

A study of the holomorphy of (r, s, t) -inverse loops

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Abstract

In this paper, the holomorphic study of inverse properties in loops is put in a more general setting. For various combinations of $r, s, t \in \mathbb{Z}$ in parities, it was established that: (i) the $A(Q)$ -holomorph $H(Q)$ of a loop Q is an (r, s, t) -inverse loop if and only if Q is an (r, s, t) -inverse loop; (ii) the $A(Q)$ -holomorph $H(Q)$ of a loop Q is an (r, s, t) -inverse loop if and only if Q is an (r, s, t) -inverse loop, $A(Q)$ is a particular kind of group (e.g abelian, Boolean) and any two elements of $A(Q)$ satisfies some autotopic conditions. Specifically, m -inverse loop (when m is odd), double weak inverse property loop (WWIPL) and weak inverse property loop were found to satisfy the case (i) while m -inverse loop (when m is even) and weak inverse property loop were found to satisfy case (ii). For a Buchsteiner loop (which is a special kind of WWIPL) Q , it was shown that the $A(Q)$ -holomorph $H(Q)$ is a Buchsteiner loop if and only if $A(Q)$ is a nuclear automorphism group. The left (right) inner

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automorphism group of a Buchsteiner loop Q was shown to be a normal subgroup of the automorphism group of Q . Existing examples of loops which are relevant to this study were cited.

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1. Introduction

A quasigroup $(Q, \cdot)$ is a non-empty set Q together with a binary operation $(\cdot)$ such that, for all $a, b \in Q$ the equations $ax = ya = b$ have unique solutions $x = a \setminus b$ and $y = b / a$, in Q . Q is called a loop, if it has a two sided identity $1 \in Q$ such that $1 \cdot x = x \cdot 1 = x$ for all $x \in Q$. For any $a \in Q$, define the left and right translation maps respectively as, $xL_a = ax$ and $xR_a = xa$, for all $x \in Q$. These maps are bijections in quasigroups and loops. For standard references on quasigroup and loop theory readers should check [6, 19, 34].

Let $(Q, \cdot)$ be a loop and let $J, J^{-1} : Q \rightarrow Q$ be permutations on Q defined by $x \mapsto xJ$ and $x \mapsto xJ^{-1}$ such that $x \cdot xJ = 1$ and $xJ^{-1} \cdot x = 1$ respectively.

The group of all permutations on Q is called the symmetric group of Q and denoted by $SYM(Q)$. A triple (U, V, W) of permutations of a loop Q (i.e. $U, V, W \in SYM(Q)$) is called a autotopism if for all $x, y \in Q$, $xU \cdot yV = (xy)W$. The set $ATP(Q)$ of all autotopisms of Q forms a group under componentwise composition of mappings in the symmetric group $SYM(Q)$ of $(Q, \cdot)$. If $U = V = W$, then, $ATP(Q)$ becomes $AUM(Q)$, the automorphism group of Q .

The variation of inverse concept ranges from cross inverse property (CIP) to weak inverse property (WIP) and their generalizations. There have been various studies of these two varieties of loops (with peculiar interest in their applications to cryptography) in the recent past and present. Among these are Jaiyéqlá [17, 18, 21, 22, 23], Jaiyéqlá and Adeniran [20], Jaiyéqlá and Smarandache [28].

Interestingly, Adeniran [1] and Robinson [35], Chiboka and Solarin [7], Bruck [4], Bruck and Paige [5], Robinson [36], Huthnance [12] and Adeniran [1] have respectively studied the holomorphs of Bol loops, conjugacy closed loops, inverse property loops, A-loops, extra loops, weak inverse property loops and Bruck loops. A set of results on the holomorph of some varieties of loops can be found in Jaiyéqlá [15, 16, 19]. Studies on

the holomorph of generalized Bol loops can be found in Adeniran et. al. [2] and Jaiyéolá and Popoola [26]. Isere et. al. [14] studied the holomorphy of Osborn loops and Jaiyéolá et al. [27] studied the holomorphy of middle Bol loops. Recently, Ogunrinade et al. [33] studied the holomorphy of self distributive quasigroup, Ilojide et al. [13] studied the holomorphy of Fenyves BCI-algebras while Effiong et al. [11] considered the holomorphy of Basarab loops. Specifically, we cite the work of Bruck [4] on inverse property loops (IPL), where he established that the holomorph of IPL is an IPL. Also, Huthnance Jr. [12] proved that the holomorph of WIP loops is a WIP loop.

The justification for this study is based on the fact that WIP is a generalization of IP and CIP. Similarly, it has been shown that the CIP implies the WIP in loops. However, just as IP and CIP are special cases of WIP, so also is WIP a special case of m -inverse property (or $(m, m+1, m)$ -inverse property), with WIP, coinciding with -1 -inverse property or $(-1, 0-1)$ -inverse when $m = -1$. The m -inverse property or $(m, m+1, m)$ -inverse property is in turn also a particular case of (r, s, t) -inverse property, thus (r, s, t) -inverse property is a generalization of inverse property in quasigroup and loop. Buchsteiner loop as studied in [9] is a 1 -inverse loop or $(1, 2, 1)$ -inverse loop, is one of numerous example of (r, s, t) -inverse loop.

Therefore, in this research work, we shall embark on a systematic study of holomorph of (r, s, t) -inverse quasigroups and loops with a view to put the earlier studies in the associated structure in a more general setting. Leveraging on our findings in the holomorph of (r, s, t) -inverse loops, the holomorph of Buchsteiner loops which happen to be doubly weak inverse property loop (WWIP) is also studied.

2. Preliminaries

Let $(Q, \cdot)$ be a loop. $(Q, \cdot)$ is called a cross inverse property loop (CIPL), weak inverse property loop (WIPL), m -inverse property loop, (r, s, t) -inverse property loop if for all $x, y \in Q$, it obeys the following inverse properties respectively:

$$(x \cdot y) \cdot xJ = y \quad \text{or its dual} \quad xJ^{-1} \cdot (y \cdot x) = y$$

cross inverse property (CIPL) (2.1)

$$x \cdot (y \cdot x)J = yJ \quad \text{or its dual} \quad (x \cdot y)J^{-1} \cdot x = yJ^{-1}$$

weak inverse property (WIPL) (2.2)

$$(x \cdot y)J^m \cdot xJ^{(m+1)} = yJ^m \text{ or its dual } xJ^{-(m+1)} \cdot (y \cdot x)J^{-m} = yJ^{-m}$$

$$m\text{-inverse property} \quad (2.3)$$

$$(x \cdot y)J^r \cdot xJ^s = yJ^t \text{ or its dual } xJ^{-s} \cdot (y \cdot x)J^{-r} = yJ^{-t}$$

$$(r, s, t)\text{-inverse property} \quad (2.4)$$

The definitions of their quasigroup versions are similar based on the existence of a permutation J .

Definition 2.1 : Let $(Q, \cdot)$ be a quasigroup. Then

1. a bijection U is called λ – regular if there exists $(U, I, U) \in ATP(Q, \cdot)$; the set of all such mappings forms a group $\Lambda(Q, \cdot)$
2. a bijection U is called ρ – regular if there exists $(I, U, U) \in ATP(Q, \cdot)$; the set of all such mappings forms a group $P(Q, \cdot)$
3. a bijection U is called μ – regular if there exists a bijection U' such that $(U, U'^{-1}, I) \in ATP(Q, \cdot)$. U' is called the adjoint of U . The set of all μ – regular mappings forms a group $\Phi(Q, \cdot) \leq SYM(Q)$. The set of all adjoint mapping forms a group $\Phi'(Q, \cdot)$.

Definition 2.2 : Let $(Q, \cdot)$ be loop such that the identity $x \setminus (xy \cdot z) = (y \cdot zx) / x$ holds, for all $x, y, z \in Q$, then $(Q, \cdot)$ is called a Buchsteiner loop.

