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Implicative equality algebras and annihilators in equality algebras

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Abstract

In this paper, the concept of an implicative equality algebra is defined and related properties are investigated. Characterizations of an implicative equality algebra and the

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relation among implicative and positive implicative equality algebras are discussed. Also, we define the notion of annihilator of equality algebras and investigate some traits of it and prove that annihilator of any nonempty set of equality algebras is a deductive system. Moreover, by using this notion, we define the operation $\circledast$ and prove that for any commutative equality algebra $\mathcal{E}$, $(\mathcal{DS}^*(\mathcal{E}), \circledast, \{1\})$ is a bounded implicative BCK-algebra.

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1. Introduction

An equality algebra is defined by Jenei in [7] and studied by other researchers in [1, 2, 3, 4, 5, 8, 13]. The basic idea for examining equations of equality is taken from EQ-algebras of Novák et al. [11]. The equality algebra has two main operations which are called meet and equivalence, and one integral element. Novák et al. [11] discussed relations among equality algebras and BCK-algebras by introducing a closure operator in the class of equality algebras. The relation between equality algebra with other algebraic structures are obtained in [12, 13] such as residuated lattice, MTL-algebra, BL-algebra and etc. They showed that under special conditions, they are equivalent. Moreover, commutative equality algebras and characterizations of them are discussed by Zebardast et al. in [13]. In [1], Aaly Kologani et al. introduced the concept of &-equality algebras, and investigated related traits. Using &-equality algebras, they induced (commutative ordered) semigroups. They provided conditions for an equality algebra to be an &-equality algebra, and also fined conditions for an equality algebra to be a commutative ones. They introduced the notion of terminal section of an element and investigated several traits of it. Using the notion of terminal section, they considered conditions for an equality algebra to be a commutative equality algebra. Given a subset of an equality algebra, they defined an $\circ$-set and $*$-set, and then they investigated several related properties. They considered conditions for the $\circ$-set (resp., $*$-set) to be deductive systems.

In this paper, the concept of an implicative equality algebra is defined, and related properties are investigated. Characterizations of an implicative equality algebra and the relation between implicative and positive implicative equality algebras are discussed. Also, we define the notion of annihilator of equality algebras and investigate some traits of it and prove that annihilator of any nonempty set of equality algebras is a deductive system. Moreover, by using this notion, we define the operation

$\circledast$ and prove that for each commutative equality algebra $\mathfrak{E}$, $(\mathcal{DS}^*(\mathfrak{E}),\ \circledast,\ \{1\})$ is a bounded implicative BCK-algebra.

2. Preliminaries

In this section, we refer to the basic concepts and features required in the field of hoop and fuzzy sets that we will use in the following sections.

Definition 2.1 : (7). An equality algebra is an algebra structure $\mathfrak{E} = (\mathfrak{E}, \wedge, \sim, 1)$ such that

(E1) $(\mathfrak{E}, \wedge, 1)$ is a commutative idempotent integral monoid (i.e., meet semilattice with the top element 1),

(E2) The operation "$\sim$" is commutative,

(E3) $(\forall \iota \in \mathfrak{E})(\iota \sim \iota = 1)$,

(E4) $(\forall \iota \in \mathfrak{E})(\iota \sim 1 = \iota)$,

(E5) $(\forall \iota, \zeta, \varepsilon \in \mathfrak{E})(\iota \preceq \zeta \preceq \varepsilon \Rightarrow \iota \sim \varepsilon \preceq \zeta \sim \varepsilon, \iota \sim \varepsilon \preceq \iota \sim \zeta)$,

(E6) $(\forall \iota, \zeta, \varepsilon \in \mathfrak{E})(\iota \sim \zeta \preceq (\iota \wedge \varepsilon) \sim (\zeta \wedge \varepsilon))$,

(E7) $(\forall \iota, \zeta, \varepsilon \in \mathfrak{E})(\iota \sim \zeta \preceq (\iota \sim \varepsilon) \sim (\zeta \sim \varepsilon))$,

where $\iota \preceq \zeta$ iff $\iota \wedge \zeta = \iota$.

Note : From now on the symbol $\mathfrak{E}$ means an equality algebra such as $(\mathfrak{E}, \wedge, \sim, 1)$.

In $\mathfrak{E}$, we define two operations "$\rightarrow$" and "$\leftrightarrows$" on $\mathfrak{E}$ as follows:

$$\iota \rightarrow \zeta := \iota \sim (\iota \wedge \zeta), \tag{2.1}$$

$$\iota \leftrightarrows \zeta := (\iota \rightarrow \zeta) \wedge (\zeta \rightarrow \iota). \tag{2.2}$$

Proposition 2.2 : [(7)]. For any $\iota, \zeta, \varepsilon \in \mathfrak{E}$, the next assertions are valid:

$$\iota \rightarrow \zeta = 1 \Leftrightarrow \iota \preceq \zeta, \tag{2.3}$$

$$\iota \rightarrow (\zeta \rightarrow \varepsilon) = \zeta \rightarrow (\iota \rightarrow \varepsilon), \tag{2.4}$$

$$1 \rightarrow \iota = \iota, \iota \rightarrow 1 = 1, \iota \rightarrow \iota = 1, \tag{2.5}$$

$$\iota \preceq \zeta \rightarrow \varepsilon \Leftrightarrow \zeta \preceq \iota \rightarrow \varepsilon, \tag{2.6}$$

$$\iota \preceq \zeta \rightarrow \iota, \tag{2.7}$$

$$\iota \preceq (\iota \rightarrow \zeta) \rightarrow \zeta, \tag{2.8}$$

$$\iota \rightarrow \zeta \preceq (\zeta \rightarrow \varepsilon) \rightarrow (\iota \rightarrow \varepsilon), \tag{2.9}$$

$$\zeta \preceq \iota \Rightarrow \iota \leftrightarrows \zeta = \iota \rightarrow \zeta = \iota \sim \zeta, \tag{2.10}$$

$$\iota \sim \zeta \preceq \iota \leftrightarrows \zeta \preceq \iota \rightarrow \zeta, \tag{2.11}$$

$$\iota \preceq \zeta \Rightarrow \begin{cases} \zeta \to \varepsilon \preceq \iota \to \varepsilon, \\ \varepsilon \to \iota \preceq \varepsilon \to \zeta \end{cases} \tag{2.12}$$

The structure $\mathfrak{E}$ is said to be *bounded* if it has the least element $0 \in \mathfrak{E}$ such that $0 \preceq \iota$ for every $\iota \in \mathfrak{E}$. If $\mathfrak{E}$ is bounded, then the operation "$\neg$" on $\mathfrak{E}$ can be defined as $\neg\iota = \iota \to 0 = \iota \sim 0$, for every $\iota \in \mathfrak{E}$.

