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Congruence relations on pseudocomplemented almost distributive lattices

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Abstract

The concept of congruence kernels and congruence cokernels are studied in pseudocomplemented almost distributive lattices. A set of equivalent conditions are derived for a given subset J of a pseudocomplemented ADL L to be a congruence kernel (resp. a congruence cokernel). Moreover, for a given congruence kernel J of L the smallest and the largest $*-$congruences on L having J as a kernel are given and characterized in various forms.

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1. Introduction

The concept of an almost distributive lattice (in short an ADL) was introduced by Swamy and Rao [11] as a common abstraction of most of the existing ring theoretic and lattice theoretic generalizations of a Boolean algebra. An ADL is an algebra with two binary operations "$\vee$" and "$\wedge$" which satisfies most of the properties of a distributive lattice with smallest element 0, except possibly the commutativity of the binary operations "$\vee$" and "$\wedge$", and the right distributivity of "$\vee$" over "$\wedge$". The notion of pseudocomplementation on ADLs was introduced in the paper [12]. Later on, Swamy, Rao and Rao [13] introduced a class of algebras called Stone ADLs as a generalization of Stone lattices.

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T. P. Speed [9] introduced two special congruences: $\Psi(S)$ and $R(J)$ on a distributive lattice L, which are defined as follows:

$$(x, y) \in \Psi(S) \textit{ iff } x \wedge a = y \wedge a \text{ for some } a \in S$$

$$(x, y) \in R(J) \textit{ iff } \text{ for all } a \in L : a \wedge x \in J \Leftrightarrow a \wedge y \in J$$

where S is a nonempty multiplicatively closed set in L and J is an ideal of L. It is shown that the congruence $R(J)$ is the largest congruence on L having J as a zero congruence class, and if S is a filter, then $\Psi(S)$ is the smallest congruence on L having S as a unit congruence class. These congruence relations are further studied by Cornish [3] in the class of pseudo-complemented distributive lattices. In his paper, Cornish defined and characterized congruence kernels and congruence cokernels in pseudo-complemented distributive lattices. Several defining properties has been given for the largest $*-$ congruence having a given congruence kernel as a kernel.

In [1], we have studied the congruence $R(J)$ and the generalization of the Glivenko's congruence to the class of ADLs. In the present paper, we study congruence kernels (resp. congruence cokernels) in a pseudo-complemented almost distributive lattice. A set of equivalent conditions are derived for a given subset J of a pseudo-complemented ADL L to be a congruence kernel (resp. a congruence cokernel). Moreover, for a given congruence kernel J of L, the smallest and the largest $*-$ congruences on L having J as a kernel are given and characterized in various forms.

2. Preliminaries

This section contains some necessary definitions and results from [1, 6, 8, 11, 12] which will be used in the text. We refer to [2, 4] for the standard concepts in distributive lattices.

Definition 2.1 : [11] An almost distributive lattice (or shortly an ADL) is an algebra $(L, \vee, \wedge, 0)$ of type $(2, 2, 0)$ satisfying the following set of axioms:

(1) $a \vee 0 = a$,

(2) $0 \wedge a = 0$,

(3) $(a \vee b) \wedge c = (a \wedge c) \vee (b \wedge c)$,

(4) $a \wedge (b \vee c) = (a \wedge b) \vee (a \wedge c)$,

(5) $a \vee (b \wedge c) = (a \vee b) \wedge (a \vee c)$,

(6) $(a \vee b) \wedge b = b$

for all $a,b,c \in L$.

The binary relation "$\leq$" on L defined by:

$$a \leq b \Leftrightarrow a \wedge b = a \Leftrightarrow a \vee b = b$$

is a partial ordering on L.

An element m in L is said to be maximal if it is maximal with respect to the partial ordering $\leq$ on L.

Lemma 2.2 : [11] *Let L be an ADL and $m \in L$. Then the following are equivalent:*

- m is maximal
- $m \vee x = m$ for all $x \in L$
- $m \wedge x = x$ for all $x \in L$

Definition 2.3 : [11] A nonempty subset I of an ADL L is called an ideal (resp. a filter) of L, if $a \vee b, a \wedge x \in I$ (resp. $a \wedge b, x \wedge a \in I$) for all $a,b \in I$ and all $x \in L$.

If I is an ideal of L, then it is observed that $x \wedge a \in I$ for all $a \in I$ and all $x \in L$. Moreover, for any $a,b \in L$, it holds that $a \wedge b \in I$ if and only if $b \wedge a \in I$.

Definition 2.4 : [11] A proper ideal P of an ADL L is called prime if:

$$a \wedge b \in P \Rightarrow a \in P \text{ or } b \in P$$

for all $a,b \in L$.

Definition 2.5 : [8] Let L be an ADL and J an ideal of L. A prime ideal P of L is said to be a minimal prime ideal belonging to J if $J \subseteq P$ and there is no prime ideal Q such that $J \subseteq Q \subset P$. That is P is minimal among the prime ideals of L containing J.

Lemma 2.6 : [8] *A prime ideal P of an ADL L containing an ideal J is a minimal prime ideal belonging to J if and only if, for each, $x \in P$, there exists $y \in L - P$ such that $x \wedge y \in J$.*

Theorem 2.7 : [1] *For any ideal J of L, if $h(J)$ denotes, the set of all minimal prime ideals belonging to J, then*

$$J = \bigcap \{P : P \in h(J)\}$$

Definition 2.8 : [6] For any nonempty subset S of L, define

$$S^* = \{x \in L : x \wedge y = 0 \text{ for all } y \in S\}$$

We call S^* the annihilator of S. If $S = \{a\}$, then we denote S^* by $(a)^*$. An ideal I of L is called annihilator ideal if $I = S^*$ for some nonempty subset S of L.

Definition 2.9 : [12] A pseudo-complemented ADL is an algebra $(L, \wedge, \vee, ^*, 0, m)$ of type $(2,2,1,0,0)$ in which $(L, \wedge, \vee, 0, m)$ is an ADL with maximal elements and the unary operation $x \mapsto x^*$ satisfies the following conditions:

(1) $a \wedge b = 0 \Rightarrow a^* \wedge b = b$

(2) $a \wedge a^* = 0$

(3) $(a \vee b)^* = a^* \wedge b^*$

for all $a, b \in L$.

