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A new ideal secret sharing scheme based on a tree

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Abstract

One of the methods used in order to protect a secret K is a *secret sharing scheme*. In this method, the secret K is distributed among a finite set of participants P by a special participant called the *dealer*, assumed to be honest, in such a way that only predefined subsets of participants can recover the secret after collaborating with their secret shares. The construction of secret sharing schemes has received a considerable attention of many researchers whose main goal was to improve the information rate. In this paper, we focused on the hierarchical secret sharing. Related to the previous works, we propose a novel simpler construction of a secret sharing scheme, where the access structure is a tree according to the hierarchical concept of companies illustrated through its organization chart. We prove that the proposed scheme is *ideal* by showing that the information rate equals 1. In order to show the efficiency of the proposed scheme, we discuss all possible kinds of attacks and prove that the security is ensured. Finally, we include a detailed didactic example of a small company organization chart.

Subject Classification: *Primary 11T71, Secondary 94A60.*

Keywords: *Hierarchical secret sharing scheme, Qualified subsets, Access structure, Interpolation, Information rate.*

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1. Introduction

The fast development of computer networks and data communication systems make the protection of secret data extremely imperative. In order to protect a secret, several methods have been applied before, one of them is to encrypt data, but this will change the problem instead of solving it, since another method is required to protect the encrypted data. Its also possible to keep the secret in one well-guarded location, but this method is very unreliable since the secret can be destroyed or become inaccessible. Another method consists in sharing the data, either by storing multiple copies of the data in different locations, which would increase security vulnerabilities, or by splitting the data into several parts and sharing them among different members of the system. This last method is called secret sharing scheme and would be very efficient in case where the reconstruction of the initial data does not require the presence of all the system members, otherwise the veto given to each member would paralyze the system [1]. Secret sharing schemes have many applications in different areas, such as access control, launching a missile, and opening a bank vault. For more details, see for instance [22, 23].

The secret sharing scheme is therefore a method of distributing a secret K among a finite set of participants P, in such a way that only predefined subsets of participants can collaborate with their secret shares to recover the secret K. These subsets are called *qualified subsets* and the set of all qualified subsets is called the *access structure* denoted Γ [8]. Each subset of participants $Y \in \Gamma$ is called *a minimal qualified subset* if $(Y' \subset Y$ and $Y' \in \Gamma)$ implies $Y' = Y$. The family of all minimal qualified subsets is noted Γ_0. In a secret sharing scheme, the secret K is chosen by a special participant, called the dealer, who is responsible for computing and distributing the shares among the set of participants P and then assumed to be honest. The share of any participant refers specifically to the information that the dealer sends in private. It is required to keep the size of shares as small as possible since the security of a system degrades as the amount of information that must be kept secret increases.

1.1 *Related literature*

Many approaches have been proposed for the construction of a secret sharing scheme [24]. The first one, called (t, n) -*threshold scheme*, was introduced independently by Shamir and Blakley [1, 6] in 1979. In a (t, n) -threshold scheme, all groups of at least t participants of n-participants are qualified and can reconstruct the secret, while those with less than

t participants are unqualified and can't have any information about the secret. The scheme proposed by Shamir is based on polynomials over a finite field $GF(q)$ since a random polynomial f is chosen by the dealer for computing and distributing the shares among the set of participants P in such a way that, each participant p_i is given an ordered pair $(x_i, f(x_i))$ as a share. This scheme is still reliable and secure even when misfortunes destroy half the pieces and security breaches expose all but one of the remaining pieces. This scheme is *perfect*, since all qualified subsets can reconstruct the secret and unqualified subsets cannot determine any information about the secret. The scheme is called *ideal*, since x_i is publicly revealed so that the share of participant p_i becomes just $f(x_i)$ and then the size of each share equals the size of the secret. The scheme proposed by Blakley is based on geometries over finite fields, it's perfect and can be modified slightly to become ideal, as explained in [8].