A $\varnothing \neq \mathcal{H} \subseteq \mathfrak{E}$ is said to be a *deductive system* of $\mathfrak{E}$ (see [8]) if:

$$1 \in \mathcal{H}, \tag{2.13}$$

$$(\forall \iota, \zeta \in \mathfrak{E})(\iota \in \mathcal{H}, \iota \preceq \zeta \Rightarrow \zeta \in \mathcal{H}), \tag{2.14}$$

$$(\forall \iota, \zeta \in \mathfrak{E})(\iota \in \mathcal{H}, \iota \sim \zeta \in \mathcal{H} \Rightarrow \zeta \in \mathcal{H}). \tag{2.15}$$

All deductive systems of $\mathfrak{E}$ is shown by $\mathcal{DS}(\mathfrak{E})$.

Lemma 2.3 : ([6]). *An $\mathcal{H} \subseteq \mathfrak{E}$ is a deductive system of $\mathfrak{E}$ iff it satisfies (2.13) and*

$$(\forall \iota, \zeta \in \mathfrak{E})(\iota \in \mathcal{H}, \iota \to \zeta \in \mathcal{H} \Rightarrow \zeta \in \mathcal{H}). \tag{2.16}$$

Definition 2.4 : ([13]). A structure $\mathfrak{E}$ is said to be commutative if it satisfies:

$$(\forall \varrho, \varpi \in \mathfrak{E})((\varrho \to \varpi) \to \varpi = (\varpi \to \varrho) \to \varrho). \tag{2.17}$$

Definition 2.5 : ([2]). A structure $\mathfrak{E}$ is said to be positive implicative if:

$$(\forall \varrho, \varpi, \theta \in \mathfrak{E})(\varrho \to (\varpi \to \theta) = (\varrho \to \varpi) \to (\varrho \to \theta)). \tag{2.18}$$

3. Implicative equality algebras

Here, we study the concept of an implicative equality algebra and survey some properties of it and the relation among implicative and positive implicative equality algebras. Also, we study that under which condition, an equality algebra can be an implicative equality algebra.

Definition 3.1 : A structure $\mathfrak{E}$ is implicative if it satisfies:

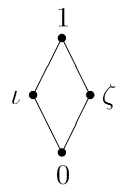
$$(\forall \varrho, \varpi \in \mathfrak{E})((\varrho \to \varpi) \to \varrho = \varrho). \tag{3.1}$$

Table 1

Cayley table for the implication " $\sim$ "

$\sim$	0	ι	ζ	1
0	1	ζ	ι	0
ι	ζ	1	0	ι
ζ	ι	0	1	ζ
1	0	ι	ζ	1

Example 3.2 : Consider $\mathfrak{E} = \{0, \iota, \zeta, 1\}$ has the next Hasse diagram.



Then define an operation $\sim$ on $\mathfrak{E}$ by Table 1.

Thus $\mathfrak{E}$ is an implicative equality algebra, and the implication $(\to)$ is given by Table 2.

Lemma 3.3 : ([2]). *An equality algebra $\mathcal{E}$ is positive implicative iff*

$$(\forall \varrho, \varpi \in \mathfrak{E})(\varrho \to (\varrho \to \varpi) = \varrho \to \varpi). \tag{3.2}$$

Lemma 3.4 : ([13]). *For any equality algebra $\mathfrak{E}$, the following are equivalent.*

(1) $\mathfrak{E}$ *is commutative,*

(2) $(\forall \varrho, \varpi \in \mathfrak{E})((\varrho \to \varpi) \to \varpi \preceq (\varpi \to \varrho) \to \varrho)$.

Theorem 3.5 : *The structure $\mathfrak{E}$ is implicative iff it is positive implicative and commutative.*

Table 2

Cayley table for the implication "$\to$"

$\to$	0	ι	ζ	1
0	1	1	1	1
ι	ζ	1	ζ	1
ζ	ι	ι	1	1
1	0	ι	ζ	1

Proof : $(\Rightarrow)$ For each $\varrho, \varpi \in \mathfrak{E}$, we have

$$\varrho \to (\varrho \to \varpi) = ((\varrho \to \varpi) \to \varrho) \to (\varrho \to \varpi) = \varrho \to \varpi,$$

for all $\varrho, \varpi \in \mathfrak{E}$. Hence $\mathfrak{E}$ is positive implicative by Lemma 3.3. Let $\varrho, \varpi \in \mathfrak{E}$. Using (2.9) and (3.1), we have

$$(\varpi \to \varrho) \to \varrho = (\varpi \to \varrho) \to ((\varrho \to \varpi) \to \varrho) \geq (\varrho \to \varpi) \to \varpi,$$

and so $\mathfrak{E}$ is commutative by Lemma 3.4.

$(\Leftarrow)$ Suppose $\varrho, \varpi \in \mathfrak{E}$. Then by using (2.5), (2.17) and (3.2), we get

$$\begin{aligned}&((\varrho \to \varpi) \to \varrho) \to \varrho \\ &= (\varrho \to (\varrho \to \varpi)) \to (\varrho \to \varpi) = (\varrho \to (\varrho \to \varpi)) \to (\varrho \to (\varrho \to \varpi)) = 1,\end{aligned}$$

and so $(\varrho \to \varpi) \to \varrho = \varrho$. Hence $\mathfrak{E}$ is an implicative equality algebra. □

Proposition 3.6 : If $\mathfrak{E}$ is implicative, then the following assertions hold:

$$(\forall \varrho, \varpi \in \mathfrak{E})((\varpi \to \varrho) \to ((\varpi \to \varrho) \to \varrho) = (\varrho \to \varpi) \to \varpi). \quad (3.3)$$

$$(\forall \varrho, \varpi \in \mathfrak{E})((\varrho \to \varpi) \to ((\varpi \to \varrho) \to \varrho) = (\varrho \to \varpi) \to \varpi). \quad (3.4)$$

$$(\forall \varrho, \varpi \in \mathfrak{E})((\varrho \to \varpi) \to ((\varrho \to \varpi) \to \varpi) = (\varpi \to \varrho) \to ((\varpi \to \varrho) \to \varrho)). \quad (3.5)$$

