x .

Proposition 4.2 : Let $\alpha \in I_*$ and $\mathcal{U} = (\mathcal{U}_\alpha)_{\alpha \in I_*} \in B^{I_*}$.

Let define the map $\Omega_\alpha : B^{I_*} \rightarrow B^{I_*}$ by $\Omega_\alpha(\mathcal{U}) = \mathcal{U}_\alpha$.

(B^{I_*}, Ω_α) is a $m\theta$ set.

Proof :

- $\forall \alpha \in I_*, \forall \mathcal{U} = (\mathcal{U}_\alpha)_{\alpha \in I_*} \in B^{I_*}$,
- $\Omega_\alpha(\mathcal{U}) = \mathcal{U}_\alpha \in B$.
- $C(B^{I_*}, \Omega_\alpha) = B \neq \emptyset$.
- $\forall \alpha, \beta \in I_*, \alpha \neq \beta \Rightarrow \Omega_\alpha \neq \Omega_\beta$.
- $\Omega_\beta \circ \Omega_\alpha = \Omega_\alpha$.
- $(\forall \alpha \in I_* (\Omega_\alpha(u) = \Omega_\alpha(v)) \Rightarrow u = v)$. □

Remark 4.2 : According to the injection $x \mapsto x_\Phi$ from L into B^{I_*} , one can identify (L, Φ_α) to a $m\theta$ sub-set of (B^{I_*}, Ω_α) .

Definition 4.3 : (B^{I_*}, Ω_α) will be defined as the $m\theta$ completion of $m\theta$ set (L, Φ_α) . Therefore $B^\theta(L, \Phi_\alpha) = (B^{I_*}, \Omega_\alpha)$.

Example 4.1 :

1. Let I be a m θ closed chain 0, 1.

Let $\alpha \in I_*$ and $F_\alpha : I \rightarrow I$ defines by $F_\alpha(x) = \begin{cases} 1 & \text{if } \alpha \leq x \\ 0 & \text{if not.} \end{cases}$

One observes that if $\alpha \leq \beta$ then $F_\beta \leq F_\alpha$. Therefore $I_\theta = (I, F_\alpha)$ is a θ -valued Lukasiewicz algebra, so a m θ set

$$C(I, F_\alpha) = \{0, 1\}, \quad B^\theta(I, F_\alpha) = (\{0, 1\}^{I_*}, \hat{F}_\alpha).$$

2. Let X be a non empty set.

Let I be a m θ closed chain 0, 1.

Let $\mathcal{P}(X \times I_*)$ be the set of sub-sets of the cartesian product $X \times I_*$.

Let denote $\wedge_I^X \subseteq \mathcal{P}(X \times I_*)$, defines by:

$A \in \wedge_I^X$ iff A verifies:

If $(x, \alpha), (y, \beta) \in A$ then necessary $x \neq y$. Thus, if $A \in \wedge_I^X$ then we can not find in A distincts elements like (x, α) and (x, β) .

If $x \in X$ and $\alpha \in I_*$, let denote the element $(x, \alpha) \in \wedge_I^X$ by $\hat{x}_\alpha$.

When one writtes $\hat{x}_\alpha$, x will be called the support of $\hat{x}_\alpha$ and α its value.

$\hat{x}_\alpha$ can be represented by $\delta_\alpha^x : X \rightarrow I$ defines as follows.

$$\forall a \in X, \delta_\alpha^x(a) = \begin{cases} \alpha & \text{if } a = x \\ 0 & \text{if } a \in X \setminus \{x\}. \end{cases}$$

By convention, one will identify $\hat{x}_1$ by x . Let $A \in \wedge_I^X$ and let set $Supp(A) = \{x \in X / \hat{x}_\alpha \in A\}$. Let $h : \wedge_I^X \rightarrow I^X$ the application defines as follows:

$$h(A)(a) = \begin{cases} \alpha & \text{if } a \in Supp(A) \text{ and } \hat{a}_\alpha \in A \\ 0 & \text{if } a \in X \setminus Supp(A). \end{cases}$$

Let $g : I^X \rightarrow \wedge_I^X$ the map defines by $\forall \hat{A} \in I^X, g(\hat{A}) = \{\hat{a}_\alpha / \hat{A}(a) = \alpha \neq 0\}$.