Note that, m in $(L, \wedge, \vee, ^*, 0, m)$ is a fixed maximal element treated as a nullary operation. It is also clear to see that $m = 0^*$ and hence $x^* \leq m$ for all $x \in L$. Furthermore, the set

$$S(L) = \{x^* : x \in L\}$$

is called the skeleton of L. It is observed that $(S(L), \wedge, \underline{\vee}, *, 0, m)$ is a Boolean algebra where the binary operation $\underline{\vee}$ is defined by:

$$a \,\underline{\vee}\, b = (a^* \wedge b^*)^*$$

for all $a, b \in S(L)$. An element $x \in L$ is called dense if $x^* = 0$. The set $D = \{x \in L : x^* = 0\}$ of all dense elements is a filter in L. The following lemma is analogues of the Lemma 1.1 (iii) of [3] in the class of almost distributive lattices.

Lemma 2.10 : *If L is a pseudo-complemented ADL, then a prime ideal P of L is a minimal prime ideal if and only if $x^{**} \in P$ for each $x \in P$; and the set of all minimal prime ideals of L endowed with the hull-kernel topology, is a compact Hausdorff space.*

Definition 2.11 : [1] Let L be an associative ADL and J and ideals of L. Define a relation $\theta(J)$ on L by:

$$(a, b) \in \theta(J) \text{ if and only if } x \vee a = x \vee b \text{ for some } x \in J$$

In an associative ADL L, it is observed that $\theta(J)$ is the smallest congruence on L containing J in a single congruence class.

Definition 2.12 : [1] For an ideal J of L and each $a \in L$ define a subset $(a)^J$ of L by:

$$(a)^J = \{x \in L : a \wedge x \in J\}$$

It is observed in [1] that, $(a)^J$ is an ideal of L containing J. Moreover, the binary relation $R(J)$ defined by:

$$(a,b) \in R(J) \text{ if and only if } (a)^J = (b)^J$$

for all $a,b \in L$, is the largest congruence on L having J as a congruence class. Furthermore, the set $D(J) = \{x \in L : (x)^J = J\}$ (if nonempty) is a filter and it is the unit congruence class of $R(J)$. For simplicity, we denote $R(\{0\})$ and $D(\{0\})$ by R and D respectively.

Lemma 2.13 : [1] *If* $h(J)$ *denotes the set of all minimal prime ideals belonging to J, then*

$$R(J) = \bigcap \{R(P) : P \in h(J)\}$$

3. Congruences in Pseudocomplemented ADLs

An equivalence relation θ on L is a congruence relation on a pseudo-coplemented ADL $(L,\vee,\wedge,*,0,m)$ if and only if it is a congruence on the ADL $(L,\vee,\wedge,0,m)$ which also has a substitution property for the "$*$" operation. For specificity it will be convenient to refer such a congruence as a $*-$ congruence. From now on, L stands for pseudo-complemented ADL $(L,\vee,\wedge,*,0,m)$ unless and otherwise it is specified. For a $*-$ congruence θ on L, $ker(\theta) = \{x \in L : x \equiv 0(\theta)\}$ and $coker(\theta) = \{x \in L : x \equiv m(\theta)\}$ respectively are called the kernel and cokernel of θ. A subset J of L is called a congruence kernel (respectively a congruence cokernel) if $J = ker(\theta)$ (respectively $J = coker(\theta)$ for some $*-$ congruence θ on L. It is easy to verify that congruence kernels (resp. congruence cokernels) of L are ideals (resp. filters) of L.

If F is a filter in an ADL L then the relation $\psi(F)$ defined by:

$$(x,y) \in \psi(F) \text{ if and only if } x \wedge f = y \wedge f \text{ for some } f \in F$$

is the smallest congruence on L having F as a congruence class. In fact, it is the special case of the congruence ψ^S generally defined for multiplicatively closed sets in [5].

Lemma 3.1 : *For any two filters F and G in an ADL L the congruence relations* $\psi(F)$ *and* $\psi(G)$ *are permutable to each other.*

Proof : It is enough if we show that $\psi(F)\circ\psi(G)=\psi(F\vee G)$. Let $(x,y)\in\psi(F)\circ\psi(G)$. Then there exists $z\in L$ such that $(x,z)\in\psi(G)$ and $(z,y)\in\psi(F)$; that is, $x\wedge b=z\wedge b$ and $z\wedge a=y\wedge a$ for some $a\in F$ and $b\in G$. This gives the equality

$$x\wedge a\wedge b=y\wedge a\wedge b=z\wedge a\wedge b$$

and $a\wedge b\in F\vee G$. This means $(x,y)\in\psi(F\vee G)$ and hence $\psi(F)\circ\psi(G)\subseteq\psi(F\vee G)$. To prove the other inclusion let $(x,y)\in\psi(F\vee G)$. There exists $a\in F$ and $b\in G$ such that $x\wedge a\wedge b=y\wedge a\wedge b$ which is equivalent to $x\wedge b\wedge a=y\wedge b\wedge a$. If we put $z=(y\wedge a)\vee(x\wedge b)$, then it can be verified that $z\wedge a=y\wedge a$ and $x\wedge b=z\wedge b$, i.e., $(z,y)\in\psi(F)$ and $(x,z)\in\psi(G)$. Therefore $(x,y)\in\psi(F)\circ\psi(G)$ and hence the equality holds. Since $F\vee G=G\vee F$, we can further conclude that $\psi(F)$ and $\psi(G)$ are permutable to each other. □

The next theorem describes congruence-cokernels in pseudo-complemented ADLs.

Theorem 3.2 : *Let* $(L,\vee,\wedge,*,0,m)$ *be a pseudo-complemented ADL. A subset F of L is a congruence cokernel if and only if F is a filter in L. Moreover, the smallest* $*-$ *congruence on L having F as its cokernel is* $\psi(F)$.

Proof : Suppose that F is a filter in L. It is known that the relation $\psi(F)$ is an ADL congruence on L having F as its cokernel. So, it is sufficient to prove that $\psi(F)$ has a substitution property for the $*-$ operation. Let $(x,y)\in\psi(F)$. Then $x\wedge a=y\wedge a$ for some $a\in F$, which gives $x^{**}\wedge a^{**}=y^{**}\wedge a^{**}$ so that $x^{*}\underline{\vee}a^{*}=y^{*}\underline{\vee}a^{*}$ in the skeleton $S(L)$ of L. Since $S(L)$ is a Boolean algebra it follows that $x^{*}\wedge a^{**}=y^{*}\wedge a^{**}$ which is equivalent to $x^{*}\wedge a=y^{*}\wedge a$. Thus $(x^{*},y^{*})\in\psi(F)$. Therefore $\psi(F)$ is a $*-$ congruence on L having F as its cokernel and it is the smallest of such $*-$ congruences. □

For an ideal J of a pseudo-complemented ADL L, let us define a set J_* as follows:

$$J_* = \{x \in L : x \wedge a^* = a^* \text{for some } a \in J\}$$

Lemma 3.3 : *For any ideal J of L, the set J_* is a filter in L.*

Theorem 3.4 : *Let J be a nonempty subset of L. Then the following conditions are equivalent:*

(1) J is a congruence kernel

(2) J is an ideal of the ADL L and $x^{**} \in J$ for all $x \in J$

(3) J is an ideal of the ADL L and each minimal prime ideal belonging to J is a minimal prime ideal.