Remark 2.1 : If we set $r = t = 1$ and $s = 2$ in equation (2.4), we obtain $(1, 2, 1)$ -inverse property. This type of property is the one Csörgo et al, [9] named Doubly Weak Inverse Property (or WWIP). Buchsteiner loops have the WWIP and Jaiyéqlá and Adéníran [25] discovered Buchsteiner loops that are Osborn loops.

Definition 2.3: Let be quasigroup (loop) and let $A(Q) \leq AUM(Q)$. Let $H(Q) = A(Q) \times Q$ and define $\circ$ on $H(Q)$ as follows $(\alpha, x) \circ (\beta, y) = (\alpha\beta, x\beta \cdot y)$ for all x, y in Q and for all $\alpha, \beta \in A(Q)$. Then, the pair $(H(Q), \circ)$ (or $H(Q)$) is called the $A(Q)$ -holomorph (or holomorph) of Q . $(H(Q), \circ)$ is a quasigroup (loop). The $A(Q)$ -holomorph $H(Q)$ of a loop Q is a semi-direct product of Q and an automorphism group $A(Q)$ of it.

The right nucleus of Q is defined by $N_\rho(Q, \cdot) = \{a \in Q \mid xy \cdot a = x \cdot ya \forall x, y \in Q\}$. The left nucleus of Q is defined by $N_\lambda(Q, \cdot) = \{a \in Q : ax \cdot y = a \cdot xy \forall x, y \in Q\}$. The middle nucleus of Q is defined by $N_\mu(Q, \cdot) = \{a \in Q : ya \cdot x = y \cdot ax \forall x, y \in Q\}$. The nucleus of Q is defined by $N(Q, \cdot) = N_\lambda(Q, \cdot) \cap N_\rho(Q, \cdot) \cap N_\mu(Q, \cdot)$. The centrum of Q is defined by $C(Q, \cdot) = \{a \in Q : ax = xa \forall x \in Q\}$. The center of Q is defined by

$Z(Q, \cdot) = N(Q, \cdot) \cap C(Q, \cdot)$. $N_\rho(Q, \cdot), N_\lambda(Q, \cdot), N_\mu(Q, \cdot), N(Q, \cdot), Z(Q, \cdot)$ are subgroups of $(Q, \cdot)$.

The group $\mathcal{M}(Q, \cdot) = \langle \{R_x, L_x, : x \in Q\} \rangle$ is called the multiplication group of $(Q, \cdot)$ and $\mathcal{M}(Q, \cdot) \leq \text{SYM}(Q)$. If $1\alpha = 1$ in a loop Q such that $\alpha \in \mathcal{M}(Q)$, then α is called an inner mapping and they form a group $\text{Inn}(Q)$ called the inner mapping group. The right, left and middle inner mappings

$$R_{(x,y)} = R_x R_y R_{xy}^{-1}, L_{(x,y)} = L_x L_y L_{yx}^{-1} \text{ and } T_x = R_x L_x^{-1} \text{ respectively,}$$

generate the left inner mapping group $\text{Inn}_\lambda(Q)$, right inner mapping group $\text{Inn}_\rho(Q)$ and the middle inner mapping group $\text{Inn}_\mu(Q)$. If $\text{Inn}_\lambda(Q) \leq \text{AUM}(Q)$, $\text{Inn}_\rho(Q) \leq \text{AUM}(Q)$, and $\text{Inn}_\mu(Q) \leq \text{AUM}(Q)$, then Q is called a left A-loop (A_λ -loop), right A-loop (A_ρ -loop) and middle A-loop (A_μ -loop) respectively. It is well known that $\text{Inn}(Q, \cdot) = \langle R_{(x,y)}, L_{(x,y)}, T_x \mid x, y \in Q \rangle$, but this was later improved to be $\text{Inn}(Q, \cdot) = \langle L_{(x,y)}, T_x \mid x, y \in Q \rangle = \langle R_{(x,y)}, T_x \mid x, y \in Q \rangle$ in Vojtěchovský [38]. If $\text{Inn}(Q, \cdot) \leq \text{AUM}(Q, \cdot)$, then $(Q, \cdot)$ is called an automorphic loop (A-loop).

Definition 2.4: ([5]) A center automorphism of a loop Q is an automorphism S such that $xs = x \cdot c(x)$, where $c(x) \in Z(Q)$.

Bruck and Paige [5] showed that if $A(Q)$ is a group of center automorphism, then, the $A(Q)$ -holomorph $(H(Q), \circ)$ is an A-loop if and only if Q is an A-loop. Abelian groups are special types of A-loops. Formula for the number of automorphisms of a finite abelian group of rank two were gotten by Sehgal et al. [37].

3. Holomorphs of (r, s, t) -inverse loop

To establish the results in this section we need the following Lemmas:

Lemma 3.1 : Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$. Then, for all $\alpha \in A(Q)$ and $x \in Q$:

1. $(\alpha, x)J^n = (\alpha^{-1}, xJ^n\alpha^{-1})$, if $n \in 2\mathbb{Z} + 1$;
2. $(\alpha, x)J^n = (\alpha, xJ^n)$, if $n \in 2\mathbb{Z}$.

Proof :

1. Observe that, the identity element of H is given by $(i, 1)$, where $i \in \text{AUM}(Q)$ is the identity mapping. Also note that $\forall x \in Q$ and $\forall \alpha \in \text{AUM}(Q)$, $(\alpha, x) \circ (\alpha, x)J = (i, 1)$. It is claimed that

$$(\alpha, x)J = (\alpha^{-1}, xJ\alpha^{-1}) \quad (3.1)$$

since $(\alpha, x) \circ (\alpha, x)J = (\alpha, x) \circ (\alpha^{-1}, xJ\alpha^{-1}) = (\alpha\alpha^{-1}, x\alpha^{-1} \cdot xJ\alpha^{-1}) = (i, (x \cdot x)J\alpha^{-1}) = (i, 1)$. In equation (3.1), let $y = xJ\alpha^{-1} \Rightarrow x = y\alpha J^{-1}$ and $\beta = \alpha^{-1} \Rightarrow \alpha = \beta^{-1}$, so that $(\alpha, x)J = (\alpha^{-1}, xJ\alpha^{-1}) = (\beta, y) \Rightarrow (\alpha, x) = (\beta, y)J^{-1} \Rightarrow (\beta, y)J^{-1} = (\beta^{-1}, y\beta^{-1}J^{-1}) \Rightarrow$

$$(\beta, y)J^{-1} = (\beta^{-1}, yJ^{-1}\beta^{-1}) \quad (3.2)$$

If we use our calculations in equations (3.1) and (3.2), then $(\alpha, x)J^{-1} = (\alpha^{-1}, xJ^{-1}\alpha^{-1})$,

$$\begin{aligned} (\alpha, x)J^{-2} &= ((\alpha, x)J^{-1})J^{-1} = (\alpha^{-1}, xJ^{-1}\alpha^{-1})J^{-1} \\ &= ((\alpha^{-1})^{-1}, (xJ^{-1}\alpha^{-1})J^{-1}(\alpha^{-1})^{-1}) = (\alpha, xJ^{-2}), \end{aligned}$$

$$(\alpha, x)J^{-3} = ((\alpha, x)J^{-2})J^{-1} = (\alpha, xJ^{-2})J^{-1} = (\alpha^{-1}, (xJ^{-2})J^{-1}\alpha^{-1}) = (\alpha^{-1}, xJ^{-3}\alpha^{-1}) \text{ and}$$

$$\begin{aligned} (\alpha, x)J^{-4} &= (\alpha, x)J^{-3}J^{-1} = (\alpha^{-1}, xJ^{-3}\alpha^{-1})J^{-1} = ((\alpha^{-1})^{-1}, (xJ^{-3}\alpha^{-1})J^{-1}(\alpha^{-1})^{-1}) \\ &= (\alpha, xJ^{-4}). \end{aligned}$$