Proof : Assume $\mathfrak{E}$ is implicative. Then $\mathfrak{E}$ is positive implicative and commutative by Theorem 3.5. Hence

$$(\varrho \to \varpi) \to \varpi = (\varpi \to \varrho) \to \varrho = ((\varpi \to \varrho) \to ((\varpi \to \varrho) \to \varrho),$$

$$(\varrho \to \varpi) \to \varpi = (\varrho \to \varpi) \to ((\varrho \to \varpi) \to \varpi) = (\varrho \to \varpi) \to ((\varpi \to \varrho) \to \varrho),$$

and

$$(\varrho \to \varpi) \to ((\varrho \to \varpi) \to \varpi) = (\varrho \to \varpi) \to \varpi = (\varpi \to \varrho) \to \varrho = (\varpi \to \varrho) \to ((\varpi \to \varrho) \to \varrho),$$

for each $\varrho, \varpi \in \mathfrak{E}$. □

Lemma 3.7 : *If an equality algebra $\mathfrak{E}$ satisfies the condition (3.3), then it satisfies the condition (3.4).*

Proof : Assume $\mathfrak{E}$ satisfies the condition (3.3). Then,

$$(\varrho \to \varpi) \to \varpi = (\varpi \to \varrho) \to ((\varpi \to \varrho) \to \varrho) \geq (\varpi \to \varrho) \to \varrho,$$

and so $(\varrho \to \varpi) \to \varpi = (\varpi \to \varrho) \to \varrho$ for all $\varrho, \varpi \in \mathfrak{E}$. Thus

$$(\varrho \to \varpi) \to \varpi = (\varrho \to \varpi) \to ((\varrho \to \varpi) \to \varpi) = (\varrho \to \varpi) \to ((\varpi \to \varrho) \to \varrho),$$

for all $\varrho, \varpi \in \mathfrak{E}$. □

Lemma 3.8 : *Consider $\mathfrak{E}$ satisfies in the condition (3.4). Then it satisfies the condition (3.5).*

Proof : Suppose $\mathfrak{E}$ satisfies in the condition (3.4). Then, for every $\rho, \varpi \in \mathfrak{E}$, by (2.7), we have,

$$(\varrho \to \varpi) \to \varpi = (\varrho \to \varpi) \to ((\varpi \to \varrho) \to \varrho) \geq (\varpi \to \varrho) \to \varrho$$

and so $(\varrho \to \varpi) \to \varpi = (\varpi \to \varrho) \to \varrho$ for all $\varrho, \varpi \in \mathfrak{E}$. Thus,

$$\begin{aligned}
(\varrho \to \varpi) \to ((\varrho \to \varpi) \to \varpi) &= (\varrho \to \varpi) \to ((\varpi \to \varrho) \to \varrho) \\
&= (\varrho \to \varpi) \to \varpi \\
&= (\varpi \to \varrho) \to \varrho \\
&= (\varpi \to \varrho) \to ((\varpi \to \varrho) \to \varrho) \\
&= (\varpi \to \varrho) \to ((\varrho \to \varpi) \to \varpi) \\
&= (\varpi \to \varrho) \to ((\varpi \to \varrho) \to \varrho)
\end{aligned}$$

Hence, the condition (3.5) holds. □

We provide a condition for an equality algebra to be implicative.

Theorem 3.9 : *If $\mathfrak{E}$ satisfies in the condition (3.5), then $\mathfrak{E}$ is implicative.*

Proof : Assume $\mathfrak{E}$ is satisfing (3.5) and $\varrho, \varpi \in \mathfrak{E}$. Then

$$\begin{aligned}
\varpi \to \varrho &= 1 \to (1 \to (\varpi \to \varrho)) \\
&= (\varrho \to (\varpi \to \varrho)) \to ((\varrho \to (\varpi \to \varrho)) \to (\varpi \to \varrho)) \\
&= ((\varpi \to \varrho) \to \varrho) \to (((\varpi \to \varrho) \to \varrho) \to \varrho) \\
&= ((\varpi \to \varrho) \to \varrho) \to (\varpi \to \varrho) \\
&= \varpi \to (((\varpi \to \varrho) \to \varrho) \to \varrho) \\
&= \varpi \to (\varpi \to \varrho),
\end{aligned}$$

which implies that

$$\begin{aligned}
(\varrho \to \varpi) \to \varpi &= (\varrho \to \varpi) \to ((\varrho \to \varpi) \to \varpi) \\
&= (\varpi \to \varrho) \to ((\varpi \to \varrho) \to \varrho) = (\varpi \to \varrho) \to \varrho.
\end{aligned}$$

This showes that $\mathfrak{E}$ is positive implicative and commutative, and so $\mathfrak{E}$ is implicative by Theorem 3.5. □

Corollary 3.10 : If $\mathfrak{E}$ satisfies in the condition (3.3) or (3.4), then $\mathfrak{E}$ is implicative.

Proof : By Lemmas 3.7, 3.8 and Theorem 3.9, the proof is clear. $\square$

4. Annihilators

Here, we define the concept of annihilator of equality algebras and peruse properties of it and prove that annihilator of any non-empty set of equality algebras is a deductive system. Moreover, by using this notion, we define the operation $\divideontimes$ and prove that for any commutative equality algebra $\mathfrak{E}$, $(\mathcal{DS}^*(\mathfrak{E}), \divideontimes, \{1\})$ is a bounded implicative BCK-algebra.

Given a subset $\mathcal{H}$ of $\mathfrak{E}$, consider the sets

$$\langle \mathcal{H} \rangle := \{\rho \in \mathfrak{E} \mid \iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varrho))) = 1 \text{ for some } \iota_1, \iota_2, \cdots, \iota_n \in \mathcal{H}\}. \quad (4.1)$$

$$Ann(\mathcal{H}) := \{\varrho \in \mathfrak{E} \mid (\varrho \to \iota) \to \iota = 1 \text{ for all } \iota \in \mathcal{H}\}. \quad (4.2)$$

If $\mathcal{H} = \{\iota\}$, then $Ann(\{\iota\})$ is simply denoted by $Ann(\iota)$, that is,

$$Ann(\iota) = \{\varrho \in \mathfrak{E} \mid (\varrho \to \iota) \to \iota = 1\}, \quad (4.3)$$

respectively. Thus $Ann(\mathcal{H}) = \bigcap_{\iota \in \mathcal{H}} Ann(\iota)$. Since $\iota \preceq \varrho \to \iota$ for all $\iota, \varrho \in \mathfrak{E}$, the sets (4.2) and (4.3) are equivalent to the following sets.

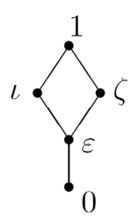
$$Ann(\mathcal{H}) := \{\varrho \in \mathfrak{E} \mid \varrho \to \iota = \iota \text{ for all } \iota \in \mathcal{H}\}. \quad (4.4)$$

and

$$Ann(\iota) = \{\varrho \in \mathfrak{E} \mid \varrho \to \iota = \iota\}, \quad (4.5)$$

respectively.