(4) $J = \cap\{P : P \text{ is a minimal prime ideal of } L\}$

Moreover, if J satisfies any one(and hence all) of the above conditions, then $\theta(J) = \psi(J_*)$. In other words, the smallest $*-$ congruence on A having J as its kernel is $\psi(J_*)$, where

$$\psi(J_*) = \{(x, y) \in L \times L : x \wedge a^* = y \wedge a^* \textit{ for some } a \in J\}$$

Proof : $(1) \Rightarrow (2)$ is clear.

$(2) \Rightarrow (1)$. Assume (2). By Lemma 3.3, J_* is a filter in L and hence by Theorem 3.2 the relation $\psi(J_*)$ is a $*-$congruence on L such that $coker(\psi(J_*)) = J_*$. It is also clear that $(x, y) \in \psi(J_*)$ if and only if $x \wedge b^* = y \wedge b^*$ for some $b \in J$. We show that $ker(\psi(J_*)) = J$. Let $x \in ker(\psi(J_*))$. Then $x \wedge b^* = 0 \wedge b^*$ for some $b \in J$; that is, $x \wedge b^* = 0$ for some $b \in J$, which gives that $b^{**} \wedge x = x$ and by our assumption $b^{**} \in J$. So that $x \in J$ and hence $ker(\psi(J_*)) \subseteq J$. To prove the other inequality, let $x \in J$. We have $x \wedge x^* = 0 = 0 \wedge x^*$, which gives $(x, 0) \in \psi(J_*)$ and hence $x \in ker(\psi(J_*))$ and the equality holds. Therefore J is a congruence kernel and $\psi(J_*) = \theta(J)$.

$(2) \Rightarrow (3)$. Assume (2) and let P be a minimal prime ideal belonging to J. If $x \in P$, then there exists $z \in L - P$ such that $x \wedge z \in J$. By (2) $x^{**} \wedge z^{**} \in J \subseteq P$. Since P is prime either $x^{**} \in P$ or $z^{**} \in P$. If $z^{**} \in P$, then $z \in P$, which is a contradiction. So $z^{**} \notin P$ and hence $x^{**} \in P$. By Lemma 2.10, P is a minimal prime ideal.

$(3) \Rightarrow (4)$. The proof follows from

$$J = \cap\{P : P \in h(J)\}$$

$(4) \Rightarrow (2)$. Suppose that

$$J = \cap\{P : P \text{ is a minimal prime ideal of } L\}$$

Let $x \in J$. Then $x \in P$ for each minimal prime ideal P of L. By Lemma 2.10, $x^{**} \in P$ for each minimal prime ideal P of L. So that $x^{**} \in J$ and (2) holds. Therefore the equivalency (1) to (4) is proved. Furthermore, if one (and hence all) of the above four conditions is satisfied, then as we see in the proof of $(2) \Rightarrow (1)$, the relation $\psi(J_*)$ is a $*-$congruence on L such that $ker(\psi(J_*)) = J$. □

Because of the previous theorem, it can be deduced that the congruence-kernels of a pseudocomplemented ADL are precisely its $\alpha-$ideals, in the sense of [7].

Let us recall an important relation η on L from [10] defined as follows:

$$(x, y) \in \eta \text{ if and only if } x \wedge y = y \text{ and } y \wedge x = x$$

It is observed that, η is the smallest ADL-congruence on L such that L/η is a distributive lattice. Since $(x \wedge y)^* = (y \wedge x)^*$ for all $x, y \in L$, it is easy to verify that η is also a $*-$congruence on L.

Theorem 3.5 : *Let L be a pseudo-complemented ADL. The following conditions are equivalent:*

(1) Every ideal in L is a congruence kernel.

(2) Every prime ideal in L is a congruence kernel.

(3) $R \subseteq \eta$.

(4) The interval $[0, m]$ *is a Boolean algebra together with the induced operations on L.*

(5) Every principal ideal in L is a congruence kernel.

Lemma 3.6 : *If* $\{J_\alpha : \alpha \in \Delta\}$ *is a nonempty family of congruence kernels in L, then* $\cap_{\alpha \in \Delta} J_\alpha$ *is a congruence kernel.*

This lemma confirms that for a given ideal J of L (not necessarily a congruence kernel) always there exists a smallest congruence kernel containing J. We denote by $k(J)$, the smallest congruence kernel containing J.

Theorem 3.7 : *For an ideal J of L,*

$$k(J) = \{x \in L : a^{**} \wedge x = x \text{ for some } a \in J\}$$

Proof : Let us put $E = \{x \in L : a^{**} \wedge x = x \text{ for some } a \in J\}$. We show that E is the smallest congruence kernel in L containing J. Clearly $J \subseteq E$. Let $x \in E$.

Then $a^{**} \wedge x = x$ for some $a \in J$, which gives $a^{**} \wedge x^{**} = x^{**}$ and so $x^{**} \in E$. That is, $x \in E$ implies that $x^{**} \in E$. To say that E is a congruence kernel, it remains to show that E is an ideal. Clearly E is nonempty. Let $x, y \in E$. Then

$$a^{**} \wedge x = x \text{ and } b^{**} \wedge y = y$$

for some $a, b \in J$. If we take $c = a \vee b$, then $c \in J$ such that $c^{**} \wedge x = x$ and $c^{**} \wedge y = y$. This implies that $c^{**} \wedge (x \vee y) = x \vee y$. So, $x \vee y \in E$. Also if $x \in E$ and $z \in L$, then there exists $a \in J$ such that $a^{**} \wedge x = x$, which gives $a^{**} \wedge (x \wedge z) = x \wedge z$ and hence $x \wedge z \in E$. Thus E is an ideal of L and therefore it is a congruence kernel. Now let I be any congruence kernel in L such that $J \subseteq I$. If $x \in E$, then $a^{**} \wedge x = x$ for some $a \in J$. Since $J \subseteq I$ and I is a congruence kernel, we get that $a^{**} \in I$ and hence $x \in I$. Thus $E \subseteq I$; that is, E is the smallest congruence kernel in L containing J and the result holds. □

In the following theorem we give another characterization for $k(J)$.