Now consider the following inductive process: $(\alpha, x)J = (\alpha^{-1}, xJ\alpha^{-1})$,

$$\begin{aligned} (\alpha, x)J^3 &= (\alpha, x)J^{2+1} = (\alpha^{-1}, xJ^{2+1}\alpha^{-1}), (\alpha, x)J^5 = (\alpha, x)J^{2+2+1} = (\alpha^{-1}, xJ^{2+2+1}\alpha^{-1}), \\ \dots, (\alpha, x)J^{2k+1} &= (\alpha^{-1}, xJ^{2k+1}\alpha^{-1}), (\alpha, x)J^{2(k+1)+1} = (\alpha^{-1}, xJ^{2(k+1)+1}\alpha^{-1}) \end{aligned}$$

since $(\alpha, x)J^{2(k+1)+1} = (\alpha, x)J^{2k+3} = (\alpha, x)J^{(2k+1)+2} = ((\alpha, x)J^{2k+1})J^2 = (\alpha^{-1}, xJ^{2k+1}\alpha^{-1})J^2 = (\alpha^{-1}, xJ^{2k+1}J^2\alpha^{-1}) = (\alpha^{-1}, xJ^{2k+3}\alpha^{-1}) = (\alpha^{-1}, xJ^{2(k+1)+1}\alpha^{-1})$.

So, $(\alpha, x)J^n = (\alpha^{-1}, xJ^n\alpha^{-1})$, whenever $n \in 2\mathbb{Z} + 1$.

2. Similarly, $(\alpha, x)J^2 = ((\alpha, x)J)J$, now using expression (3.1), then, $(\alpha, x)J^2 = ((\alpha, x)J)J = (\alpha^{-1}, xJ\alpha^{-1})J = ((\alpha^{-1})^{-1}, (xJ\alpha^{-1})J(\alpha^{-1})^{-1}) = (\alpha, xJ^2)$.

Now, consider the following induction: $(\alpha, x)J^2 = (\alpha, x)J^{2 \cdot 1} = (\alpha, xJ^{2 \cdot 1})$, $(\alpha, x)J^{2 \cdot 2} = (\alpha, xJ^{2 \cdot 2})$, $(\alpha, x)J^6 = (\alpha, x)J^{2 \cdot 3} = (\alpha, xJ^{2 \cdot 3})$, $\dots$, $(\alpha, x)J^{2 \cdot k} = (\alpha, xJ^{2 \cdot k})$, $(\alpha, x)J^{2(k+1)} = (\alpha, xJ^{2(k+1)})$ since $(\alpha, x)J^{2(k+1)} = (\alpha, x)J^{2k+2} = (\alpha, x)J^{2k}J^2 = ((\alpha, x)J^{2k})J^2 = ((\alpha, xJ^{2k})J^2) = (\alpha, xJ^{2k}J^2) = (\alpha, xJ^{2(k+1)})$. Therefore, we have $(\alpha, x)J^n = (\alpha, xJ^n)$ for all $n \in 2\mathbb{Z}$. $\square$

We now turn to the main results of this section.

Theorem 3.1 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$ and $r, s, t \in \mathbb{Z}$. Then, $(H(Q), \circ)$ is a (r, s, t) -inverse loop if and only if Q is a (r, s, t) -inverse loop, $A(Q)$ is a Boolean group and $(\alpha, i, i) \in \text{ATP}(Q)$ for all $\alpha \in A(Q)$, whenever r, s and t are odd.*

Proof: Suppose H is a (r, s, t) -inverse loop. Since $(Q, \cdot)$ is isomorphic to a subloop of H , then Q is a (r, s, t) -inverse loop. Now choose $(\alpha, x), (\beta, y) \in H$, having in mind that H is a (r, s, t) -inverse loop, then

$$[(\alpha, x) \circ (\beta, y)]J^r \circ (\alpha, x)J^s = (\beta, y)J^t \Rightarrow (\alpha\beta, x\beta \cdot y)J^r \circ (\alpha, x)J^s = (\beta, y)J^t.$$

Given that r, s and t are all odd and using result from Lemma 3.1, we obtain

$$\begin{aligned} & ((\alpha\beta)^{-1}, (x\beta \cdot y)J^r (\alpha\beta)^{-1}) \circ (\alpha^{-1}, xJ^s \alpha^{-1}) = (\beta^{-1}, yJ^t \beta^{-1}) \\ \Rightarrow & ((\alpha\beta)^{-1} \alpha^{-1}, (x\beta \cdot y)J^r (\alpha\beta)^{-1} \alpha^{-1} \cdot xJ^s \alpha^{-1}) = (\beta^{-1}, yJ^t \beta^{-1}) \text{ which is true } \Leftrightarrow \end{aligned}$$

$$\alpha^2 = i \text{ and } (x\beta \cdot y)J^r (\alpha\beta)^{-1} \alpha^{-1} \cdot xJ^s \alpha^{-1} = yJ^t \beta^{-1} \quad (3.3)$$

Thus, by the first equation in (3.3), $A(Q)$ is a Boolean group, which settles the first assertion. Moreover, using the fact that H is a (r, s, t) -inverse loop and the second equation in (3.3), we obtain

$$\begin{aligned} (x\beta \cdot y)J^r (\alpha\beta)^{-1} \alpha^{-1} \cdot xJ^s \alpha^{-1} &= yJ^t \beta^{-1} \Rightarrow (xy)J^r \beta^{-1} \cdot x\beta^{-1} J^s \alpha^{-1} \\ &= yJ^t \beta^{-1} \text{ (by (3.3) and } x \mapsto x\beta^{-1}) \\ \Rightarrow x\beta^{-1} J^s \alpha^{-1} J^{-s} \cdot [(xy)J^r \beta^{-1} \cdot x\beta^{-1} J^s \alpha^{-1}] J^{-t} &= (xy)J^r \beta^{-1} J^{-r} \\ &= x\beta^{-1} J^s \alpha^{-1} J^{-s} \cdot yJ^t \beta^{-1} J^{-t} \quad (3.4) \end{aligned}$$

$$\Rightarrow (xy)J^r \beta^{-1} J^{-r} = x\beta^{-1} J^s \alpha^{-1} J^{-s} \cdot yJ^t \beta^{-1} J^{-t} \quad (3.5)$$

Equation (3.5) implies that $(\beta^{-1} \alpha^{-1}, \beta^{-1}, \beta^{-1}) = (\beta^{-1}, \beta^{-1}, \beta^{-1})(\alpha^{-1}, i, i) \in ATP(Q)$, and this gives the required result for the second assertion, since $\beta \in A(Q)$.

Conversely, suppose Q is a (r, s, t) -inverse loop in which $A(Q)$ is a Boolean group and $(\alpha^{-1}, i, i) \in ATP(Q)$ for all $\alpha \in AUM(Q)$. Then, for any $x, y \in Q$, using this autotopism, $x\alpha^{-1} \cdot y = xy$. Setting $x = x\beta$ and $y = y\beta$ and with understanding that $\beta \in AUM(Q)$ then, $x\beta^{-1} \alpha^{-1} \cdot y\beta^{-1} = (xy)\beta^{-1}$. Thus,

$$\begin{aligned} [x\beta^{-1} \alpha^{-1} \cdot y\beta^{-1}]J^r \cdot x\beta^{-1} \alpha^{-1} J^s &= (xy)\beta^{-1} J^r \cdot x\beta^{-1} \alpha^{-1} J^s \\ \Rightarrow y\beta^{-1} J^t &= (xy)\beta^{-1} J^r \cdot x\beta^{-1} \alpha^{-1} J^s \\ \Rightarrow yJ^t \beta^{-1} \alpha^{-1} &= (xy)J^r \beta^{-1} \alpha^{-1} \cdot xJ^s \beta^{-1} \alpha^{-1} \alpha^{-1} \\ \Rightarrow yJ^t \beta^{-1} \alpha^{-1} &= (xy)J^r \beta^{-1} \alpha^{-1} \cdot xJ^s \beta^{-1} \\ \Rightarrow yJ^t \beta^{-1} \alpha^{-1} &= [(xy)J^r \beta^{-1} \cdot xJ^s \beta^{-1} \alpha] \alpha^{-1} \\ \Rightarrow yJ^t \beta^{-1} &= (xy)J^r \beta^{-1} \cdot xJ^s \beta^{-1} \alpha. \end{aligned}$$