One has $g \circ h = Id_{\wedge_I^X}$ and $h \circ g = Id_{I^X}$. Accordingly, it follows that g and h are the reciprocal bijections.

As I^X can be endowed with the structure of the θ -valued Lukasiewicz algebra $I_\theta = (I, F_\alpha)$ defines as precedently, one can conclud by the bijection h that $\wedge_I^X$ is a θ -valued Lukasiewicz algebra so a m θ set.

Let denote $X_\theta = (\wedge_I^X, \Phi_\alpha)$ the $m\theta$ set defined as above.

By convention $C(X_\theta) = C(\wedge_I^X, \Phi_\alpha) = \mathcal{P}(X)$ the set of sub-sets of X .

Therefore $B^\theta(X_\theta) = ((\mathcal{P}(X))^{I_*}, \Omega_\alpha)$ is the $m\theta$ completion of X_θ .

Notation 4.1 : In what follows, we will denote the $m\theta$ completion $B^\theta(I_\theta)$ by $(2^\theta, \omega_\alpha)$.

Definition 4.4 : Let X^θ be the set of sub-sets of $X \times 2_*^\theta$ defined as follows:
 $2_*^\theta = 2^\theta \setminus \{0\}$.

If $A \subseteq X \times 2_*^\theta$, then $A \in X^\theta$ iff A verifies: if (x, α) and (y, μ) are two elements of A then $x \neq y$. We will denote again $\hat{x}_\tau = (x, \tau) \in X \times 2_*^\theta$. One will call again x the support and τ the value of the element $\hat{x}_\alpha$.

If $A \in X^\theta$ and $\alpha \in I_*$, one defines the map $\bar{\Omega}_\alpha : X^\theta \rightarrow X^\theta$ by:

$$\bar{\Omega}_\alpha(A) = \{\omega_\alpha(\hat{x}_\tau) = \hat{x}_{\omega_\alpha(\tau)} / \omega_\tau(\tau) = 1 \text{ and } \hat{x}_\tau \in A\}.$$

By definition $\bar{\Omega}_\alpha(A) \in \mathcal{P}(X)$ the set of sub-sets of X . As I^X and $\wedge_I^X$ are in bijection according to the example that preceeds, X^θ and $(\mathcal{P}(X))^{I_*}$ are also in bijection then $\bar{\Omega}_\alpha$ are $m\theta$ morphism imported from $B^\theta(X_\theta)$ on X^θ by this bijection.

Remark 4.3 : $B^\theta(X_\theta)$ is the $m\theta$ completion whom elements are chrysippian expression of elements of X^θ .

Proposition 4.3 : Let X be a non empty set.

Let $x \in X$ and $x^\theta = \{\hat{x}_\tau / \tau \in 2^\theta\}$ where $(2^\theta, \omega_\alpha) = B^\theta(I_\theta)$.

Let for $\alpha \in I_*$, $F_\alpha : x^\theta \rightarrow x^\theta$ by $F_\alpha(\hat{x}_\tau) = \hat{x}_{\omega_\alpha(\tau)}$.

(x^θ, F_α) is a $m\theta$ set and $C(x^\theta, F_\alpha) = \{\hat{x}_1, \hat{x}_0\} = \{x, \hat{x}_0\}$.

Proof : Let $(2^\theta, \omega_\alpha)$ be the $m\theta$ completion of $I_\theta = (I, F_\alpha)$.

$$\begin{aligned} \text{Let } \gamma : 2^\theta &\rightarrow x^\theta \\ \tau &\mapsto \hat{x}_\tau \end{aligned}$$

γ is a $m\theta$ bijection from $(2^\theta, \omega_\alpha)$ into (x^θ, F_α) . Indeed, $\forall \tau \in 2^\theta$, one has $F_\alpha \circ \gamma(\tau) = F_\alpha(\hat{x}_\tau) = \hat{x}_{\omega_\alpha(\tau)} = \gamma \circ \omega_\alpha(\tau)$, $\forall \alpha \in I_*$.

Thus γ transports on x^θ , the structure of $m\theta$ completion of $(2^\theta, \omega_\alpha)$.

□

Definition 4.5 : If X is a non empty set and $x \in X$.

One will call the $m\theta$ set (x^θ, F_α) an atom of θ -valent structure set of X .

Definition 4.6 : Let (E, F_α) be a $m\theta$ set.