Example 4.1 : Let $\mathfrak{E} = \{0, \iota, \zeta, \varepsilon, 1\}$ has the next Hasse diagram.



Then define a binary operation $\sim$ on $\mathfrak{E}$ by Table 3.

Table 3

Cayley table for the implication "$\sim$"

$\sim$	0	ι	ζ	ε	1
0	1	0	0	0	0
ι	0	1	ε	ζ	ι
ζ	0	ε	1	ι	ζ
ε	0	ζ	ι	1	ε
1	0	ι	ζ	ε	1

Thus, $\mathcal{E} := (\mathfrak{E}, \wedge, \sim, 1)$ is an equality algebra, and the implication "$\to$" is given by Table 4.

Table 4

Cayley table for the implication "$\to$"

$\to$	0	ι	ζ	ε	1
0	1	1	1	1	1
ι	0	1	ζ	ζ	1
ζ	0	ι	1	ι	1
ε	0	1	1	1	1
1	0	ι	ζ	ε	1

If $X = \{0, \iota\}$, then $Ann(X) = \{\zeta, 1\}$. If $X = \{\iota, \zeta\}$ and $Y = \{\varepsilon\}$, then $\langle X\rangle = \{\iota, \zeta, 1\}$ and $\langle Y\rangle = \{\iota, \zeta, \varepsilon, 1\}$.

Theorem 4.2 : *Given a subset $\mathcal{H}$ of $\mathfrak{E}$, the set $\langle\mathcal{H}\rangle$ is the smallest deductive system of $\mathfrak{E}$ which contains $\mathcal{H}$.*

We call $\langle\mathcal{H}\rangle \in \mathcal{DS}(\mathfrak{E})$ which is generated by $\mathcal{H}$.

Proof : Clearly $1 \in \langle\mathcal{H}\rangle$. Assume $\varrho, \varpi \in \mathfrak{E}$ such that $\varrho \to \varpi \in \langle\mathcal{H}\rangle$ and $\varrho \in \langle\mathcal{H}\rangle$. Then there exist $\iota_1, \iota_2, \cdots, \iota_n, \zeta_1, \zeta_2, \cdots, \zeta_m \in \mathcal{H}$ such that

$$\iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varrho))\cdots) = 1$$

and $\zeta_m \to (\cdots \to (\zeta_2 \to (\zeta_1 \to (\varrho \to \varpi)))\cdots) = 1$. Using (2.8) and (2.12), we have

$$1 = \iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varrho))\cdots) \preceq \iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to ((\varrho \to \varpi) \to \varpi)))\cdots)$$

and so $\iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to ((\varrho \to \varpi) \to \varpi)))\cdots) = 1$. From (2.4), we obtain

$$(\varrho \to \varpi) \to (\iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varpi))\cdots)) = 1,$$

that is, $\varrho \to \varpi \preceq \iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varpi))\cdots)$. Hence

$$\begin{aligned}1 &= \zeta_m \to (\cdots \to (\zeta_2 \to (\zeta_1 \to (\varrho \to \varpi)))\cdots)\\ &\preceq \zeta_m \to (\cdots \to (\zeta_2 \to (\zeta_1 \to (\iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varpi))\cdots))))\cdots),\end{aligned}$$

and so

$$\zeta_m \to (\cdots \to (\zeta_2 \to (\zeta_1 \to (\iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varpi))\cdots))))\cdots) = 1,$$

which shows that $\varpi \in \langle \mathcal{H} \rangle$. Therefore, $\langle \mathcal{H} \rangle \in \mathcal{DS}(\mathfrak{E})$. Clearly, $\mathcal{H} \subseteq \langle \mathcal{H} \rangle$. Consider $\mathcal{Q} \in \mathcal{DS}(\mathfrak{E})$ such that $\mathcal{H} \subseteq \mathcal{Q}$. Suppose $\rho \in \langle \mathcal{H} \rangle$. Then there exist $\iota_1, \iota_2, \cdots, \iota_n \in \mathcal{H}$ such that $\iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varrho))\cdots) = 1$. Thus, $\varrho \in \mathcal{Q}$ since $\mathcal{Q} \in \mathcal{DS}(\mathfrak{E})$ and $\mathcal{H} \subseteq \mathcal{Q}$. Therefore, $\langle \mathcal{H} \rangle$ is the smallest deductive system of $\mathfrak{E}$ which contains F. □

Proposition 4.3 : Given subsets $\mathcal{H}$ and $\mathcal{Q}$ of $\mathfrak{E}$, we have

(1) $\mathcal{H} \subseteq \mathcal{Q} \;\Rightarrow\; Ann(\mathcal{Q}) \subseteq Ann(\mathcal{H})$ and $Ann(Ann(\mathcal{H})) \subseteq Ann(Ann(\mathcal{Q}))$.

(2) If $\mathfrak{E}$ is commutative, then $\mathcal{H} \subseteq Ann(Ann(\mathcal{H}))$.

(3) If $\mathfrak{E}$ is commutative, then $Ann(Ann(Ann(\mathcal{H}))) = Ann(\mathcal{H})$.

(4) If $\mathfrak{E}$ is commutative, then

$$(\forall \varrho \in \mathfrak{E})(\forall \iota \in \mathcal{H})(\varrho \in Ann(\mathcal{H}) \Leftrightarrow \varrho \to \iota = \iota, \iota \to \varrho = \varrho). \tag{4.6}$$

Proof :

(1) Straightforward.

(2) Let $\iota \in \mathcal{H}$. Then $(\iota \to \varrho) \to \varrho = (\varrho \to \iota) \to \iota = 1$ for all $\varrho \in Ann(\mathcal{H})$. Hence $\iota \in Ann(Ann(\mathcal{H}))$, and so $\mathcal{H} \subseteq Ann(Ann(\mathcal{H}))$.

(3) If we replace $\mathcal{H}$ by $Ann(\mathcal{H})$ in (2), then $Ann(\mathcal{H}) \subseteq Ann(Ann(Ann(\mathcal{H})))$. Combining (1) and (2) induces $Ann(Ann(Ann(\mathcal{H}))) \subseteq Ann(\mathcal{H})$. Thus $Ann(Ann(Ann(\mathcal{H}))) = Ann(\mathcal{H})$.