Setting $x\beta = x$ and $\alpha = \alpha^{-1}$ in the last expression gives $yJ^t\beta^{-1} = (x\beta \cdot y)J^r\beta^{-1} \cdot xJ^s\alpha^{-1}$. Compare the last equation with equations in (3.3). Then, doing the reverse from (3.3) using Lemma 3.1, we arrive at $A(Q)$ -holomorph H is a (r, s, t) -inverse loop. $\square$

Theorem 3.2 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$ and $r, s, t \in \mathbb{Z}$. Then, $(H(Q), \circ)$ is a (r, s, t) -inverse loop if and only if Q is a (r, s, t) -inverse loop, whenever r is odd, s is even and t is odd.*

Proof : Suppose H is an (r, s, t) -inverse loop, then Q is an (r, s, t) -inverse loop. Now choose (α, x) and (β, y) in H , then $[(\alpha, x) \circ (\beta, y)]J^r \circ (\alpha, x)J^s = (\beta, y)J^t \Rightarrow (\alpha\beta, x\beta \cdot y)J^r \circ (\alpha, x)J^s = (\beta, y)J^t$. Since r, t are odd and s even, Lemma 3.1 can be applied to the last equation to obtain

$$\begin{aligned} & ((\alpha\beta)^{-1}, (x\beta \cdot y)J^r(\alpha\beta)^{-1}) \circ (\alpha, x)J^s = (\beta^{-1}, yJ^t\beta^{-1}) \\ \Rightarrow & ((\alpha\beta)^{-1}\alpha, (x\beta \cdot y)J^r(\alpha\beta)^{-1}\alpha \cdot xJ^s) = (\beta^{-1}, yJ^t\beta^{-1}) \end{aligned} \quad (3.6)$$

(3.6) can only be true if only $(\alpha\beta)^{-1}\alpha = \beta^{-1} \Leftrightarrow (\alpha\beta)^{-1} = \beta^{-1}\alpha^{-1}$ and $(x\beta \cdot y)J^r(\alpha\beta)^{-1}\alpha \cdot xJ^s = yJ^t\beta^{-1}$. Now, combining those last two equations, we get $(x\beta \cdot y)J^r\beta^{-1} \cdot xJ^s = yJ^t\beta^{-1} \Rightarrow (xJ^s)J^{-s} \cdot [(x\beta \cdot y)J^r\beta^{-1} \cdot xJ^s]J^{-t} = (xJ^s)J^{-s} \cdot (yJ^t\beta^{-1})J^{-t} \Rightarrow (x\beta \cdot y)J^r\beta^{-1}J^{-r} = x \cdot yJ^t\beta^{-1}J^{-t} \Rightarrow (x \cdot y)J^r\beta^{-1}J^{-r} = x\beta^{-1} \cdot yJ^t\beta^{-1}J^{-t}$. The last expression in autotopic form gives $(\beta^{-1}, J^t\beta^{-1}J^{-t}, J^r\beta^{-1}J^{-r}) = (\beta^{-1}, \beta^{-1}, \beta^{-1}) \in ATP(Q)$. This fact is obvious since β^{-1} is an automorphism.

Conversely, suppose Q is an (r, s, t) -inverse loop, then the steps above can be reversed to obtain the sufficient proof. $\square$

Corollary 3.1 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$ and $m \in \mathbb{Z}$. Then, $(H(Q), \circ)$ is a $(m, m+1, m)$ -inverse loop if and only if Q is a $(m, m+1, m)$ -inverse loop, whenever m is odd.*

Proof : When $r = t = m$ odd, $s = m+1$ is even, we have a special case of Theorem 3.2. $\square$

Corollary 3.2 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$. Then, $(H(Q), \circ)$ is a $(1, 2, 1)$ -inverse loop if and only if Q is a $(1, 2, 1)$ -inverse loop.*

Proof : This is a case where m is odd i.e $m = 1$, and the proof is immediate from Theorem 3.2. $\square$

Remark 3.1 : Loops with $(1, 2, 1)$ -inverse property are the ones referred to as doubly weak inverse property (WWIP) by Csörgö et al. in [9] and by Corollary 3.2, their holomorph is also a WWIP. Theorem 3.2, Corollary 3.1 and Corollary 3.2 are the generalizations of Huthnance's [12] result on the holomorph of weak inverse property loop.

Corollary 3.3 : Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$. Then, $(H(Q), \circ)$ is a WIP loop if and only if Q is a WIP loop.

Proof : WIP loop is a case where either $r = t = -1$ and $s = 0$ or when m is odd i.e -1 , and the proof is immediate from Theorem 3.2. $\square$

Theorem 3.3 : Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$ and $r, s, t \in \mathbb{Z}$. Then, $(H(Q), \circ)$ is a (r, s, t) -inverse loop if and only if Q is a (r, s, t) -inverse loop, $\alpha^2 \beta^2 = i$ and $(\alpha\beta, i, \beta^{-1}) \in ATP(Q)$ for all $\alpha, \beta \in A(Q)$, whenever r is odd, s is odd and t is even.

Proof : Suppose H is a (r, s, t) -inverse loop. Since $(Q, \cdot)$ is isomorphic to a subloop of H , then Q is a (r, s, t) -inverse loop. Now choose $(\alpha, x), (\beta, y) \in H$, having in mind that H is a (r, s, t) -inverse loop, then

$$[(\alpha, x) \circ (\beta, y)]J^r \circ (\alpha, x)J^s = (\beta, y)J^t \Rightarrow (\alpha\beta, x\beta \cdot y)J^r \circ (\alpha, x)J^s = (\beta, y)J^t$$

given that r and s are odd, and t is even and using result from Lemma 3.1, to have

$$\begin{aligned} ((\alpha\beta)^{-1}, (x\beta \cdot y)J^r(\alpha\beta)^{-1}) \circ (\alpha^{-1}, xJ^s\alpha^{-1}) &= (\beta, yJ^t) \Rightarrow ((\alpha\beta)^{-1}\alpha^{-1}, \\ & (x\beta \cdot y)J^r(\alpha\beta)^{-1}\alpha^{-1} \cdot xJ^s\alpha^{-1}) = (\beta, yJ^t). \end{aligned}$$

The last expression is true if and only if

$$(\alpha\beta)^{-1}\alpha^{-1} = \beta \Leftrightarrow \alpha^2\beta^2 = i \text{ and } (x\beta \cdot y)J^r(\alpha\beta)^{-1}\alpha^{-1} \cdot xJ^s\alpha^{-1} = yJ^t \quad (3.7)$$

With (3.7), the first assertion is accomplished. Moreover, the second equation in (3.7) implies that

$(x\beta \cdot y)J^r(\alpha\beta)^{-1}\alpha^{-1} \cdot xJ^s\alpha^{-1} = yJ^t \Rightarrow (xJ^s\alpha^{-1})J^{-s} \cdot [(x\beta \cdot y)J^r(\alpha\beta)^{-1}\alpha^{-1} \cdot xJ^s\alpha^{-1}]J^{-t} = (xJ^s\alpha^{-1})J^{-s} \cdot (yJ^t)J^{-t} \Rightarrow xJ^s\alpha^{-1}J^{-s} \cdot y = (x\beta \cdot y)J^r(\alpha\beta)^{-1}\alpha^{-1}J^{-r}$. Setting $x \mapsto x\beta$ and (3.7) in the last expression, then $x\beta^{-1}J^s\alpha^{-1}J^{-s} \cdot y = (x \cdot y)J^r\beta J^{-r}$ is obtained. Expressing this in term of autotopism gives $(\beta^{-1}J^s\alpha^{-1}J^{-s}, i, J^r\beta J^{-r}) = (\beta^{-1}\alpha^{-1}, i, \beta) \in ATP(Q)$. Reversing the steps above will produce the sufficiency proof. $\square$