Let $X \subseteq E$ such that $X \neq \emptyset$.

Let $x \in E$ and $(2^\theta, \omega_\alpha)$ be the $m\theta$ completion of $I_\theta = (I, F_\alpha)$.

1. One calls the $m\theta$ belonging of x to X , the element of 2^θ denoted $v_x^\theta(x)$ and defined as follows:

$$\omega_\alpha(v_x^\theta(x)) = \begin{cases} 1 & \text{if } F_\alpha x \in F_\alpha X \neq C(E, F_\alpha) \\ 0 & \text{if not } \begin{cases} F_\alpha x \notin F_\alpha X \neq C(E, F_\alpha) \text{ or } \\ F_\alpha X = C(E, F_\alpha) \end{cases} \end{cases}$$

2. One calls $m\theta$ characteristic function of X the $m\theta$ morphism from (E, F_α) to $(2^\theta, \omega_\alpha)$ denoted v_X^θ and defines as follows.

(i) If $\forall \alpha \in I_*$, $F_\alpha(X) = C(E, F_\alpha)$, then $v_X^\theta = \chi_X$ the characteristic function of X .

(2i) If not $\exists \mu \in I_*$, $F_\mu(X) \neq C(E, F_\alpha)$, then v_X^θ is the application from E into 2^θ such that the value in x , $x \in E$ is $v_X^\theta(x)$.

Definition 4.7 : Let be X a non empty set and (E, F_α) a $m\theta$ set .

X is a $m\theta$ sub-set of (E, F_α) if $X \subseteq E$; $(X, F_{\alpha|_X})$ is a $m\theta$ set.

Theorem 4.1 : Let consider (E, F_α) and (E', F'_α) two $m\theta$ sets.

Let set

- $(E, F_\alpha) \times (E', F'_\alpha) = (E \times E', F_\alpha \times F'_\alpha)$
- $F_\alpha \times F'_\alpha = (F_\alpha, F'_\alpha) :: E \times E' \rightarrow E \times E'$
 $(x, y) \mapsto (F_\alpha x, F'_\alpha y)$

$(E \times E', F_\alpha \times F'_\alpha)$ is a $m\theta$ set.

Proof :

$$\begin{aligned} 1^*) \quad C(E \times E', F_\alpha \times F'_\alpha) &= \{(x, y) \in E \times E' / \forall \alpha \in I_*, F_\alpha x = x, F'_\alpha y = y\} \\ &= \{(x, y) \in E \times E' / x \in C(E, F_\alpha), y \in C(E', F'_\alpha)\} = C(E, F_\alpha) \times C(E', F'_\alpha) \neq \emptyset. \end{aligned}$$

$$2^*) \alpha \neq \beta \Rightarrow (F_\alpha \neq F_\beta \text{ and } F'_\alpha \neq F'_\beta) \Rightarrow F_\alpha \times F'_\alpha \neq F_\beta \times F'_\beta.$$

$$3^*) F_\beta \times F'_\beta \circ F_\alpha \times F'_\alpha = (F_\beta, F'_\beta) \circ (F_\alpha, F'_\alpha) = (F_\beta \circ F_\alpha, F'_\beta \circ F'_\alpha) = (F_\alpha, F'_\alpha) = F_\alpha \times F'_\alpha.$$

$$4^*) \forall \alpha \in I_*, F_\alpha \times F'_\alpha(x, y) = F_\alpha \times F'_\alpha(a, b) \Rightarrow (F_\alpha x, F'_\alpha y) = (F_\alpha a, F'_\alpha b) \Rightarrow F_\alpha x = F_\alpha a \text{ and } F'_\alpha y = F'_\alpha b \Rightarrow (x = a \text{ and } y = b) \Rightarrow (x, y) = (a, b).$$

□

5. The category of $m\theta$ sets $\text{ENS } \theta$

5.1 Definition of $\text{ENS } \theta$

Let denoted by $\text{ENS } \theta$ the category of $m\theta$ sets.

- The objects are $m\theta$ sets (E, F_α) .
- A morphism from (E, F_α) to (F, F'_α) is a $m\theta$ morphism i.e an application $f: E \rightarrow F$ such that $\forall \alpha \in I_*, F'_\alpha \circ f = f \circ F_\alpha$.
- The composition of morphisms is defined as follows.