(4) Assume that $\mathfrak{E}$ is commutative. If $\varrho \in Ann(\mathcal{H})$, then $(\iota \to \varrho) \to \varrho = (\varrho \to \iota) \to \iota = 1$ for all $\iota \in \mathcal{H}$. By (2.7), we get $\iota \preceq \varrho \to \iota \preceq \iota$ and $\iota \to \varrho \preceq \varrho$. Hence $\varrho \to \iota = \iota$ and $\iota \to \varrho = \varrho$. The converse is obvious. □

Lemma 4.4 : *If* $\mathcal{H} \subseteq \mathfrak{E}$, *then* $Ann(\mathcal{H}) \in \mathcal{DS}(\mathfrak{E})$.

Proof: Obviosly, $1 \in Ann(\mathcal{H})$. Suppose $\rho \in Ann(\mathcal{H})$ and $\rho \to \varpi \in Ann(\mathcal{H})$. Then $\varrho \to \iota = \iota$ and $(\varrho \to \varpi) \to \iota = \iota$ for all $\iota \in \mathcal{H}$. By (2.4), (2.7) and (2.9) we consequence

$$\iota \preceq \varpi \to \iota \preceq (\varrho \to \varpi) \to (\varrho \to \iota) = (\varrho \to \varpi) \to \iota = \iota.$$

Hence $\varpi \to \iota = \iota$, and so $\varpi \in Ann(\mathcal{H})$. Therefore, $Ann(\mathcal{H}) \in \mathcal{DS}(\mathfrak{E})$. □

Theorem 4.5 : *If $\mathcal{H} \in \mathcal{DS}(\mathfrak{E})$, then $Ann(\mathcal{H})$ is the greatest deductive system of $\mathfrak{E}$ such that $\mathcal{H} \cap Ann(\mathcal{H}) = \{1\}$.*

Proof : Let $\mathcal{H} \in \mathcal{DS}(\mathfrak{E})$. Then $Ann(\mathcal{H}) \in \mathcal{DS}(\mathfrak{E})$ by Lemma 4.4, and obviously $\{1\} \subseteq \mathcal{H} \cap Ann(\mathcal{H})$. Let $\varpi \in \mathcal{H} \cap Ann(\mathcal{H})$. Then $\varpi \to \iota = \iota$ for all $\iota \in \mathcal{H}$. Since $\varpi \in \mathcal{H}$, we obtain $\varpi = \varpi \to \varpi = 1$. Hence $\mathcal{H} \cap Ann(\mathcal{H}) = \{1\}$. Suppose $\mathcal{Q} \in \mathcal{DS}(\mathfrak{E})$ such that $\mathcal{H} \cap \mathcal{Q} = \{1\}$. Let $\varrho \in \mathcal{Q}$. If $\varrho = 1$, then $\varrho \in Ann(\mathcal{H})$. Assume that $\varrho \neq 1$. Then $\varrho \notin \mathcal{H}$ since $\mathcal{H} \cap \mathcal{Q} = \{1\}$. From $\varrho \preceq (\varrho \to \iota) \to \iota$ and $\iota \preceq (\varrho \to \iota) \to \iota$, for all $\iota \in \mathcal{H}$, $\mathcal{H}, \mathcal{Q} \in \mathcal{DS}(\mathfrak{E})$, thus $(\varrho \to \iota) \to \iota \in \mathcal{H} \cap \mathcal{Q} = \{1\}$. Hence, $\varrho \in Ann(\mathcal{H})$, and so $\mathcal{Q} \subseteq Ann(\mathcal{H})$. Therefore, $Ann(\mathcal{H})$ is the greatest deductive system of $\mathfrak{E}$. □

Lemma 4.6 : *If $\mathfrak{E}$ is commutative and $\emptyset \neq \mathcal{H} \subseteq \mathfrak{E}$, then $Ann(\mathcal{H}) = Ann(\langle \mathcal{H} \rangle)$.*

Proof: Since $\mathcal{H} \subseteq \langle \mathcal{H} \rangle$, we have $Ann(\langle \mathcal{H} \rangle) \subseteq Ann(\mathcal{H})$ by Proposition 4.3(1). Let $\rho \in Ann(\mathcal{H})$. Then $\varrho \to \iota = \iota$ and $\iota \to \varrho = \varrho$ for all $\iota \in \mathcal{H}$ by Proposition 4.3(4). Suppose $\varpi \in \langle \mathcal{H} \rangle$. Then there exist $\iota_n, \cdots, \iota_2, \iota_1 \in \mathcal{H}$ such that

$$\iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varpi))\cdots) = 1.$$

Hence

$$1 = \iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varpi))\cdots) \preceq \iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to ((\varpi \to \varrho) \to \varrho)))\cdots),$$

by (2.8) and (2.12). By (2.4) and (2.17) we consequence

$$\begin{aligned} 1 \quad &= \iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to ((\varpi \to \varrho) \to \varrho)))\cdots) \\ &= \iota_n \to (\cdots \to (\iota_2 \to ((\varpi \to \varrho) \to (\iota_1 \to \varrho)))\cdots) \\ &= \iota_n \to (\cdots \to (\iota_2 \to ((\varpi \to \varrho) \to \varrho))\cdots) \\ &\vdots \\ &= (\varpi \to \varrho) \to \varrho = (\varrho \to \varpi) \to \varpi. \end{aligned}$$

Thus $\rho \in Ann(\langle \mathcal{H} \rangle)$, and therefore $Ann(\mathcal{H}) = Ann(\langle \mathcal{H} \rangle)$. □

Table 5

Cayley table for the implication " $\sim$ "

$\sim$	0	ι	ζ	1
0	1	0	0	0
ι	0	1	ι	ι
ζ	0	ι	1	ζ
1	0	ι	ζ	1

Table 6

Cayley table for the implication "$\to$"

$\to$	0	ι	ζ	1
0	1	1	1	1
ι	0	1	1	1
ζ	0	ι	1	1
1	0	ι	ζ	1

The following example shows that Lemma 4.6 is false when the commutativity of $\mathfrak{E}$ is missed.

Example 4.7 : Consider $\mathfrak{E} = \{0, \iota, \zeta, 1\}$ is a toset where $0 \preceq \iota \preceq \zeta \preceq 1$. Define "$\sim$" and "$\to$" on $\mathfrak{E}$ by Tables 5 and 6, respectively.

Routine calculation shows that $\mathfrak{E} := (\mathfrak{E}, \wedge, \sim, 1)$ is an equality algebra where $\rho \wedge \varpi = \min\{\rho, \varpi\}$ for all $\rho, \varpi \in \mathfrak{E}$. Since $(\iota \to 0) \to 0 \neq (0 \to \iota) \to \iota$, it is not commutative. If $\mathcal{H} = \{\iota\}$, then $Ann(\mathcal{H}) = \{\zeta, 1\} \neq \{1\} = Ann(\langle \mathcal{H} \rangle)$.