Theorem 3.4 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$ and $r, s, t \in \mathbb{Z}$. Then, $(H(Q), \circ)$ is a (r, s, t) -inverse loop if and only if Q is a (r, s, t) -inverse loop, $A(Q)$ is a Boolean group and $\beta \in \Lambda(Q)$ for all $\beta \in A(Q)$, , whenever r is odd, s is even and t is even.*

Proof : Suppose H is a (r, s, t) -inverse loop. Since $(Q, \cdot)$ is isomorphic to a subloop of H , then Q is a (r, s, t) -inverse loop. Now choose $(\alpha, x), (\beta, y) \in H$, having in mind that H is a (r, s, t) -inverse loop, then

$$[(\alpha, x) \circ (\beta, y)]J^r \circ (\alpha, x)J^s = (\beta, y)J^t \Rightarrow (\alpha\beta, x\beta \cdot y)J^r \circ (\alpha, x)J^s = (\beta, y)J^t$$

given that r is odd, s and t is even and using result from Lemma 3.1, to have

$$((\alpha\beta)^{-1}, (x\beta \cdot y)J^r (\alpha\beta)^{-1}) \circ (\alpha, xJ^s) = (\beta, yJ^t) \Rightarrow ((\alpha\beta)^{-1}\alpha, (x\beta \cdot y)J^r (\alpha\beta)^{-1}\alpha \cdot xJ^s) = (\beta, yJ^t)$$

The last expression is true if and only if

$$(\alpha\beta)^{-1}\alpha = \beta \text{ and } (x\beta \cdot y)J^r (\alpha\beta)^{-1}\alpha \cdot xJ^s = yJ^t \quad (3.8)$$

The first equation in (3.8) implies $A(Q)$ is a Boolean group. Similarly, combining the equations in (3.8), setting $x = x\beta$ and bearing in mind that H is an (r, s, t) -inverse loop, then the expression $x\beta^{-1} \cdot y = (xy)J^r \beta J^{-r}$ is obtained. Expressing this in term of autotopism gives the necessary proof.

Conversely if the argument above is done in reversed direction the proof of the sufficiency is also obtained. $\square$

Theorem 3.5 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$ and $r, s, t \in \mathbb{Z}$. Then, $(H(Q), \circ)$ is a (r, s, t) -inverse loop if and only if Q is a (r, s, t) -inverse loop, $\beta\alpha\beta = \alpha$ and $(\beta^{-1}, \alpha\beta, i) \in ATP(Q)$ for all $\alpha, \beta \in A(Q)$, whenever r is even, s is odd and t is odd.*

Proof : Suppose H is a (r, s, t) -inverse loop. Since $(Q, \cdot)$ is isomorphic to a subloop of H , then Q is a (r, s, t) -inverse loop. Now choose $(\alpha, x), (\beta, y) \in H$, having in mind that H is a (r, s, t) -inverse loop, then

$$[(\alpha, x) \circ (\beta, y)]J^r \circ (\alpha, x)J^s = (\beta, y)J^t \Rightarrow (\alpha\beta, x\beta \cdot y)J^r \circ (\alpha, x)J^s = (\beta, y)J^t$$

given that r is even, s and t is odd and using result from Lemma 3.1, to have

$$(\alpha\beta, (x\beta \cdot y)J^r) \circ (\alpha^{-1}, xJ^s\alpha^{-1}) = (\beta^{-1}, yJ^t\beta^{-1}) \Rightarrow (\alpha\beta\alpha^{-1}, (x\beta \cdot y)J^r\alpha^{-1} \cdot xJ^s\alpha^{-1}) \\ = (\beta^{-1}, yJ^t\beta^{-1}).$$

The last expression is true if and only if $\alpha\beta\alpha^{-1} = \beta^{-1} \Leftrightarrow \alpha\beta = \beta^{-1}\alpha \Leftrightarrow \beta\alpha\beta = \alpha$ and $(x\beta \cdot y)J^r\alpha^{-1} \cdot xJ^s\alpha^{-1} = yJ^t\beta^{-1} \Leftrightarrow xJ^s\alpha^{-1}J^{-s} \cdot yJ^t\beta^{-1}J^{-t} = (x\beta \cdot y)J^r\alpha^{-1}J^{-r}$ setting $x = x\beta$ and expressing in term of autotopism gives $(\beta^{-1}J^s\alpha^{-1}J^{-s}, J^t\beta^{-1}J^{-t}, J^r\alpha^{-1}J^{-r}) = (\beta\alpha^{-1}, \beta^{-1}, \alpha^{-1})$. This gives the required results for the necessary condition.

These steps that obtained the proof of necessary condition can be worked in reverse direction to obtain the sufficient condition proof. $\square$

Theorem 3.6 : Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$ and $r, s, t \in \mathbb{Z}$. Then, $(H(Q), \circ)$ is a (r, s, t) -inverse loop if and only if Q is a (r, s, t) -inverse loop, $A(Q)$ is an abelian group, $\beta \in \Phi(Q)$ and $\beta' = \alpha \in \Phi'(Q)$ whenever r is even, s is odd and t is even.

Proof : Suppose H is a (r, s, t) -inverse loop. Since $(Q, \cdot)$ is isomorphic to a subloop of H , then Q is a (r, s, t) -inverse loop. Now choose $(\alpha, x), (\beta, y) \in H$, having in mind that H is a (r, s, t) -inverse loop, then

$$[(\alpha, x) \circ (\beta, y)]J^r \circ (\alpha, x)J^s = (\beta, y)J^t \Rightarrow (\alpha\beta, x\beta \cdot y)J^r \circ (\alpha, x)J^s = (\beta, y)J^t$$

Given that r is even s is odd, and t is even and using result from Lemma 3.1, we have

$$(\alpha\beta, (x\beta \cdot y)J^r) \circ (\alpha^{-1}, xJ^s\alpha^{-1}) = (\beta, yJ^t) \Rightarrow (\alpha\beta\alpha^{-1}, (x\beta \cdot y)J^r\alpha^{-1} \cdot xJ^s\alpha^{-1}) = (\beta, yJ^t).$$

The last expression is true if and only if $\alpha\beta\alpha^{-1} = \beta \Leftrightarrow \alpha\beta = \beta\alpha$ which ensures commutativity for $A(Q)$. And

$$(x\beta \cdot y)J^r\alpha^{-1} \cdot xJ^s\alpha^{-1} = yJ^t \Leftrightarrow x\beta^{-1} \cdot yJ^t\alpha J^{-t} = xy \\ \Leftrightarrow (\beta^{-1}, J^t\alpha J^{-t}, i) = (\beta^{-1}, \alpha, i) \in ATP(Q) \quad (3.9)$$

Conversely, suppose Q is an (r, s, t) -inverse loop, the autotopism (3.9) holds in Q and that $A(Q)$ is abelian, then we shall show that the $A(Q)$ -holomorph H is also an inverse loop. Thus, for any x, y in Q ,

$$x\beta^{-1} \cdot yJ^t\alpha J^{-t} = xy \Leftrightarrow [x\beta^{-1} \cdot yJ^t\alpha J^{-t}]J^r \cdot x\beta^{-1}J^s = (x \cdot y)J^r \cdot x\beta^{-1}J^s \\ \Leftrightarrow (x \cdot y)J^r \cdot x\beta^{-1}J^s = yJ^t\alpha \Leftrightarrow (x\beta \cdot y)J^r \cdot x\beta^{-1}J^s = yJ^t\alpha \quad (3.10) \\ (x\beta \cdot y)J^r\alpha^{-1} \cdot xJ^s\alpha^{-1} = yJ^t$$