If $f: (E, F_\alpha) \rightarrow (F, F'_\alpha)$ and $g: (F, F'_\alpha) \rightarrow (G, F''_\alpha)$ are two morphisms, then $g \circ f: (E, F_\alpha) \rightarrow (G, F''_\alpha)$ is defined by: $g \circ f: E \rightarrow G$ such that $\forall \alpha \in I_*, F''_\alpha \circ (g \circ f) = (g \circ f) \circ F_\alpha$.

- The identity morphism $i: (E, F_\alpha) \rightarrow (E, F_\alpha)$ is the identity map: $Id_E: E \rightarrow E$. Indeed, $\forall \alpha \in I_*, Id_E \circ F_\alpha = F_\alpha \circ Id_E$.

Indeed, if $f: (E, F_\alpha) \rightarrow (F, F'_\alpha)$ and $g: (F, F'_\alpha) \rightarrow (G, F''_\alpha)$ are two morphisms of $\text{ENS } \theta$, we have $\forall \alpha \in I_*, F'_\alpha \circ f = f \circ F_\alpha$ and $F''_\alpha \circ g = g \circ F'_\alpha$.

Let $\alpha \in I_*$,

$$\begin{aligned} F''_\alpha \circ (g \circ f) &= (F''_\alpha \circ g) \circ f \\ &= (g \circ F'_\alpha) \circ f \text{ because } F''_\alpha \circ g = g \circ F'_\alpha \\ &= g \circ (F'_\alpha \circ f) \\ &= g \circ (f \circ F_\alpha) \text{ because } F'_\alpha \circ f = f \circ F_\alpha \\ &= (g \circ f) \circ F_\alpha. \end{aligned}$$

The composition “ $\circ$ ” of morphisms is associative and we also have

$$Id_F \circ f = f = f \circ Id_E.$$

Definition 5.1 : Let (X, F_α) and (Y, F'_α) be two $m\theta$ sets.

1. One says that $f \in \text{End}_{\text{ENS}\theta}((X, F_\alpha)) = \text{Hom}_{\text{ENS}\theta}((X, F_\alpha), (X, F_\alpha))$ is an $m\theta$ endomorphism.

2. Let $f : (X, F_\alpha) \rightarrow (Y, F'_\alpha)$ be a $m\theta$ morphism.

A $m\theta$ retraction or a left $m\theta$ inverse of f is a $m\theta$ morphism $g : (Y, F'_\alpha) \rightarrow (X, F_\alpha)$ such that $g \circ f = Id_X$.

A $m\theta$ section or a right $m\theta$ inverse of f is a $m\theta$ $g : (Y, F'_\alpha) \rightarrow (X, F_\alpha)$ such that $f \circ g = Id_Y$.

3. If $f : (X, F_\alpha) \rightarrow (Y, F'_\alpha)$ has at once a $m\theta$ retraction g and a $m\theta$ section h , then these are unique and $h = g$. One then says that f is a $m\theta$ isomorphism and we write $f : (X, F_\alpha) \xrightarrow{\sim\theta} (Y, F'_\alpha)$

Let set $f^{-1} = g$, it is the $m\theta$ inverse of f .

4. $f : (X, F_\alpha) \rightarrow (Y, F'_\alpha)$ is a $m\theta$ monomorphism if for every $g, h : (Z, F''_\alpha) \rightarrow (X, F_\alpha)$ such that $f \circ g = f \circ h$, then $g = h$.
5. $f : (X, F_\alpha) \rightarrow (Y, F'_\alpha)$ is an $m\theta$ epimorphism if for every $g, h : (Y, F'_\alpha) \rightarrow (Z, F''_\alpha)$ such that $g \circ f = h \circ f$, then $g = h$.

Proposition 5.1 :

1. $m\theta$ isomorphisms are $m\theta$ monomorphisms and $m\theta$ epimorphisms.
2. The composition of $m\theta$ monomorphisms (resp. $m\theta$ epimorphism) is a $m\theta$ monomorphism (resp. an $m\theta$ epimorphism.)
3. If $g \circ f$ is a $m\theta$ monomorphism (resp. $g \circ f$ is an $m\theta$ epimorphism), then f is a $m\theta$ monomorphism (resp. g is an $m\theta$ epimorphism.)
4. A $m\theta$ section is a $m\theta$ monomorphism.
5. A $m\theta$ retraction is an $m\theta$ epimorphism.