Theorem 3.8 : *For an ideal J of L,*

$$k(J) = \{x \in L : x^{*} = a^{*} \text{ forsome } a \in J\}$$

Proof : Let us put

$$E = \{x \in L : a^{**} \wedge x = x \text{ for some } a \in J\} \text{ and } G = \{x \in L : a^{*} = x^{*} \text{for some } a \in J\}$$

It is enough to show that $E = G$. Let $x \in E$. Then there exists $a \in J$ such that $a^{**} \wedge x = x$, which gives that $a^{**} \wedge x^{**} = x^{**}$; that is, $(a \wedge x)^{**} = x^{**}$, which is equivalent to $(a \wedge x)^{*} = x^{*}$. If we put $b = a \wedge x$, then $b \in J$ such that $b^{*} = x^{*}$. Thus $x \in G$ and hence $E \subseteq G$. Also if $x \in G$, then $x^{*} = a^{*}$ for some $a \in J$. So, $x^{**} = a^{**}$ and that $a^{**} \wedge x = x^{**} \wedge x = x$. Thus $x \in E$ and the equality holds. □

Lemma 3.9 : *The following holds*

(1) $J \subseteq k(J)$

(2) $I \subseteq J$ *implies* $k(I) \subseteq k(J)$

(3) $k(k(J)) = k(J)$

It follows from this lemma that the map $J \mapsto k(J)$ is a closure operator on the class of all ideal of L and the congruence kernels of L are precisely those closed elements of the class $I(L)$ of ideals of L with respect to this closure operator.

Lemma 3.10 : *The set $CK(L)$ of all congruence-kernels of a pseudocomplemented ADL L is a complete distributive lattice, where the meet is set-theoretic intersection and the join of a family $\{J_\lambda : \lambda \in \Delta\}$ of congruence-kernels is*

$$\begin{aligned}\underline{\vee}\{J_\lambda : \lambda \in \Delta\} &= k(\vee\{J_\lambda : \lambda \in \Delta\}) \\ &= \{x \in L : (a_{\lambda_1} \vee \ldots \vee a_{\lambda_n})^{**} \wedge x = x \\ &\quad \text{forsome } a_{\lambda_1} \in J_{\lambda_1}, \ldots, a_{\lambda_n} \in J_{\lambda_n} \text{ and } \lambda_1, \ldots, \lambda_n \in \Delta\}\end{aligned}$$

Theorem 3.1 : *Let $I(S(L))$ be the lattice of ideals of the skeleton $S(L)$. If $\pi : CK(L) \to I(S(L))$ is defined by:*

$$\pi(J) = \{b \in S(L) : b = x^{**} \text{for some } x \in J\}$$

for all $J \in CK(L)$, then it is a lattice-isomorphism between $CK(L)$ and the lattice of ideals of the skeleton $S(L)$.

Proof : We first show that π is well defined. For this, it is enough to show that $\pi(J)$ is an ideal of $S(L)$ for all $J \in CK(L)$. Let $J \in CK(L)$. Clearly $\pi(J)$ is a nonempty subset of $S(L)$. Let $a, b \in \pi(J)$. Then $a = x^{**}$ and $b = y^{**}$ for some $x, y \in J$. If we put $z = x \vee y$, then $z \in J$ and $a \underline{\vee} b = z^{**}$, which gives that $a \underline{\vee} b \in \pi(J)$. Also let $a \in \pi(J)$ and $b \in S(L)$. Then $a = x^{**}$ for some $x \in J$ and $b = y^{*}$ for some $y \in L$. If we take $z = x \wedge b$, then $z \in J$ and $a \wedge b = z^{**}$ so that $a \wedge b \in \pi(J)$; that is, $\pi(J)$ is an ideal of $S(L)$ and hence π is well defined. Next we show that π is a lattice homomorphism. For each $I, J \in CK(L)$, it is clear that $\pi(I \cap J) = \pi(I) \cap \pi(J)$. It remains to show that $\pi(I \underline{\vee} J) = \pi(I) \vee \pi(J)$. Remember that the supremum in the left hand is the supremum in $CK(L)$ and the supremum in the right hand is the supremum of ideals in $S(L)$. Let $z \in \pi(I \underline{\vee} J)$. Then $z = x^{**}$ for some $x \in I \underline{\vee} J$. By Lemma 3.10 $x = (a \vee b)^{**} \wedge x$ for some $a \in I$ and $b \in J$. Now we have:

$$z = x^{**} = ((a \wedge x)^{**} \underline{\vee} (b \wedge x)^{**} \quad in S(L)$$

If we put $x_1 = a \wedge x$ and $x_2 = b \wedge x$, then $x_1 \in I$, $x_2 \in J$ and $z = x_1^{**} \underline{\vee} x_2^{**}$, which gives that $z \in \pi(I) \vee \pi(J)$. Thus $\pi(I \underline{\vee} J) \subseteq \pi(I) \vee \pi(J)$. To prove the other inclusion let $z \in \pi(I) \vee \pi(J)$. Then $z = z_1 \underline{\vee} z_2$ for some $z_1 \in \pi(I)$ and $z_2 \in \pi(J)$; that is, $z_1 = a^{**}$ for some $a \in I$ and $z_2 = b^{**}$ for some $b \in J$. If we put $x = a \vee b$, then $x \in I \vee J \subseteq I \underline{\vee} J$ in $S(L)$ such that

$z = x^{**}$. Thus $z \in \pi(I \underline{\vee} J)$ and hence the equality holds. Therefore π is a lattice homomorphism. To prove that π is one-one let us take $I, J \in CK(L)$ such that $\pi(I) = \pi(J)$.

Claim. $I = J$. Let $x \in I$. Then $x^{**} \in \pi(I) = \pi(J)$ so that $x^{**} = y^{**}$ for some $y \in J$; that is, $x \in k(J)$. Since J is a congruence kernel we have $k(J) = J$ and hence $x \in J$. Thus $I \subseteq J$. By symmetry, $J \subseteq I$ and so the equality holds. Therefore π is one-one. It remains to show that π is onto. Let J be an ideal of $S(L)$. Put

$$I = \{x \in L : x^{**} \in J\}$$

It is clear that I is a nonempty subset of L. Our aim is to show that $I \in CK(L)$. We first show that I is an ideal of L. Let $x, y \in I$. Then $x^{**}, y^{**} \in J$. Since J is an ideal of $S(L)$, we get $x^{**} \underline{\vee}\, y^{**} \in J$ which is equivalent to $(x \vee y)^{**} \in J$ so that $x \vee y \in I$. Also let $x \in I$ and $y \in L$. Then $x^{**} \in J$ and $y^{**} \in S(L)$, which gives $(x \wedge y)^{**} = x^{**} \wedge y^{**} \in J$; that is, $x \wedge y \in I$ and hence I is an ideal of L. Moreover, if $x \in I$, then $x^{****} = x^{**} \in J$ and so $x^{**} \in I$; mean that I is a congruence kernel in L. It is also routine to verify that $\pi(I) = J$ and hence π is onto. Therefore π is a lattice isomorphism. Furthermore, if $\overleftarrow{\pi} : I(S(L)) \to CK(L)$ is defined by:

$$\overleftarrow{\pi}(I) = \{x \in L : x^{**} \in I\}$$

for all $I \in I(S(L))$, then one can verify that π and $\overleftarrow{\pi}$ are mutually inverse lattice-isomorphisms between $CK(L)$ and $I(S(L))$. ☐