Since $A(Q)$ is abelian, then for any α, β in $A(Q)$, it implies that $\alpha\beta = \beta\alpha \Leftrightarrow \alpha\beta\alpha^{-1} = \beta$. Combining equations (3.10) and the last equation,

$$\begin{aligned}
(\alpha\beta\alpha^{-1}, (x\beta \cdot y)J^r\alpha^{-1} \cdot xJ^s\alpha^{-1}) &= (\beta, yJ^t) \Rightarrow (\alpha\beta, (x\beta \cdot y)J^r) \circ (\alpha^{-1}, xJ^s\alpha^{-1}) = (\beta, yJ^t) \\
&\Rightarrow (\alpha\beta, (x\beta \cdot y))J^r \circ (\alpha, x)J^s = (\beta, y)J^t \quad (\because r \text{ and } t \text{ are even, } s \text{ odd}) \\
&\text{odd} \Rightarrow ((\alpha, x) \circ (\beta, y))J^r \circ (\alpha, x)J^s = (\beta, y)J^t \quad (3.11)
\end{aligned}$$

Equation (3.11) implies the (r, s, t) -inverse identity relative to the holomorph H of Q . Thus the result is proved. $\square$

Corollary 3.4 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$ and $m \in \mathbb{Z}$. Then, $(H(Q), \circ)$ is a $(m, m+1, m)$ -inverse loop if and only if Q is a $(m, m+1, m)$ -inverse loop, $A(Q)$ is an abelian group and, $\beta \in \Phi(Q)$ and $\beta' = \alpha \in \Phi'$, whenever m is even.*

Proof : This case is similar to the above Theorem 3.6, where $r = t = m$ and $s = m+1$ for even m . $\square$

Corollary 3.5 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$. Then, $(H(Q), \circ)$ is a CIP loop if and only if Q is a CIP loop, $A(Q)$ is an abelian and, $\beta \in \Phi(Q)$ and $\beta' = \alpha \in \Phi'$.*

Proof : This case is a special case of Corollary 3.5, where $m = 0$. $\square$

Theorem 3.7 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$ and $r, s, t \in \mathbb{Z}$. Then, $(H(Q), \circ)$ is a (r, s, t) -inverse loop if and only if Q is a (r, s, t) -inverse loop, $(\alpha\beta)^2 = i$ and $(\beta^{-1}, \beta^{-1}, \alpha) \in ATP(Q)$ for all $\alpha, \beta \in A(Q)$, whenever r is even, s is even and t is odd.*

Proof : Suppose H is a inverse loop. Since $(Q, \cdot)$ is isomorphic to a subloop of H , then Q is a inverse loop. Now choose $(\alpha, x), (\beta, y) \in H$, having in mind that H is a inverse loop, then

$$[(\alpha, x) \circ (\beta, y)]J^r \circ (\alpha, x)J^s = (\beta, y)J^t \Rightarrow (\alpha\beta, x\beta \cdot y)J^r \circ (\alpha, x)J^s = (\beta, y)J^t$$

Given that r and s are even, and t is odd and using result from Lemma 3.1, we have

$$(\alpha\beta, (x\beta \cdot y)J^r) \circ (\alpha, x)J^s = (\beta^{-1}, yJ^t\beta^{-1}) \Rightarrow (\alpha\beta\alpha, (x\beta \cdot y)J^r\alpha \cdot xJ^s) = (\beta^{-1}, yJ^t\beta^{-1}).$$

The last expression is true if and only if $\alpha\beta\alpha = \beta^{-1} \Leftrightarrow \alpha\beta = (\alpha\beta)^{-1} \Leftrightarrow (\alpha\beta)^2 = i$ and

$$\begin{aligned}
(x\beta \cdot y)J^r\alpha \cdot xJ^s &= yJ^t\beta^{-1} \Leftrightarrow x\beta^{-1} \cdot yJ^t\beta J^{-t} = (xy)J^r\alpha J^{-r} \\
&\Leftrightarrow (\beta^{-1}, J^t\beta J^{-t}, J^r\alpha J^{-r}) = (\beta^{-1}, \beta, \alpha) \text{ in } ATP(Q).
\end{aligned}$$

Using this same method but in reverse order of the prove the necessary condition, the sufficient condition is obtained. $\square$

Theorem 3.8 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$ and $r, s, t \in \mathbb{Z}$. Then, $(H(Q), \circ)$ is a (r, s, t) -inverse loop if and only if Q is a (r, s, t) -inverse loop, $\alpha\beta\alpha = \beta$ in $A(Q)$ and $(\beta, i, \alpha^{-1}) \in ATP(Q)$ for all $\alpha, \beta \in A(Q)$, whenever r, s and t are even.*

Proof : Suppose H is a (r, s, t) -inverse loop. Since $(Q, \cdot)$ is isomorphic to a subloop of H , then Q is a (r, s, t) -inverse loop. Now choose $(\alpha, x), (\beta, y) \in H$, having in mind that H is a (r, s, t) -inverse loop, then

$$[(\alpha, x) \circ (\beta, y)] J^r \circ (\alpha, x) J^s = (\beta, y) J^t \Rightarrow (\alpha\beta, x\beta \cdot y) J^r \circ (\alpha, x) J^s = (\beta, y) J^t$$

given that r, s and t are even and using result from Lemma 3.1, to have

$$(\alpha\beta, (x\beta \cdot y) J^r) \circ (\alpha, x J^s) = (\beta, y J^t) \Rightarrow (\alpha\beta\alpha, (x\beta \cdot y) J^r \alpha \cdot x J^s) = (\beta, y J^t).$$

The last expression is true if and only if $\alpha\beta\alpha = \beta$ and $(x\beta \cdot y) J^r \alpha \cdot x J^s = y J^t$. Setting $x = x\beta$ and bearing in mind that H is (r, s, t) -inverse loop, then the equation implies

$$x\beta^{-1} \cdot y = (xy) J^r \alpha J^{-r} \Leftrightarrow (\beta^{-1}, i, J^r \alpha J^{-r}) \in ATP(Q).$$

The converse of this argument is also obtained, by reversing the steps of the necessary prove. $\square$

4. Holomorphs of Buchsteiner Loops

In this section, holomorphy theory of Buchsteiner loops is given attention.

Theorem 4.1 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$. Then, $(H(Q), \circ)$ is a Buchsteiner loop if and only if $(L_{x\phi}^{-1}, R_x, L_{x\phi}^{-1} R_x)$ is an autotopism of $(Q, \cdot)$, $\forall x \in Q$ and $\forall \phi \in AUM(Q)$.*

Proof : Suppose the $AUM(Q)$ -holomorph $(H, \circ)$ of Q is a Buchsteiner loop, then we have the following:

$$\begin{aligned} & \{(\beta, y) \circ [(\gamma, z) \circ (\alpha, x)]\} / (\alpha, x) = (\alpha, x) \setminus [(\alpha, x) \circ (\beta, y)] \circ (\gamma, z) \} \\ \Rightarrow & [(\beta, y) \circ (\gamma\alpha, z\alpha \cdot x)] / (\alpha, x) = (\alpha, x) \setminus [(\alpha\beta, x\beta \cdot y) \circ (\gamma, z)] \\ \Rightarrow & (\beta \cdot \gamma\alpha, y\gamma\alpha \cdot (z\alpha \cdot x)) / (\alpha, x) = (\alpha, x) \setminus (\alpha\beta \cdot \gamma, (x\beta \cdot y)\gamma \cdot z) \end{aligned} \quad (4.1)$$