Proof : As in the category $\text{ENS}\theta$ a $m\theta$ morphism is first an arrow of the category of classical sets ENS , the proof is done as in ENS . $\square$

Proposition 5.2 : The following properties are equivalent:

1. f is an $m\theta$ isomorphism.
2. f is a $m\theta$ retraction and a $m\theta$ monomorphism.

Proof : It is done as in *ENS*

□

Observation 5.1 :

1. If $\theta = 2$, then $ENS\theta = ENS$ the category of classical sets.
2. If $\theta > 2$, then for every (A, F_α) object of $ENS\theta$, $|A| > 2$.

Indeed,

if $\theta = 2$, then $I = \{0, 1\}$, and $I_* = \{1\}$, so for every

$(A, F_\alpha) \in |ENS\theta|$, $F_\alpha = F_1 = Id_A$, therefore $C(A, F_1) = A \in |ENS|$

If $\theta > 2$, then $|I_*| \geq 2$.

Let $\alpha, \beta \in I_*$ such that $\alpha \neq \beta$, we have, $F_\alpha \neq F_\beta \Rightarrow \exists x \in A$ such that $F_\alpha(x) \neq F_\beta(x)$; as $F_\alpha(x)$ and $F_\beta(x)$ are in $C(A, F_\alpha) \subsetneq A$ then $|A| > 2$.

Thus, there not exist terminal or initial objects as defined in the category of classical sets *ENS*.

5.2 The $m\theta$ topos $ENS\theta$

Let suppose that $\theta > 2$ in what follows.

Let recall that if (A, F_α) is an object of $ENS\theta$, then A is an object of *ENS* the category of classical sets.

Definition 5.2 : Let $((X_i, (F_\alpha^{(i)})_{\alpha \in I_*}))_{i \in \wedge}$ be a family objects of $ENS\theta$.

A $m\theta$ product of $((X_i, (F_\alpha^{(i)})_{\alpha \in I_*}))_{i \in \wedge}$ is an object $(X, (F_\alpha)_{\alpha \in I_*})$ with $m\theta$ projections $P_i : (X, (F_\alpha)_{\alpha \in I_*}) \rightarrow (X_i, (F_\alpha^{(i)})_{\alpha \in I_*})$ such that for each $f_i : (Y, (F'_\alpha)_{\alpha \in I_*}) \rightarrow (X_i, (F_\alpha^{(i)})_{\alpha \in I_*})$, there exists a unique $m\theta$ morphism $f : (Y, (F'_\alpha)_{\alpha \in I_*}) \rightarrow (X, (F_\alpha)_{\alpha \in I_*})$ fulfilling $\forall i \in \wedge, P_i \circ f = f_i$. The dual concept is the $m\theta$ sum.

Definition 5.3 : Let (A, F_α) and (B, F'_α) be two objects of $ENS\theta$.

Let f and g be two $m\theta$ morphisms from (A, F_α) into (B, F'_α) .

The equaliser of f and g is the tuple $((K, F_\alpha), k)$ such that $K = \{a \in A / f(a) = g(a)\}$ and $k : K \rightarrow A$ is the natural inclusion.

Proposition 5.3 :

1. K is a $m\theta$ sub-set of (A, F_α) .
2. k is $m\theta$ morphism from $(K, F_{\alpha|_K})$ into (A, F_α) .

Proof :

1. $K \subseteq A$ by definition.

Let $(a, \alpha) \in K \times I_*$ and let show that $F_\alpha(a) \in K$. As f and g are $m\theta$ morphisms from (A, F_α) into (B, F'_α) , then

$$\forall \alpha \in I_*, F'_\alpha \circ f = f \circ F_\alpha \text{ and } F'_\alpha \circ g = g \circ F_\alpha.$$

$$a \in K \Rightarrow f(a) = g(a) \text{ (accordind the definition of K)}$$

$$\Rightarrow F'_\alpha(f(a)) = F'_\alpha(g(a)), \forall \alpha \in I_*$$

$$\Rightarrow (f \circ F_\alpha)(a) = (g \circ F_\alpha)(a), \forall \alpha \in I_*$$

$$\Rightarrow f(F_\alpha(a)) = g(F_\alpha(a)), \forall \alpha \in I_*$$

$$\Rightarrow F_\alpha(a) \in K, \forall \alpha \in I_*.$$

Thus $(K, F_{\alpha|_K})$ is a $m\theta$ set.