Lemma 3.12 : *Let θ be a $*-$ congruence on a pseudo-complemented ADL L and let $\theta : L \to L/\theta$ be the canonical epimorphism. Then the set-function induced by θ and the inverse image map $\overleftarrow{\theta}$ are mutually inverse lattice isomorphisms between the interval $[ker(\theta), L] = \{J \in CK(L) : \ker(\theta) \leq J\}$ of the lattice $CK(L)$ of congruence-kernels of L and the lattice $CK(L/\theta)$ of congruence-kernels of L/θ. Hence the map $P \mapsto \theta(P)$ is a bijection of the set of minimal prime ideals of L which contain $ker(\theta)$ onto the set of minimal prime ideals of L/θ.*

4. The congruence $R(J)$

In this section, we give several characterization for congruence kernels in pseudo-complemented ADLs. For a given congruence kernel J, we give a better description of the largest $*-$ congruence $R(J)$ having J as its kernel.

For an ideal J of L, we begin by defining the following binary relations on L which will be used to characterize congruence kernels.

(1) Φ is the largest $*-$ congruence on L such that $ker(\Phi) = J$.

(2) For $x, y \in L$, $(x, y) \in \Phi_1$ if and only if $x^{**} \wedge a^* = y^{**} \wedge a^*$ for some $a \in J$.

(3) For $x, y \in L$, $(x, y) \in \Phi_2$ if and only if $x \wedge b \wedge a^* = y \wedge b \wedge a^*$ for some $a \in J$ and some $b \in D$.

(4) For $x, y \in L$, $(x, y) \in \Phi_3$ if and only if $x^{**} \wedge y^* \in J$ and $x^* \wedge y^{**} \in J$.

Theorem 4.1 : *An ideal J of L is a congruence kernel if and only if* $R(J)$ *is a* $*-$ *congruence on L. In this case* $R(J) = \Phi$.

Proof : For any ideal J of L, it is shown in [1] that $R(J)$ is the largest ADL-congruence on L having J as its kernel. If $R(J)$ is a $*-$ congruence, then it is clear that J is a congruence kernel. Suppose that J is a congruence kernel. It remains to show that $R(J)$ is compatible with the unary operation $*$. By Lemma 2.13, we have:

$$R(J) = \cap\{R(P) : P \in h(J)\}$$

Let P be a minimal prime ideal belonging to J. By the equivalency in Theorem 3.4, P is a minimal prime ideal of L. Let $(x, y) \in R(P)$. Then $(x)^P = (y)^P$. We show that $(x^*)^P = (y^*)^P$. Let $a \in (x^*)^P$. Then $x^* \wedge a \in P$, and so either $x^* \in P$ or $a \in P$. If $a \in P$, then $y^* \wedge a \in P$ holds and hence $(x^*)^P \subseteq (y^*)^P$ and by symmetry the equality holds. Suppose that, $a \notin P$. Then $x^* \in P$. We use contradiction. Assume that $y^* \notin P$. Then $y \in P$ and so $(y)^P = L = (x)^P$, implying that $x \in P$, Since $x^* \in P$, we get $x \vee x^* \in P$. By Lemma 2.10 we have $(x \vee x^*)^{**} \in P$, which gives that $0^* \in P$ which is a contradiction. Therefore $y^* \in P$ and hence the equality holds. This gives $(x^*, y^*) \in R(P)$; that is, $R(P)$ is a $*-$ congruence for each $P \in h(J)$. Since the lattice of $*-$ congruences on L is complete, $R(J)$ is a $*-$ congruences on L. □

Theorem 4.2 : *For an ideal J of L,* $\Phi_1 = \Phi_2$.

Proof : Let $x, y \in L$ and suppose that $(x, y) \in \Phi_1$. Then there exists $a \in J$, such that

$$x^{**} \wedge a^* = y^{**} \wedge a^*$$

If $b=(x\vee x^*)\wedge(y\vee y^*)$, then $b\in D$. By operating b from the left with the binary operation $\wedge$ on both sides of the above equality, we get

$$x^{**}\wedge b\wedge a^*=y^{**}\wedge b\wedge a^*$$

We know that $x^{**}\wedge(x\vee x^*)=x$. Now consider the following:

$$\begin{aligned}x\wedge b\wedge a^* &= x^{**}\wedge(x\vee x^*)\wedge b\wedge a^*\\ &= x^{**}\wedge(x\vee x^*)\wedge(x\vee x^*)\wedge(y\vee y^*)\wedge a^*\\ &= x^{**}\wedge(x\vee x^*)\wedge(y\vee y^*)\wedge a^*\\ &= x^{**}\wedge b\wedge a^*\end{aligned}$$

Similarly we can verify that $y\wedge b\wedge a^*=y^{**}\wedge b\wedge a^*$. Now consider;

$$x\wedge b\wedge a^*=x^{**}\wedge b\wedge a^*=y^{**}\wedge b\wedge a^*=y\wedge b\wedge a^*$$

Therefore $(x,y)\in\Phi_2$ and so $\Phi_1\subseteq\Phi_2$. To prove the other inequality, let $(x,y)\in\Phi_2$. Then $x\wedge b\wedge a^*=y\wedge b\wedge a^*$ for some $a\in J$ and some $b\in D$. Since b is dense element, b^{**} is maximal and we get $x^{**}\wedge b^{**}\wedge a^*=y^{**}\wedge b^{**}\wedge a^*$ which is equivalent to $x^{**}\wedge a^*=y^{**}\wedge a^*$. Thus $(x,y)\in\Phi_1$ and hence the equality holds. □