Now for any $(\phi, p) \in H$, using equation (4.1), we have

$$(\beta \cdot \gamma \alpha, y \gamma \alpha \cdot (z \alpha \cdot x)) / (\alpha, x) = (\phi, p) = (\alpha, x) \setminus (\alpha \beta \cdot \gamma, (x \beta \cdot y) \gamma \cdot z) \quad (4.2)$$

The implications of equation (4.2) are $(\beta \cdot \gamma \alpha, y \gamma \alpha \cdot (z \alpha \cdot x)) / (\alpha, x) = (\phi, p)$ and $(\phi, p) = (\alpha, x) \setminus (\alpha \beta \cdot \gamma, (x \beta \cdot y) \gamma \cdot z)$. Solving these two equations give $(\beta \gamma, [y \gamma \cdot (z \cdot x \alpha^{-1})] / x \alpha^{-1}) = (\beta \gamma, x \beta \gamma \setminus [(x \beta \cdot y) \gamma \cdot z])$ which is true if $\beta \gamma = \beta \gamma$ and $[y \gamma \cdot (z \cdot x \alpha^{-1})] / x \alpha^{-1} = x \beta \gamma \setminus [(x \beta \gamma \cdot y \gamma) \cdot z]$. If we set $x = x \alpha$ and $y = y \gamma$ in the last expression, we obtain $(y \cdot z x) / x = x \alpha \beta \gamma \setminus [(x \alpha \beta \gamma \cdot y) z]$. Putting this equation in autotopic form gives the required result if $\alpha \beta \gamma = \phi$.

Conversely, suppose the autotopism holds in $(Q, \cdot)$, then doing the calculations above in reverse order shows that $A(Q)$ -holomorph $(H, \circ)$ is a Buchstainer loop. $\square$

Theorem 4.2 : *Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$. Then, $(H(Q), \circ)$ is a Buchsteiner loop if and only if $(Q, \cdot)$ is a Buchsteiner loop and each $\phi \in A(Q)$ is a nuclear automorphism of $(Q, \cdot)$.*

Proof : Suppose $(H, \circ)$ is a Buchsteiner loop. Considering the fact that $(Q, \cdot)$ is isomorphic to a subloop of $(H, \circ)$, then $(Q, \cdot)$ is a Buchsteiner loop. From Theorem 4.1, we see that,

$$A = (L_x^{-1}, R_x, L_x^{-1} R_x), B = (L_{x\phi}^{-1}, R_x, L_{x\phi}^{-1} R_x) \in ATP(Q)$$

$\forall x \in Q$ and $\forall \phi \in A(Q)$. Then, $B^{-1}A = (L_{x\phi}^{-1}, R_x, L_{x\phi}^{-1} R_x)^{-1} (L_x^{-1}, R_x, L_x^{-1} R_x)$, which gives

$$B^{-1}A = (L_{x\phi} L_x^{-1}, i, R_x^{-1} L_{x\phi} L_x^{-1} R_x) \in ATP(Q) \quad (4.3)$$

$\forall x \in Q$ and $\forall \phi \in A(Q)$. Thus, for any $a, b \in Q$, we have from (4.3) that $a L_{x\phi} L_x^{-1} \cdot b i = (ab) R_x^{-1} L_{x\phi} L_x^{-1} R_x$, if we set $b = 1$, we have $L_{x\phi} L_x^{-1} = R_x^{-1} L_{x\phi} L_x^{-1} R_x$. If we put this into (4.3) we obtained

$$B^{-1}A = (L_{x\phi} L_x^{-1}, i, L_{x\phi} L_x^{-1}) \quad (4.4)$$

Repeating the same process using (4.4) we have $L_{x\phi} L_x^{-1} = L_{x \setminus x\phi}$, and so (4.4) gives $B^{-1}A = (L_{x \setminus x\phi}, i, L_{x \setminus x\phi})$, for all x in Q and for all ϕ in $A(Q)$. Thus, $x \setminus x\phi \in N_\lambda$ and by [Corollary 1.6, [9]], $x \setminus x\phi \in N(Q)$ for all x in Q and for all ϕ in $A(Q)$, hence ϕ is nuclear.

Conversely, suppose $(Q, \cdot)$ is a Buchsteiner loop, and each ϕ in $A(Q)$ is nuclear. Then for each ϕ in $A(Q)$ and each x in Q the element $x \setminus x\phi \in N_\mu$ and $x\phi \cdot y = (x \cdot (x \setminus x\phi))y = x((x \setminus x\phi)y) \Rightarrow y L_{x\phi} = y L_{x \setminus x\phi} L_x$ which therefore means that $L_{x\phi} L_x^{-1} = L_{x \setminus x\phi}$. Since the nucleus of $(Q, \cdot)$ coincide, we have $C = (L_{x \setminus x\phi}, i, L_{x \setminus x\phi}) = (L_{x\phi} L_x^{-1}, i, L_{x\phi} L_x^{-1})$. Therefore,

$$AC^{-1} = (L_x^{-1}, R_x, L_x^{-1}R_x)(L_{x\phi}L_x^{-1}, i, L_{x\phi}L_x^{-1})^{-1} = (L_{x\phi}^{-1}, R_x, L_x^{-1}R_xL_{x\phi}L_x^{-1}).$$

Now, for any $y, z \in Q$, $yL_{x\phi}^{-1} \cdot zR_x = (yz)L_x^{-1}R_xL_{x\phi}L_x^{-1}$, if we put $z = 1$, we have $yL_{x\phi}^{-1}R_x = yL_x^{-1}R_xL_{x\phi}L_x^{-1} \Rightarrow L_{x\phi}^{-1}R_x = L_x^{-1}R_xL_{x\phi}L_x^{-1}$ thus, $AC^{-1} = (L_{x\phi}^{-1}, R_x, L_{x\phi}^{-1}R_x)$, and the result is proved. $\square$

Corollary 4.1 : Let Q be any loop with $A(Q)$ -holomorph $(H(Q), \circ)$. If $(H(Q), \circ)$ is a Buchsteiner loop, then, $x \setminus x\phi \in N(Q)$ for all $x \in Q$ and for all $\phi \in A(Q)$.

Proof : By Theorem 4.2, the result follows. $\square$

4.1 Nuclear Automorphism of Buchsteiner Loop

Lemma 4.1 : Let $(Q, \cdot)$ be a Buchsteiner loop, the right and left inner mappings $R_{(x,y)}, L_{(x,y)}$ are nuclear automorphisms for any $x, y \in Q$.

Proof : Let Q be a Buchsteiner loop, then by [[9], Corollary 1.8], $N(Q)$ is a normal subgroup of Q and hence Q/N is an abelian group by [[9], Theorem 5.3]. By [[9], Corollary 5.4], $R_{(x,y)}, L_{(x,y)} \in AUM(Q)$ for any $x, y \in Q$. Now, let $\theta: Q \rightarrow Q/N$ be a natural homomorphism. Then, $\ker \theta = N(Q)$. Suppose $x, y, z, w \in Q$, then $w = z \setminus zR_{(x,y)} \Leftrightarrow zx \cdot y = zw \cdot xy$. Thus, $(zx \cdot y)\theta = (zw \cdot xy)\theta \Rightarrow z\theta x\theta y\theta = z\theta w\theta x\theta y\theta \Rightarrow w\theta = e_{Q/N} \Rightarrow w \in \ker \theta \Rightarrow w \in N(Q)$. Thus, $w = z \setminus zR_{(x,y)} \in N(Q)$ for all $x, y, z \in Q$. Hence $R_{(x,y)}$ is a nuclear automorphism. A similar proof goes for $L_{(x,y)}$. $\square$

Remark 4.1 : In a Buchsteiner loop $(Q, \cdot)$, $R_{(x,y)} = L_{(y,x)}^{-1}$ according to [9, Theorem 1.4]. Furthermore, $\text{Inn}_\rho(Q) = \text{Inn}_\lambda(Q)$.