2. $\forall \alpha \in I^*, F_\alpha \circ k = k \circ F_{\alpha|_K}$, so k is $m\theta$ morphism from $(K, F_{\alpha|_K})$ into (A, F_α) .

□

Proposition 5.4 : Let $(A, F_\alpha), (B, F'_\alpha) \in |ENS\theta|$. Let $(B, F'_\alpha)^{(A, F_\alpha)} = \{f : A \rightarrow B / \forall \alpha \in I_*, F'_\alpha(f) = f(F_\alpha)\}$.

Let $F''_\alpha : (B, F'_\alpha)^{(A, F_\alpha)} \rightarrow (B, F'_\alpha)^{(A, F_\alpha)}$ defined by: $\forall \alpha \in I_*, \forall f \in (B, F'_\alpha)^{(A, F_\alpha)}$
 $F''_\alpha(f) = F'_\alpha(f) = f(F_\alpha)$. $((B, F'_\alpha)^{(A, F_\alpha)}, F''_\alpha)$ is a $m\theta$ set.

Proof : Let $\alpha \in I_*, F''_\alpha(f) = F'_\alpha(f) : A \xrightarrow{f} B \xrightarrow{F'_\alpha} B$

$F'_\alpha(f) \in B^A$ and $F'_\alpha(F'_\alpha(f)) = F'_\alpha(f)(F_\alpha)$. Therefore, $\forall \alpha \in I_*, \forall f \in (B, F'_\alpha)^{(A, F_\alpha)}$. One has $F''_\alpha(f) \in (B, F'_\alpha)^{(A, F_\alpha)}$. As (A, F_α) and (B, F'_α) are $m\theta$ set, then $C(A, F_\alpha) \neq \emptyset$ and $C(B, F'_\alpha) \neq \emptyset$.

Let $f \in B^A$ such that $\text{Im}(f) \in C(B, F'_\alpha)$, $f|_{C(A, F_\alpha)} \in C((B, F'_\alpha)^{(A, F_\alpha)}, F''_\alpha)$, thus $C((B, F'_\alpha)^{(A, F_\alpha)}, F''_\alpha) \neq \emptyset$; $\forall \alpha, \beta \in I_*, F''_\alpha \circ F''_\beta = F''_\beta$ and $\alpha \neq \beta \Rightarrow F''_\alpha \neq F''_\beta$. □

Notation 5.1 : If (A, F_α) and (B, F'_α) are $m\theta$ set then in what follows, we will denoted $(B, F'_\alpha)^{(A, F_\alpha)} = B^{A^\theta}$ and $(B^{A^\theta}, F''_\alpha) = ((B, F'_\alpha)^{(A, F_\alpha)}, F''_\alpha)$.

Proposition 5.5 : Let (A, F_α) and (B, F'_α) be two $m\theta$ sets.

Let $e: A \times B^{A^\theta} \rightarrow B$ be the classical evaluation .
 $(a, f) \mapsto f(a)$

e is a $m\theta$ morphism.

Proof: Let recall that if (A, F_α) and (B, F'_α) are $m\theta$ sets then $(B^{A^\theta}, F''_\alpha)$ and $(A \times B^{A^\theta}, F_\alpha \times F''_\alpha)$ are $m\theta$ sets . Let show that $\forall \alpha \in I_*, F'_\alpha \circ e = e \circ (F_\alpha \times F''_\alpha)$.
 Let $\alpha \in I_*$ and $(a, f) \in A \times B^{A^\theta}$.

$$\begin{aligned} F'_\alpha \circ e(a, f) &= F'_\alpha(f(a)) \\ &= f(F_\alpha(a)) \text{ because } \forall \alpha \in I_*, F'_\alpha \circ f = f \circ F_\alpha \\ &= f(F_\alpha(F_\alpha(a))) \text{ because } F_\alpha \circ F_\alpha = F_\alpha \\ &= e(F_\alpha(a), f(F_\alpha(a))) = e(F_\alpha(a), F'_\alpha(f)) = e(F_\alpha(a), F''_\alpha(f)) \\ &= e \circ (F_\alpha \times F''_\alpha)(a, f). \end{aligned}$$

□

Definition 5.4 : The application e defined as above will be called the $m\theta$ evaluation.