Theorem 4.3 : *If J is a congruence kernel, then $\Phi_1=\Phi_3$.*

Proof : Let $(x,y)\in\Phi_1$. Then $x^{**}\wedge a^*=y^{**}\wedge a^*$ for some $a\in J$. Operating x^* with a binary operation $\wedge$ on both sides of this equality gives $x^*\wedge y^{**}\wedge a^*=0$ and so $a^{**}\wedge(x^*\wedge y^{**})=(x^*\wedge y^{**})$. Since by our assumption J is a congruence kernel, it follows from Theorem 3.4 that $a^{**}\in J$ and hence $x^*\wedge y^{**}\in J$. Similarly we get $x^{**}\wedge y^*\in J$ and therefore $(x,y)\in\Phi_3$. Thus $\Phi_1\subseteq\Phi_3$. To prove the other inclusion, let $(x,y)\in\Phi_3$. Then $x^*\wedge y^{**}\in J$ and $x^{**}\wedge y^*\in J$. If we put $a=(x^*\wedge y^{**})\vee(x^{**}\wedge y^*)$, then $a\in J$ and $a^*=(x^*\wedge y^{**})^*\wedge(x^{**}\wedge y^*)^*$. Moreover, we have $x^{**}\wedge(x^*\wedge y^{**})^*=x^{**}$ and $x^{**}\wedge(x^{**}\wedge y^*)^*=x^{**}\wedge y^{**}$. Now consider the following:

$$x^{**}\wedge a^*=x^{**}\wedge(x^*\wedge y^{**})^*\wedge(x^{**}\wedge y^*)^*=x^{**}\wedge y^{**}$$

Similarly, we get $y^{**}\wedge a^*=x^{**}\wedge y^{**}$, i.e., $x^{**}\wedge a^*=y^{**}\wedge a^*$ where $a\in J$ and hence $(x,y)\in\Phi_1$. Therefore the equality holds. □

Theorem 4.4 : *For an ideal J of L, Φ_2 (and hence Φ_1 and Φ_3 are) is a $*-$ congruence on L.*

Proof : Clearly Φ_2 is reflexive and symmetric. We show that it is transitive. Let $x,y,z \in L$ such that $(x,y),(y,z) \in \Phi_2$. Then there exist $a_1,a_2 \in J$ and $b_1,b_2 \in D$ such that:

$$x \wedge b_1 \wedge a_1^* = y \wedge b_1 \wedge a_1^* \text{ and } y \wedge b_2 \wedge a_2^* = z \wedge b_2 \wedge a_2^*$$

If we put $a = a_1 \vee a_2$ and $b = b_1 \wedge b_2$, then $a \in J$, $b \in D$ and we get the following:

$$x \wedge b \wedge a^* = y \wedge b \wedge a^* = z \wedge b \wedge a_2^*$$

which gives that $(x,z) \in \Phi_2$. So that Φ_2 is transitive. Similarly we can show that Φ_2 is compatible with the binary operations $\wedge$ and $\vee$; that is, Φ_2 is an ADL-congruence on L. It remain to show that it is compatible with the unary operation $*-$. Let $(x,y) \in \Phi_2$. By Theorem 4.3, there exists $a \in J$ such that $x^{**} \wedge a^* = y^{**} \wedge a^*$, which gives $(x^{**} \wedge a^*)^* = (y^{**} \wedge a^*)^*$. This is equivalent to $x^* \underline{\vee} a^{**} = y^* \underline{\vee} a^{**}$ in the skeleton $S(L)$. Since $S(L)$ is a Boolean algebra it follows that $x^* \wedge a^* = y^* \wedge a^*$; that is, $(x^*,y^*) \in \Phi_2$. Therefore Φ_2 is a $*-$ congruence on L. □

Theorem 4.5 : *An ideal J of L a congruence kernel if and only if* $R(J) = \Phi_1 (= \Phi_2)$.

Proof : If we are assuming that $R(J) = \Phi_1 (= \Phi_2)$, then it follows from Theorem 4.2 and Theorem 4.4 that $R(J)$ is a $*-$ congruence on *L* and hence *J* is a congruence kernel. Suppose that *J* is a congruence kernel. It is enough to prove $R(J) = \Phi_1$. Suppose that *J* is a congruence kernel. It is enough to prove $R(J) = \Phi_1$ only. By Theorem 3.1 and 4.4 respectively, both $R(J)$ and Φ_2 are $*-$ congruences on *L*. We first show that $ker(\Phi_1) = J$. Let $x \in ker(\Phi_1)$. Then $(x,0) \in \Phi_1$ and hence $x^{**} \wedge a^* = 0$ for some $a \in J$. This gives $a^{**} \wedge x^{**} = x^{**}$ and $a^{**} \in J$. So, $x^{**} \in J$ and hence $x \in J$. Therefore $ker(\Phi_1) \subseteq J$. To prove the other inclusion let $x \in J$. It is clear that $x^{**} \wedge x^* = 0 = 0^{**} \wedge x^*$ and hence $(x,0) \in \Phi_1$. Thus $x \in ker(\Phi_1)$ and so the equality holds. That is Φ_1 is a $*-$ congruence on *L* having *J* as its kernel. Since $R(J)$ is the largest $*-$ congruence with this property we get that $\phi_1 \subseteq R(J)$. To prove the other side of the inequality, we first show that; for each minimal prime ideal *P* belonging to *J*, $(x,y) \in R(P)$ if and only if $x^{**} \wedge a^* = y^{**} \wedge a^*$ for some $a \in P$. Let *P* be any minimal prime ideal belonging to *J*. By Theorem 3.4 *P* is a minimal prime ideal of *L* containing *J*. Let $(x,y) \in R(P)$. Then $(x)^P = (y)^P$; mean that, either $x,y \in P$ or $x,y \in L-P$. Assume the first case and put $a = x \vee y$. Then $a \in P$ and $x^{**} \wedge a^* = 0$. Similarly, $y^{**} \wedge a^* = 0$ and hence $x^{**} \wedge a^* = y^{**} \wedge a^*$. On the other hand, assume that $x,y \in L-P$. Then $x \wedge y \notin P$, which gives that $(x \wedge y)^* \in P$. If we put $a = (x \wedge y)^*$, then $a \in P$

and $x^{**} \wedge a^{*} = x^{**} \wedge y^{**} = y^{**} \wedge a^{*}$. Thus $x^{**} \wedge a^{*} = y^{**} \wedge a^{*}$. If we denote by Φ_P the binary relation:

$$\Phi_P = \{(x,y) \in L \times L : x^{**} \wedge a^{*} = y^{**} \wedge a^{*} \text{for some } a \in P\}$$

Then it is a $*-$ congruence on L having P as its kernel. Since $R(P)$ is the largest $*-$ congruence having P as its kernel, we get that $\Phi_P \subseteq R(P)$.