Theorem 4.3 : Let $(Q, \cdot)$ be a Buchsteiner loop and let $\text{NAUT}(Q) = \{\phi \in AUM(Q) \mid x \setminus x\phi \in N(Q) \forall x \in Q\}$, the set of all nuclear automorphisms of $(Q, \cdot)$. Then, $\text{Inn}_\rho(Q) = \text{Inn}_\lambda(Q) \trianglelefteq \text{NAUT}(Q) \trianglelefteq AUM(Q, \cdot)$.

Proof : $\text{NAUT}(Q) \neq \emptyset$ since $x \setminus xI = e \in N(Q)$. Recall from the converse proof of the Theorem 4.2 that $L_{u \setminus u\phi} = L_{u\phi}^{-1}L_u^{-1}$ for all $u \in Q$ and $\forall \phi \in \text{NAUT}(Q)$. Thus,

$$\begin{aligned} \text{NAUT}(Q) &= \{\phi \in A(Q) \mid D(\phi, x) = (L_{x \setminus x\phi}, I, L_{x \setminus x\phi}) \in \text{AUT}(Q) \forall x \in Q\} \\ &= \{\phi \in A(Q) \mid D(\phi, x) = (L_{x\phi}L_x^{-1}, I, L_{x\phi}L_x^{-1}) \in \text{AUT}(Q) \forall x \in Q\}. \end{aligned}$$

Let $\phi, \sigma \in \text{NAUT}(Q)$, then, for all $u \in Q$,

$$\begin{aligned}
D(\phi, u) &= (L_{u \setminus u\phi}, I, L_{u \setminus u\phi}) = (L_{u\phi} L_u^{-1}, I, L_{u\phi} L_u^{-1}) \in \text{AUT}(Q) \text{ and} \\
D(\sigma, u) &= (L_{u \setminus u\sigma}, I, L_{u \setminus u\sigma}) = (L_{u\sigma} L_u^{-1}, I, L_{u\sigma} L_u^{-1}) \in \text{AUT}(Q). \text{ So} \\
D(\sigma, u\phi) D(\phi, u) &= (L_{u\phi\sigma} L_{u\phi}^{-1}, I, L_{u\phi\sigma} L_{u\phi}^{-1}) (L_{u\phi} L_u^{-1}, I, L_{u\phi} L_u^{-1}) \in \text{AUT}(Q) \\
&\Rightarrow D(\sigma, u\phi) D(\phi, u) = (L_{u\phi\sigma} L_u^{-1}, I, L_{u\phi\sigma} L_u^{-1}) \in \text{AUT}(Q) \quad (4.5) \\
&\Rightarrow D(\beta, u\alpha) D(\alpha, u) = (L_{u \setminus u\phi\sigma}, i, L_{u \setminus u\phi\sigma}) \in \text{AUT}(Q) \quad (4.6)
\end{aligned}$$

From equation (4.6), $u \setminus u\phi\sigma \in N(Q)$. Thus $\phi\sigma \in \text{NAUT}(Q)$. If $\phi \in \text{NAUT}(Q)$, then for all $u \in Q$,

$$\begin{aligned}
D(\phi, u) &= (L_{u\phi} L_u^{-1}, I, L_{u\phi} L_u^{-1}) \in \text{AUT}(Q) \Rightarrow \\
D(\phi, u)^{-1} &= (L_u L_{u\phi}^{-1}, I, L_u L_{u\phi}^{-1}) \in \text{AUT}(Q) \Rightarrow \\
D(\phi, u\phi^{-1})^{-1} &= (L_{u\phi^{-1}} L_{u\phi^{-1}\phi}^{-1}, I, L_{u\phi^{-1}} L_{u\phi^{-1}\phi}^{-1}) \in \text{AUT}(Q) \Rightarrow \\
D(\phi, u\phi^{-1})^{-1} &= (L_{u\phi^{-1}} L_u^{-1}, I, L_{u\phi^{-1}} L_u^{-1}) \in \text{AUT}(Q) \Rightarrow \phi^{-1} \in \text{NAUT}(Q).
\end{aligned}$$

Thus, $\text{NAUT}(Q) \leq A(Q, \cdot)$. Let $\phi \in \text{NAUTS}(Q)$, then $u \setminus u\phi \in N(Q, \cdot)$, $\forall u \in Q$ and so $(u \setminus u\phi)xy = (u \setminus u\phi)x \cdot y \forall u, x, y \in Q$, if ϕ is an automorphism of $(Q, \cdot)$, then we have $\{u\phi \setminus u\phi\phi\}(x\phi \cdot y\phi) = \{u\phi \setminus u\phi\phi\}x\phi \cdot y\phi$ for all $u, x, y \in Q$, and if we replace u by $u\phi^{-1}$ in the last equation, we have

$$\begin{aligned}
\{u\phi^{-1}\phi \setminus u\phi^{-1}\phi\phi\}(x\phi \cdot y\phi) &= \{u\phi^{-1}\phi \setminus u\phi^{-1}\phi\phi\}x\phi \cdot y\phi \Rightarrow \\
\{u \setminus u\phi^{-1}\phi\phi\}(x\phi \cdot y\phi) &= \{u \setminus u\phi^{-1}\phi\phi\}x\phi \cdot y\phi \Rightarrow u \setminus u\phi^{-1}\phi\phi \in N(Q, \cdot) \\
&\Rightarrow \phi^{-1}\phi\phi \in \text{NAUT}(Q)
\end{aligned}$$

for all $u \in Q$ and $\phi, \phi \in \text{NAUT}(Q)$. So, $\text{NAUT}(Q) \trianglelefteq \text{AUM}(Q)$. Furthermore, by Lemma 4.1, $\text{Inn}_\rho(Q) = \text{Inn}_\lambda(Q) \trianglelefteq \text{NAUT}(Q)$. $\square$

Remark 4.2 : Based on the holomorphic characterization of Buchsteiner loop in this work, some special types of Buchsteiner loops Q for which there exists a non-trivial group $A(Q) \leq \text{NAUT}(Q)$ exist.

1. In [10], it was shown that there exists a nilpotent Buchsteiner loop Q of nilpotency class 3 and order 128 such that $\text{Inn}(Q)$ is an abelian group. Thus, $\text{Inn}_\rho(Q) = \text{Inn}_\lambda(Q)$ is an abelian group. By Theorem 4.3, $\text{Inn}_\rho(Q) = \text{Inn}_\lambda(Q) \trianglelefteq \text{NAUT}(Q)$ and so, we can take $A(Q) = \text{Inn}_\rho(Q) = \text{Inn}_\lambda(Q)$ so that $A(Q)$ -holomorph $(H(Q), \circ)$ is a Buchsteiner loop in accordance to Theorem 4.2.
2. In [8], the authors described the structure of Buchsteiner loops Q with abelian inner mapping groups. So, we can take $A(Q) = \text{Inn}_\rho(Q) = \text{Inn}_\lambda(Q)$

such that the $A(Q)$ -holomorph $(H(Q), \circ)$ is a Buchsteiner loop in accordance to Theorem 4.2.

3. In [[9], Lemma 2.3], it was shown that an extra loop Q is a Buchsteiner loop. Going by [[32], Lemma 2.7], $Inn_{\rho}(Q)$ and $Inn_{\lambda}(Q)$ are Boolean groups. So, we can take $A(Q) = Inn_{\rho}(Q)$ or $A(Q) = Inn_{\lambda}(Q)$ such that the $A(Q)$ -holomorph $(H(Q), \circ)$ is a Buchsteiner loop in accordance to Theorem 4.2.

More generally, based on the holomorphic characterization of some (r, s, t) -inverses loops in this work, some special types of (r, s, t) -inverses loops Q for which there exist non-trivial group $A(Q)$ which is a Boolean group or abelian group so that their $A(Q)$ -holomorph $(H(Q), \circ)$ is (r, s, t) -inverse loop exist; Jaiyéṣá and Adéníran [24] showed that there exist A-loops with some commutative inner mappings and others of order 2.

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