Proposition 5.6 : Let $(A, F_\alpha), (B, F'_\alpha)$ and (X, F_α^X) be three $m\theta$ sets.

Let $(A \times X, F_\alpha \times F_\alpha^X)$ be the product $m\theta$ set of (A, F_α) and (X, F_α^X) .

Let f be a $m\theta$ morphism from $(A \times X, F_\alpha \times F_\alpha^X)$ into (B, F'_α) .

There exists a single $m\theta$ morphism $\bar{f}$ from (X, F_α^X) into $(B^{A^\theta}, F''_\alpha)$ such that $f = e \circ (1_A \times \bar{f})$.

Proof : f is a $m\theta$ morphism from $(A \times X, F_\alpha \times F_\alpha^X)$ into (B, F'_α) , so $\forall \alpha \in I_*, F'_\alpha \circ f = f \circ (F_\alpha \times F_\alpha^X)$. As the category ENS is a topos, then there exists a single morphism $\bar{f}: X \rightarrow B^{A^\theta}$ such that $f = e \circ (1_A \times \bar{f})$. Let show that $\bar{f}$ is a $m\theta$ morphism i.e $\forall \alpha \in I_*, F''_\alpha \circ \bar{f} = \bar{f} \circ F_\alpha^X$. Let $\alpha \in I_*, x \in X$ and $a \in A$.

$$\begin{aligned}
(F''_\alpha \circ \bar{f}(x))(F_\alpha(a)) &= e(F_\alpha(a), F''_\alpha(\bar{f}(x))) \\
&= e \circ (F_\alpha \times F''_\alpha)(a, \bar{f}(x)) \\
&= F'_\alpha \circ e(a, \bar{f}(x)) \text{ because } e \circ (F_\alpha \times F''_\alpha) = F'_\alpha \circ e \\
&\quad \text{according to the above proposition} \\
&= F'_\alpha \circ f(a, x) \text{ because } f = e \circ (1_A \times \bar{f}) \\
&= f \circ (F_\alpha, F_\alpha^X)(a, x) \text{ because } F'_\alpha \circ f = f \circ (F_\alpha \times F_\alpha^X) \\
&= f(F_\alpha(a), F_\alpha^X(x)) \\
&= e \circ (1_A \times \bar{f})(F_\alpha(a), F_\alpha^X(x)) = e \circ (F_\alpha(a), \bar{f}(F_\alpha^X(x))) \\
&= \bar{f}(F_\alpha^X(x))(F_\alpha(a)).
\end{aligned}$$

Thus $F''_\alpha \circ \bar{f} = \bar{f} \circ F_\alpha^X$. Therefore $\bar{f}$ is a m θ morphism. $\square$

Observation 5.2 : Let (x^θ, F_α) be an atom of $\{x\}$.

Let $(2^\theta, \omega_\alpha)$ be the m θ completion of $I_\theta = (I, F_\alpha)$.

Let recall that $\gamma : \begin{matrix} 2^\theta & \rightarrow & x^\theta \\ \tau & \mapsto & \hat{x}_\tau \end{matrix}$ is a m θ bijection.

Thus, $\gamma^{-1} : x^\theta \rightarrow 2^\theta$ is also a m θ bijection.

According to the m θ chrysippian logic, $(2^\theta, \omega_\alpha)$ is considered as the m θ set of m θ truth values. So, if $\hat{x}_\tau \in x^\theta$, then $\gamma^{-1}(\hat{x}_\tau) = \tau$ is considered as the m θ truth value of $\hat{x}_\tau$.

Proposition 5.7 : Let (A, Ω_α) be a m θ set and (x^θ, F_α) defined as above.

Let f_A be the map defined from A into x^θ such that for every $a \in A$ $f_A(a) = x_\tau$ such that $\forall \beta \in I_*$, $\Omega_\beta(a) = x_{\omega_\beta(\tau)}$. f_A is a m θ morphism.