Conversely, let $(x,y) \in R(J)$ and $h(J) = \{P_\lambda : \lambda \in \Delta\}$ be the set of all minimal prime ideals containing J. By Lemma 2.13, $(x,y) \in R(P_\lambda)$ for all $\lambda \in \Delta$. Then for each $\lambda \in \Delta$ there exists $a_\lambda \in P_\lambda$ such that $x^{**} \wedge a_\lambda^{*} = y^{**} \wedge a_\lambda^{*}$. For each $a \in L$ let us consider a subset $g(a)$ of the space of minimal prime ideals of L defined by:

$$g(a) = \{Q : Q \text{ is a minimal prime ideal and } a \notin Q\}$$

Remember that; sets of the form $g(a)$ for $a \in L$ are basic open sets and the set $h(J)$ is a closed subset of the space of minimal prime ideals endowed with the hull-kernel topology. By Lemma 2.10, $h(J)$ is compact. Since each $a_\lambda \in P_\lambda$ and hence $a_\lambda^{*} \notin P_\lambda$, we get $P_\lambda \in g(a_\lambda^{*})$; that is, the family

$$\{g(a_\lambda^{*}) : \lambda \in \Delta\}$$

is an open cover for the closed subset $h(J)$. Then there exist $\lambda_1, \ldots, \lambda_n \in \Delta$ such that

$$h(J) \subseteq \bigcup_{i=1}^{n} g(a_{\lambda_i}^{*}) = g(a_{\lambda_1}^{*} \vee a_{\lambda_2}^{*} \vee \ldots \vee a_{\lambda_n}^{*})$$

Put $b = a_{\lambda_1}^{*} \vee \ldots \vee a_{\lambda_n}^{*}$. Then $h(J) \subseteq g(b)$ and $b^{*} \in \cap\{Q : Q \in g(b)\}$. To verify this, $Q \in g(b)$ implies $b \notin Q$ and hence $b^{*} \in Q$. Since $h(J) \subseteq g(b)$, we have the inequality

$$\cap\{Q : Q \in g(b)\} \subseteq \cap\{Q : Q \in h(J)\}$$

Since $J = \cap\{Q : Q \in h(J)\}$, it follows that $b^{*} = a_{\lambda_1}^{**} \wedge \ldots a_{\lambda_n}^{**} \in J$. Again if we put, for simiplicity, $a = b^{*} = a_{\lambda_1}^{**} \wedge \ldots a_{\lambda_n}^{**}$, then $a \in J$ and $a^{*} = a_{\lambda_1}^{*} \underline{\vee} \ldots \underline{\vee} a_{\lambda_n}^{*}$ in the skeleton $S(L)$ of L. Moreover, $x^{**} \wedge a^{*} = y^{**} \wedge a^{*}$. That is, $x^{**} \wedge a^{*} = y^{**} \wedge a^{*}$ where $a \in J$. So that $(x,y) \in \Phi_1$ and hence $R(J) \subseteq \Phi_1$. Therefore $R(J) = \Phi_1$.

Theorem 4.6 : *J is a congruence kernel if and only if $\Phi_1 = \theta(J) \vee R$; where the supremum is taken in the lattice of $*-$ congruence relations on L.*

Proof : Suppose that J is a congruence kernel. We first show that $R = \psi(D)$. Let $(x,y) \in R$. Then $(x)^{*} = (y)^{*}$. Since L is pseudo-complemented it is equivalent to that $(x^*] = (y^*]$, which gives $x^* \wedge m = y^* \wedge m$ for all maximal elements m of L. In particular, $x^* \wedge 0^* = y^* \wedge 0^*$. So $x^* = y^*$ and hence $x^{**} = y^{**}$. If we put $b = (x \vee x^*) \wedge (y \vee y^*)$, then $b \in D$ and $x^{**} \wedge b = b = y^{**} \wedge b$. By applying the same procedure as in Theorem 4.3 we get that $x \wedge b = y \wedge b$, where $b \in D$. This means that $(x, y) \in \psi(D)$. Thus $R \subseteq \psi(D)$. To prove the other side of the inequality, let $(x, y) \in \psi(D)$. Then $x \wedge b = y \wedge b$ for some $b \in D$, which gives $x^{**} \wedge b^{**} = y^{**} \wedge b^{**}$. Since b is dense element $b^{**} = 0^*$ and we get that $x^{**} = y^{**}$ which is equivalent to $x^* = y^*$. Therefore $(x, y) \in R$ and hence the equality holds. By Theorem 3.2, $\psi(D)$ and $\psi(J_*)$ are $*-$congruences on L. Moreover, it is observed in Theorem 3.4 that $\theta(J) = \psi(J_*)$. Now consider the following:

$$\theta(J) \vee R = \psi(J_*) \vee \psi(D) = \psi(J_* \vee D)$$

where $J_* \vee D = \{a \wedge b : a \in J_*, b \in D\}$. Now it remains to show that $\Phi_2 = \psi(J_* \vee D)$. Let $(x, y) \in \Phi_2$. Then $x \wedge b \wedge a^* = y \wedge b \wedge a^*$ for some $b \in D$ and some $a \in J$. If we take $z = b \wedge a^*$, then $z \in J_* \vee D$ and $x \wedge z = y \wedge z$, which implies that $(x, y) \in \psi(J_* \vee D)$. Also let $(x, y) \in \psi(J_* \vee D)$. Then $x \wedge z = y \wedge z$ for some $z \in J_* \vee D = D \vee J_*$; that is, $z = b \wedge a$ for some $b \in D$ and some $a \in J^*$. There is some $i \in J$ such that $a \wedge i^* = i^*$. This gives $x \wedge b \wedge i^* = y \wedge b \wedge i^*$, where $b \in D$ and $i \in J$. Therefore $(x, y) \in \Phi_2$ and hence the equality holds. □

Corollary 4.7 : *J is a congruence kernel if and only if* $R(J) = \theta(J) \vee R$.

Theorem 4.8 : *J is a congruence kernel if and only if* $D \subseteq D(J)$.

Proof : Suppose that J is a congruence kernel and let $x \in D$. Then $x^{**} = 0^*$. Let $y \in L$ such that $x \wedge y \in J$. Then $x^{**} \wedge y^{**} \in J$, which gives that $y^{**} \in J$ and hence $y \in J$. This means $x \in D(J)$ and therefore $D \subseteq D(J)$. Conversely suppose that the inclusion holds. Let $x \in J$. By Theorem 3.4 it is enough to show that $x^{**} \in J$. Put $y = x \vee x^*$. Then $y \in D$ and hence by our assumption $y \in D(J)$; that is, for any $z \in L$, $z \wedge y \in J$ implies that $z \in J$. Since $x^{**} \wedge y = x \in J$, we get that $x^{**} \in J$. This completes the proof. □

Theorem 4.9 : *J is a congruence kernel if and only if* $D(J) = J_* \vee D$ *in the lattice of filters.*

Proof : Suppose that J is a congruence kernel. Let $x \in D(J)$. Then for each $a \in L$, $x \wedge a \in J$ implies that $x \in J$. Since $x \wedge x^* = 0 \in J$, we get $x^* \in J$. If we put $a = x^{**}$ and $b = x \vee x^*$, then $a \in J_*$ and $b \in D$ such that $x = a \wedge b$

and so that $x \in J_* \vee D$. Thus $D(J) \subseteq J_* \vee D$. To prove the other inclusion, let $x \in J_* \vee D$. Then $x = y \wedge z$ for some $y \in J_*$ and $z \in D$; i.e., $y \wedge a^* = a^*$ for some $a \in J$ and $z^* = 0$. Let $b \in L$ such that $x \wedge b \in J$; that is, $a^* \wedge z \wedge b \in J$, which gives that $a^* \wedge b^{**} \in J$. Since $a \in J$, we get $a \vee (a^* \wedge b^{**}) \in J$ which implies that $b^{**} \in J$ and hence $b \in J$. Therefore $x \in D(J)$ and hence the equality holds. The converse part follows from Theorem 4.8. □

Theorem 4.10 : *J is a congruence kernel if and only if* $R(J) = \psi(D(J))$.