Theorem 4.8 : *If* $\mathfrak{E} := (\mathfrak{E}, \wedge, \sim, 1)$ *is commutative, then*

$$Ann(\mathcal{H}) = \{\rho \in \mathfrak{E} \mid \langle \rho \rangle \cap \langle \mathcal{H} \rangle = \{1\}\}$$

for any subset $\mathcal{H}$ *of* $\mathfrak{E}$.

Proof : If $\varrho \notin Ann(\mathcal{H})$, then $(\varrho \to \iota) \to \iota \neq 1$ for some $\iota \in \mathcal{H}$. Since $\iota \preceq (\varrho \to \iota) \to \iota$ and $\varrho \preceq (\varrho \to \iota) \to \iota$, we obtain $1 \neq (\varrho \to \iota) \to \iota \in \langle \varrho \rangle \cap \langle \mathcal{H} \rangle$, and so

$$\{\varrho \in \mathfrak{E} \mid \langle \varrho \rangle \cap \langle \mathcal{H} \rangle = \{1\}\} \subseteq Ann(\mathcal{H}).$$

Let $\varrho \in Ann(\mathcal{H})$. Then $\langle \varrho \rangle \subseteq Ann(\mathcal{H})$. If $\varpi \in Ann(\mathcal{H}) \cap \langle \mathcal{H} \rangle$, then there exist $\iota_1, \iota_2, \cdots, \iota_n \in \mathcal{H}$ such that

$$\iota_n \to (\cdots \to (\iota_2 \to (\iota_1 \to \varpi))\cdots) = 1 \tag{4.7}$$

and

$$(\forall \iota \in \mathcal{H})((\iota \to \varpi) \to \varpi = (\varpi \to \iota) \to \iota = 1). \tag{4.8}$$

The condition (4.8) implies that $\iota \to \varpi = \varpi$ for all $\iota \in \mathcal{H}$. From (4.7), we get $\varpi = 1$. Therefore

$$\langle \varrho \rangle \cap \langle \mathcal{H} \rangle \subseteq Ann(\mathcal{H}) \cap \langle \mathcal{H} \rangle = \{1\}.$$

This completes the proof. □

Proposition 4.9 : If $\mathcal{H}$ and $\mathcal{Q}$ are two subsets of $\mathfrak{E}$, then

$$Ann(\mathcal{H} \cup \mathcal{Q})) = Ann(\mathcal{H}) \cap Ann(\mathcal{Q}). \tag{4.9}$$

Proof : Since $\mathcal{H}, \mathcal{Q} \subseteq \mathcal{H} \cup \mathcal{Q}$, by Proposition 4.3(1), we have $Ann(\mathcal{H} \cup \mathcal{Q}) \subseteq Ann(\mathcal{H}) \cap Ann(\mathcal{Q})$. Now, suppose $\rho \in Ann(\mathcal{H}) \cap Ann(\mathcal{Q})$. Then for any $\iota \in \mathcal{H}$ and $\zeta \in \mathcal{Q}$, we have $\varrho \to \iota = \iota$ and $\varrho \to \zeta = \zeta$. Let $\alpha \in \mathcal{H} \cup \mathcal{Q}$. Then $\alpha \in \mathcal{H}$ or $\alpha \in \mathcal{Q}$. Thus $\varrho \to \alpha = \alpha$. Hence $\varrho \in Ann(\mathcal{H} \cup \mathcal{Q})$. Therefore, $Ann(\mathcal{H} \cup \mathcal{Q}) = Ann(\mathcal{H}) \cap Ann(\mathcal{Q})$. □

Proposition 4.10 : If $\mathfrak{E}$ is commutative and $\mathcal{H}, \mathcal{Q} \in \mathcal{DS}(\mathfrak{E})$, then $\mathcal{H} \cap \mathcal{Q} = \{1\}$ iff $\mathcal{H} \subseteq Ann(\mathcal{Q})$.

Proof : If $\mathcal{H} \subseteq Ann(\mathcal{Q})$, then $\mathcal{H} \cap \mathcal{Q} \subseteq \mathcal{Q} \cap Ann(\mathcal{Q}) = \{1\}$ by Theorem 4.5. Conversely, if $\mathcal{H} \cap \mathcal{Q} = \{1\}$, then $\mathcal{H} \subseteq Ann(\mathcal{Q})$. □

For any $\iota, \varrho \in \mathfrak{E}$ and any natural number n, we denote

$$[\iota, \varrho]^n := \iota \to (\cdots \to (\iota \to (\iota \to \varrho))\cdots) \tag{4.10}$$

where ι occurs n times. Using (2.7), we know that

$$\varrho \preceq [\iota, \varrho] \preceq [\iota, \varrho]^2 \preceq \cdots \preceq [\iota, \varrho]^n \preceq \cdots.$$

Definition 4.11 : An equality algebra $\mathfrak{E}$ is said to satisfy Decending Chain Condition (DCC) if there is $n \in \mathbb{N}$ such that $[\iota, \varrho]^{n+1} = [\iota, \varrho]^n$.

Example 4.12 : Every positive implicative equality algebra satisfies the DCC since

$$[\iota, \varrho]^2 = \iota \to (\iota \to \varrho) = \iota \to \varrho = [\iota, \varrho]$$

for all $\iota, \varrho \in \mathfrak{E}$ by Lemma 3.3. Therefore any implicative equality algebra satisfies the DCC.

Theorem 4.13 : Suppose $\mathfrak{E}$ is commutative such that it is satisfying the DCC. If $\mathcal{H} \in \mathcal{DS}(\mathfrak{E})$, then $\mathcal{H} = Ann(Ann(\mathcal{H}))$.