Proof : Let $a \in A$ and $\lambda \in I_*$.

$$F_\lambda(f_A(a)) = F_\lambda(x_\tau) = x_{\omega_\lambda(\tau)} = f_A(\Omega_\lambda(a)). \quad \text{Therefore } \forall \lambda \in I_*, F_\lambda \circ f_A = f_A \circ \Omega_\lambda. \quad \square$$

Corollary 5.1 : Let $Im(f_A) = f_A(A)$ the image of the map f_A .

$$(Im(f_A), F_{a|_{Im(f_A)}}) \text{ is a m}\theta \text{ sub-set of } (x^\theta, F_\alpha).$$

Definition 5.5 : Let denote $1^{(A, \Omega_\alpha)} = (Im(f_A), F_{\alpha|_{Im(f_A)}})$. One will call $1^{(A, \Omega_\alpha)}$ the A - $m\theta$ terminal object.

Proposition 5.8 : Let (B, F_α^B) be a the $m\theta$ sub-set of the $m\theta$ set (A, F_α^A) .

The following diagram is a pullback diagram.

$$\begin{array}{ccc}
 (B, F_\alpha^B) & \xrightarrow{f_B} & 1^{(B, F_\alpha^B)} \\
 \downarrow i & & \downarrow \gamma_{Im(f_B)}^{-1} \\
 (A, F_\alpha^A) & \xrightarrow{\chi_B^\theta, \omega_\alpha} & 1^{(A, F_\alpha^A)}
 \end{array}$$

where χ_B^θ is ch θ -characteristic function of B .

Proof : Let $y \in B$,

Let recall that $f_B(y) = x_\tau$ such that $\forall \lambda \in I_*, F_\lambda^B(y) = x_{\omega_\lambda(\tau)}$.

$$\gamma_{Im(f_B)}^{-1}(f_B(y)) = \gamma_{Im(f_B)}^{-1}(x_\tau) = \tau = \chi_B^\theta(y) = \chi_B^\theta \circ i(y).$$

Thus $\gamma_{Im(f_B)}^{-1} \circ f_B = \chi_B^\theta \circ i$, and we have the commutativity of the diagram.

Let set $g = \gamma_{Im(f_B)}^{-1}$. As g is injective, then there exists an $m\theta$ isomorphism between $1^{(B, F_\alpha^B)}$ and $(Im(g), \omega_{\alpha|_{Im(g)}})$. As $(\chi_B^\theta)^{-1}(Im(g)) = B$, then accor-dingly the commuting diagram above is a pullback. $\square$

Remark 5.1 : f_B is a unique $m\theta$ morphism such that the above diagram is pullback diagram.

Indeed, let h_B be a $m\theta$ morphism such that the following diagram is a pullback.

$$\begin{array}{ccc}
 (B, F_\alpha^B) & \xrightarrow{h_B} & 1^{(B, F_\alpha^B)} \\
 \downarrow i & & \downarrow \gamma_{Im(f_B)}^{-1} \\
 (A, F_\alpha^A) & \xrightarrow{\chi_B^\theta, \omega_\alpha} & 1^{(A, F_\alpha^A)}
 \end{array}$$

Where $1^{(B, F_\alpha^B)} = (Im(h_B), F_{\alpha|_{Im(h_B)}}^B)$.

We have $g \circ h_B = \chi_B^\theta \circ i = g \circ f_B \Rightarrow g \circ h_B = g \circ f_B \Rightarrow h_B = f_B$ because $g = \gamma_{lm(f_B)}^{-1}$ is injective.

Definition 5.6 : Let (A, F_α^A) be an object of $ENS \theta$.

Let $X = \{x\}$ an non empty set of one element. Let (x^θ, F_α) be an atom of $X = \{x\}$. The $m\theta$ set (x^θ, F_α) will be called the $m\theta$ extension of A - $m\theta$ terminal objects.

Definition 5.7 : Every $m\theta$ set that is $m\theta$ isomorphic to $(2^\theta, \omega_\alpha)$ will be called the $m\theta$ subobjects classifier in $ENS \theta$.

Definition 5.8 : We call the category $ENS \theta$ a $m\theta$ topos.

6. Conclusion

In this paper we have presented the notion of $m\theta$ set as defined in [1] by F.AYISSI ETEME. It appears that the $m\theta$ sets are objects of a category denoted $ENS \theta$. We have investigated if properties of toposes are verified in the category $ENS \theta$. It appears that the category $ENS \theta$ is not a topos as in the classical view, but it looks like a topos in the $m\theta$ approach. Thus, we have defined notions of $m\theta$ terminal object, of $m\theta$ subobjects classifier and of $m\theta$ extension of A - $m\theta$ terminal objects. So the category $ENS \theta$ is a $m\theta$ topos.

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