Proof : Suppose that *J* is a congruence kernel. It follows from Corollary 4.7 and Theorem 4.9 that

$$R(J) = \theta(J) \vee R = \psi(J_* \vee D) = \psi(D(J))$$

Conversely suppose that $R(J) = \psi(D(J))$. By Theorem 3.2 $\psi(D(J)$ is a $*-$ congruence on *L*, and hence $R(J)$ is a $*-$ congruence. This completes the proof. □

Theorem 4.11 : *J is a congruence kernel if and only if* $R \subseteq R(J)$.

Proof : If *J* is a congruence kernel, then it follows from Corollary 4.7 that $R \subseteq R(J)$. Conversely suppose that $R \subseteq R(J)$. Let $x \in J$. Since $x^* = x^{***}$, we have $(x, x^{**}) \in R \subseteq R(J)$. Then for each $a \in L$, $a \wedge x \in J$ if and only if $a \wedge x^{**} \in J$. If we take *a* to be x^{**} we get $x^{**} \in J$. This completes the proof. □

Theorem 4.12 *For an ideal J of L, the following are equivalent:*

(1) J is a congruence kernel.

(2) For any filter F, the congruences $R(J)$ *and* $\psi(F)$ *are permutable.*

(3) $R(J)$ *and R are permutable.*

Proof : $(1) \Rightarrow (2)$. Suppose that *J* is a congruence kernel. By Theorem 4.10 $R(J)$ is a $*-$ congruence on *L* such that $R(J) = \psi(D(J))$ and it is known that $D(J)$ is a filter in *L*. If *F* is any filter in *L*, then it follows from Lemma 3.1 that $R(J)$ and $\psi(F)$ are permutable. Since $R = \psi(D)$, $(2) \Rightarrow (3)$ is clear and we proceed to prove $(3) \Rightarrow (1)$. Assume that $R(J)$ and *R* are permutable and let $x \in J$. Then $(0, x) \in R(J)$. Also, since $x^{***} = x^*$ we have $(x, x^{**}) \in R$ which gives that $(0, x^{**}) \in R \circ R(J) = R(J) \circ R$. Then there exists $y \in L$ such that $(0, y) \in R$ and $(y, x^{**}) \in R(J)$; that is, $y^* = 0^*$, which gives $y = 0$ and hence $x^{**} \in J$. Therefore *J* is a congruence kernel. □

Theorem 4.13 : *If J is a congruence kernel, then $R(J)$ is the smallest (or equivalently the unique) $*-$ congruence on L such that $ker(R(J)) = J$ and the quotient $L / R(J)$ is a Boolean algebra.*

Proof : Suppose that J is a congruence kernel. Then by Theorem 4.1 $R(J)$ is a $*-$ congruence on L having J as its kernel. We show that the quotient $L / R(J)$ is a Boolean algebra. Clearly $L / R(J)$ is a bounded distributive lattice with least element $ker(R(J)) = J$ and with unit element $coker(R(J)) = D(J)$. To verify that $L / R(J)$ is a lattice it suffices to show $\eta \subseteq R(J)$. Now for each $x \in L$, consider:

$$(x \vee x^*)^* = x^* \wedge x^{**} = 0$$

so that $x \vee x^* \in D$. By Theorem 4.8 we have $D \subseteq D(J)$ and hence $x \vee x^* \in D(J)$. This confirms that the quotient $L / R(J)$ is a Boolean algebra. Furthermore, suppose that ϕ is any $*-$ congruence on L such that $ker(\phi) = J$ and L / ϕ is a Boolean algebra. Then $\eta \subseteq \phi$ and $x \vee x^* \in coker(\phi) = \{x \in L : (x, 0^*) \in \phi\}$ for all $x \in L$; that is, $D \subseteq coker(\phi)$. Since every maximal element m is dense, it holds that $m \in coker(\phi)$; that is, $(m, 0^*) \in \phi$ (or $m \equiv_\phi 0^*$). For each $x \in L$, always we can find a maximal element $m_x \in L$ such that $x \leq m_x$. This can be obtained by using Zorn's lemma. Now let $(x, y) \in R(J)$. By the equality in Theorem 4.5 we get the following:

$$x \wedge b \wedge a^* = y \wedge b \wedge a^*$$

for some $b \in D$ and some $a \in J$. Since $D \subseteq coker(\phi)$, $b \equiv_\phi 0^*$ and since $J = ker(\phi)$, we have $a^* \equiv_\phi 0^*$. It follows that $(x, y) \in \phi$ and hence $R(J) \subseteq \phi$. Therefore $R(J)$ is the smallest $*-$ congruence on L such that $ker(R(J)) = J$ and the quotient $L / R(J)$ is a Boolean algebra. Furthermore, since $R(J)$ is the largest congruence having J as a kernel it is equivalent that $R(J)$ is the unique $*-$ congruence on L with the given property. □

We summarize all the above results in the following theorem.

Theorem 4.14 : *Let J be an ideal of F. Then the following conditions are equivalent.*

- J is a congruence kernel
- $R(J)$ *is a* $*-$ *congruence on L.*
- For $x, y \in L$, $(x, y) \in R(J)$ if and only if $x^{**} \wedge a^* = y^{**} \wedge a^*$ for some $a \in J$.

- For $x, y \in L$, $(x, y) \in R(J)$ if and only if $x \wedge b \wedge a^{*} = y \wedge b \wedge a^{*}$ for some $a \in J$ and some $b \in D$.
- $R(J) = \psi(D(J))$.
- $D \subseteq D(J)$
- $D(J) = J_{*} \vee D$
- $R(J) = \theta^{J} \vee R$ in the lattice of $*$-congruences.
- $R \subseteq R(J)$
- For any filter F, the congruences $R(J)$ and $\psi(F)$ are permutable.
- $R(J)$ and R are permutable.

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