Proof : Suppose $\mathcal{H} \in \mathcal{DS}(\mathfrak{E})$. Then $\mathcal{H} \subseteq Ann(Ann(\mathcal{H}))$ by Proposition 4.3(2). Let $\rho \notin \mathcal{H}$. Then $\iota \to \varrho \notin \mathcal{H}$ for all $\iota \in \mathcal{H}$, and so $[\iota, \varrho]^n \notin \mathcal{H}$ for $n \in \mathbb{N}$. Since $\mathfrak{E}$ satisfies DCC, there is $k \in \mathbb{N}$ such that $[\iota, \varrho]^{k+1} = [\iota, \varrho]^k$. From Proposition 4.3(4) that $[\iota, \varrho]^k \in Ann(\mathcal{H})$. Hence $[\iota, \varrho]^k \notin Ann(Ann(\mathcal{H}))$ since $Ann(\mathcal{H}) \cap Ann(Ann(\mathcal{H})) = \{1\}$. Since $\iota \in \mathcal{H} \subseteq Ann(Ann(\mathcal{H}))$, we have $[\iota, \varrho]^{k-1} \notin Ann(Ann(\mathcal{H}))$. Continuing this process, we get $\varrho \notin Ann(Ann(\mathcal{H}))$, and so $Ann(Ann(\mathcal{H})) \subseteq \mathcal{H}$. Therefore, $\mathcal{H} = Ann(Ann(\mathcal{H}))$. □

Corollary 4.14 : *Consider $\mathfrak{E}$ is implicative such that it is satisfying the DCC. Then $\mathcal{H} = Ann(Ann(\mathcal{H}))$ for any $\mathcal{H} \in \mathcal{DS}(\mathfrak{E})$.*

Proposition 4.15 : If $\mathfrak{E}$ is commutative such that

$$(\forall \mathcal{H} \in \mathcal{DS}(\mathfrak{E}))(\mathcal{H} = Ann(Ann(\mathcal{H}))), \tag{4.11}$$

then $Ann(Ann(\mathcal{Q})) = \langle \mathcal{Q} \rangle$ for all subset $\mathcal{Q}$ of $\mathfrak{E}$.

Proof : Let $\mathcal{Q}$ be a subset of $\mathfrak{E}$. Since $\mathcal{Q} \subseteq \langle \mathcal{Q} \rangle$, we have

$$Ann(Ann(\mathcal{Q})) \subseteq Ann(Ann(\langle \mathcal{Q} \rangle)) = \langle \mathcal{Q} \rangle.$$

by Proposition (4.3)(1) and (4.11). Using Proposition 4.3(2), we get $\mathcal{Q} \subseteq Ann(Ann(\mathcal{Q}))$, and so $\langle \mathcal{Q} \rangle \subseteq Ann(Ann(\mathcal{Q}))$ since $Ann(Ann(\mathcal{Q})) \in \mathcal{DS}(\mathfrak{E})$. This completes the proof.

Proposition 4.16 : Assume $\mathfrak{E}$ is commutative which satisfies the condition (4.11), then $Ann(\mathcal{H} \cap \mathcal{Q}) = \langle Ann(\mathcal{H}) \cup Ann(\mathcal{Q}) \rangle$ for all $\mathcal{H}, \mathcal{Q} \in \mathcal{DS}(\mathfrak{E})$.

Proof : Since $\mathcal{H} \cap \mathcal{Q} \subseteq \mathcal{H}, \mathcal{Q}$, by Proposition (4.3)(1) that $Ann(\mathcal{H}) \subseteq Ann(\mathcal{H} \cap \mathcal{Q})$ and $Ann(\mathcal{Q}) \subseteq Ann(\mathcal{H} \cap \mathcal{Q})$. Hence $Ann(\mathcal{H}) \cup Ann(\mathcal{Q}) \subseteq Ann(\mathcal{H} \cap \mathcal{Q})$, and so

$$\langle Ann(\mathcal{H}) \cup Ann(\mathcal{Q}) \rangle \subseteq Ann(\mathcal{H} \cap \mathcal{Q}).$$

Since $Ann(\mathcal{H}) \subseteq Ann(\mathcal{H}) \cup Ann(\mathcal{Q}) \subseteq \langle Ann(\mathcal{H}) \cup Ann(\mathcal{Q}) \rangle$ and $Ann(\mathcal{Q}) \subseteq Ann(\mathcal{H}) \cup Ann(\mathcal{Q}) \subseteq \langle Ann(\mathcal{H}) \cup Ann(\mathcal{Q}) \rangle$, We have

$$Ann(\langle Ann(\mathcal{H}) \cup Ann(\mathcal{Q}) \rangle) \subseteq Ann(Ann(\mathcal{H})) = \mathcal{H}$$

and

$$Ann(\langle Ann(\mathcal{H}) \cup Ann(\mathcal{Q}) \rangle) \subseteq Ann(Ann(\mathcal{Q})) = \mathcal{Q}$$

Hence

$$Ann(\langle Ann(\mathcal{H}) \cup Ann(\mathcal{Q})\rangle) \subseteq \mathcal{H} \cap \mathcal{Q}$$

which implies from Proposition (4.3)(1) and (4.11) that

$$Ann(\mathcal{H} \cap \mathcal{Q}) \subseteq Ann(Ann(\langle Ann(\mathcal{H}) \cup Ann(\mathcal{Q})\rangle)) = \langle Ann(\mathcal{H}) \cup Ann(\mathcal{Q})\rangle$$

This completes the proof. □

Proposition 4.17 : Consider $\mathfrak{E}$ is commutative. If $\mathcal{H}, \mathcal{Q} \in \mathcal{DS}(\mathfrak{E})$, then $\mathcal{H} \cap Ann(\mathcal{H} \cap \mathcal{Q}) \subseteq Ann(\mathcal{Q})$.

Proof : If $\mathcal{H}, \mathcal{Q} \in \mathcal{DS}(\mathfrak{E})$, then

$$\{1\} = (\mathcal{H} \cap \mathcal{Q}) \cap Ann(\mathcal{H} \cap \mathcal{Q}) = (\mathcal{H} \cap Ann(\mathcal{H} \cap \mathcal{Q})) \cap \mathcal{Q},$$

which implies from Proposition 4.10 that $\mathcal{H} \cap Ann(\mathcal{H} \cap \mathcal{Q}) \subseteq Ann(\mathcal{Q})$. □

Assume $\mathfrak{E}$ commutative and

$$\mathcal{DS}^*(\mathfrak{E}) := \{\mathcal{H} \in \mathcal{DS}(\mathfrak{E}) \mid Ann(Ann(\mathcal{H})) = \mathcal{H}\}.$$

Clearly $\{1\}, \mathfrak{E} \in \mathcal{DS}^*(\mathfrak{E})$. Define a binary operation "$\divideontimes$" and an order "$\preceq$" on $\mathcal{DS}^*(\mathfrak{E})$ by

$$(\forall \mathcal{H}, \mathcal{Q} \in \mathcal{DS}^*(\mathfrak{E}))(\mathcal{H} \divideontimes \mathcal{Q} = \mathcal{H} \cap Ann(\mathcal{Q})) \tag{4.12}$$

and

$$(\forall \mathcal{H}, \mathcal{Q} \in \mathcal{DS}^*(\mathfrak{E}))(\mathcal{H} \preceq \mathcal{Q} \Leftrightarrow \mathcal{H} \subseteq \mathcal{Q}), \tag{4.13}$$

respectively. We will show that $(\mathcal{DS}^*(\mathfrak{E}), \divideontimes, \{1\})$ is a BCK-algebra.

Lemma 4.18 : *Suppose $\mathfrak{E}$ is commutative. The set $\mathcal{DS}^*(\mathfrak{E})$ is closed under the operation "$\divideontimes$".*

Proof : Let $\mathcal{H}, \mathcal{Q} \in \mathcal{DS}^*(\mathfrak{E})$. Using Proposition 4.16, we have

$$\begin{aligned} Ann(Ann(\mathcal{H} \divideontimes \mathcal{Q})) &= Ann(Ann(\mathcal{H} \cap Ann(\mathcal{Q}))) \\ &= Ann(\langle Ann(\mathcal{H}) \cup Ann(Ann(\mathcal{Q}))\rangle) \\ &= Ann(\langle Ann(\mathcal{H}) \cup \mathcal{Q})\rangle) \\ &= Ann(Ann(\mathcal{H} \cap Ann(\mathcal{Q}))) \\ &= \mathcal{H} \cap Ann(\mathcal{Q}) \\ &= \mathcal{H} \divideontimes \mathcal{Q}, \end{aligned}$$

and so $\mathcal{H} \divideontimes \mathcal{Q} \in \mathcal{DS}^*(\mathfrak{E})$. □

Lemma 4.19 : *Assume* $\mathfrak{E}$ *is commutative. Then*

$$(\forall \mathcal{H}, \mathcal{Q} \in \mathcal{DS}^*(\mathfrak{E}))(\mathcal{H} \preceq \mathcal{Q} \Leftrightarrow \mathcal{H} \circledast \mathcal{Q} = \{1\}). \tag{4.14}$$

Proof **:** If $\mathcal{H} \preceq \mathcal{Q}$, then $\mathcal{H} \subseteq \mathcal{Q} = Ann(Ann(\mathcal{Q}))$ and so $\mathcal{H} \circledast \mathcal{Q} = \mathcal{H} \cap Ann(\mathcal{Q}) = \{1\}$ by Proposition 4.10. Conversely, if $\mathcal{H} \circledast \mathcal{Q} = \{1\}$, then $\mathcal{H} \cap Ann(\mathcal{Q}) = \{1\}$ and so $\mathcal{H} \subseteq Ann(Ann(\mathcal{Q})) = \mathcal{Q}$ by Proposition 4.10. Hence $\mathcal{H} \preceq \mathcal{Q}$. □

Theorem 4.20 : If $\mathfrak{E}$ is commutative, then $(\mathcal{DS}^*(\mathfrak{E}), \circledast, \{1\})$ is a bounded implicative BCK-algebra.

***Proof* :** Let $\mathcal{H}, \mathcal{Q}, K \in \mathcal{DS}^*(\mathfrak{E})$. Then

$$\begin{aligned}
(\mathcal{H} \circledast \mathcal{Q}) \circledast (\mathcal{H} \circledast K) &= (\mathcal{H} \cap Ann(\mathcal{Q})) \cap Ann(\mathcal{H} \cap Ann(K)) \\
&= (\mathcal{H} \cap Ann(\mathcal{H} \cap Ann(K))) \cap Ann(\mathcal{Q}) \\
&\subseteq Ann(Ann(K)) \cap Ann(\mathcal{Q}) \\
&\subseteq K \cap Ann(\mathcal{Q}) = K \circledast \mathcal{Q},
\end{aligned}$$

and so $(\mathcal{H} \circledast \mathcal{Q}) \circledast (\mathcal{H} \circledast K) \preceq K \circledast \mathcal{Q}$. Using Proposition 4.17, we have

$$\mathcal{H} \circledast (\mathcal{H} \circledast \mathcal{Q}) = \mathcal{H} \cap Ann(\mathcal{H} \circledast \mathcal{Q}) = \mathcal{H} \cap Ann(\mathcal{H} \cap Ann(\mathcal{Q})) \subseteq Ann(Ann(\mathcal{Q})) = \mathcal{Q},$$

and thus $\mathcal{H} \circledast (\mathcal{H} \circledast \mathcal{Q}) \preceq \mathcal{Q}$. Obviously $\mathcal{H} \preceq \mathcal{H}$ and $\{1\} \preceq \mathcal{H}$ for all $\mathcal{H} \in \mathcal{DS}^*(\mathfrak{E})$. Let $\mathcal{H}, \mathcal{Q} \in \mathcal{DS}^*(\mathfrak{E})$ such that $\mathcal{H} \preceq \mathcal{Q}$ and $\mathcal{Q} \preceq \mathcal{H}$. Then $\mathcal{H} \subseteq \mathcal{Q}$ and $\mathcal{Q} \subseteq \mathcal{H}$, and so $\mathcal{H} = \mathcal{Q}$. Thus $\mathfrak{E} \in \mathcal{DS}^*(\mathfrak{E})$ and $\mathcal{H} \preceq \mathfrak{E}$ for all $\mathcal{H} \in \mathcal{DS}^*(\mathfrak{E})$. For any $\mathcal{H}, \mathcal{Q} \in \mathcal{DS}^*(\mathfrak{E})$, we have $\mathcal{Q} \cap Ann(\mathcal{H}) \subseteq Ann(\mathcal{H})$, and so $\mathcal{H} = Ann(Ann(\mathcal{H})) \subseteq Ann(\mathcal{Q} \cap Ann(\mathcal{H}))$. Hence $\mathcal{H} = \mathcal{H} \cap Ann(\mathcal{Q} \cap Ann(\mathcal{H})) = \mathcal{H} \circledast (\mathcal{Q} \circledast \mathcal{H})$. Therefore $(\mathcal{DS}^*(\mathfrak{E}), \circledast, \{1\})$ is a bounded implicative BCK-algebra. □

5. Conclusion

In this paper, the concept of an implicative equality algebra is introduced, and some properties of it are investigated. Characterizations of an implicative equality algebra and the relation between implicative and positive implicative equality algebras are discussed. Also, the notion of annihilator of equality algebras are defined and some properties of it are investigated and proved that annihilator of any nonempty set of equality algebras is a deductive system. Moreover, by using this notion, the new operation $\circledast$ is introduced and proved that for any commutative equality algebra $\mathfrak{E}$, $(\mathcal{DS}^*(\mathfrak{E}), \circledast, \{1\})$ is a bounded implicative BCK-algebra.

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