Ito et al. have generalized the concept of threshold scheme and showed that, given any *monotone access structure* Γ, i.e., for $Y \in \Gamma$, if $Y \subset Y'$ then $Y' \in \Gamma$, there exists a perfect secret sharing scheme to realize the structure [14, 13]. Benaloh and Leichter proposed a different algorithm that has a lower *information rate* than Ito's et al. construction [15]. In both constructions, the information rate decreases exponentially as a function of the number of participants $n = |P|$. The information rate, noted ρ, is considered as a measure of the efficiency of a secret sharing scheme. It is defined as the ratio between the secret size and the maximum size of the shares S, that is, $\rho = \frac{\log_2(|K|)}{\log_2(|S|)}$ [8]. Other measures can also be considered such as *the average information rate*, which is defined as the ratio between the length of the secret and the arithmetic mean of the length of all shares and expressed as follow $\tilde{\rho} = \frac{n\log_2(|K|)}{\sum_{i=1}^{n}\log_2(|S_i|)}$ [16].

Many researchers have been interested to propose new secret sharing schemes with an improved information rate. Recently, Montoya and Mejia have studied in [17] the information rates that can be achieved by linear, abelian and nonabelian schemes and proved that nonabelian schemes outperform linear schemes. By providing some strong evidence, Montoya and Mejia conjecture that abelian schemes also outperform linear schemes.

1.2 *Hierarchical scheme*

Shamir had specified that one of the useful properties of the proposed threshold scheme is that by using tuples of polynomial values as parts

[1], it is possible to get a hierarchical scheme in which the number of parts needed to determine the secret depends on the importance of the participants. He also brought a brief explanation based on an example of a company's check signature. This was a motivation for another line of work, consisting in construction of ideal secret sharing scheme for families of access structures with interesting and special properties, which was introduced by Simmons in 1988. In [12], Simmons proposed two families of access structure: *the multilevel and the compartmented access structures* which are *multipartite*. In such access structures, participants are divided into several parts (levels and compartments) and the participants of each part play an equivalent role into the structure. In a multilevel access structure, each participant is assigned according to their importance. Participants are then hierarchically ordered and those in the higher level are more powerful than the ones in lower levels. In [8] Brickell shown that given any multilevel access structure, there exists q_0 such that for any q, a prime power with $q > q_0$, there is an ideal secret sharing scheme realizing this access structure over $GF(q)$.

In [28], Tassa has interested in problems in which the presence of a higher level participants is imperative to allow the reconstruction of the secret. Tassa proposed an ideal hierarchical threshold secret sharing scheme in which the access structure is determined by a sequence of threshold requirements. Unlike secret sharing schemes proposed before, Tassa have used polynomial derivatives to generate lesser shares for participants of lower levels and then Birkhoff interpolation for the reconstruction of the initial polynomial. Tassa and Dyn proposed in [29] constructions of ideal secret sharing schemes for several families of multipartite access structures which are based on bivariate interpolation. Later, Seluk et al. [26] have proposed a much simpler scheme that achieves the same results as the ones proposed by Tassa and Dyn. Moreover, a complete characterization of the ideal hierarchical access structures based on matroids was presented in [9].

1.3 *Visual secret sharing scheme*

Another type of secret sharing schemes, called *visual secret sharing (VSS) scheme*, was first proposed by Naor and Shamir in [18]. In such schemes, the shared secret is an image which is broken into shadow images so that humans can visually reconstruct the secret with their eyes by superposing a qualified combination of visual shares each printed on a transparency. The reconstruction of the secret can therefore be done

without the knowledge of cryptography and cryptographic computations. Many researchers have proposed different constructions of such scheme, see for instance [25, 19, 12]. later, the conception of conventional visual secret sharing schemes was enhanced by the development of *multi-secret sharing schemes* bases on visual cryptography, for more details see for instance [33, 27, 10, 31, 11]. Recently, Mishra and Gupta propose in [2] an (2, n, m) multi secret sharing scheme in which exactly n shares are generated to encode m secret images and secret images are revealed if shares are taken from qualified set. Unlike previous schemes, the technique proposed by Mishra and Gupta don't require perfect alignment of shares between shares to obtain secret image.

1.4 *Schemes based on graph access structure*

There are also other approaches based on *graph access structure* that have received a considerable attention. In the most of these approaches, many researchers have proposed different constructions of a perfect secret sharing scheme based on uniform access structures, which contain qualified subsets all of the same cardinality. In these constructions, participants are represented by the vertices of a graph G, the uniform access structure Γ is based on the concept of adjacent vertices and represented by the edges, for more details see for instance [5, 30, 4, 21, 7, 20]. In [3], a novel approach to design a graph access structure, which is based on the concept of non-adjacent vertices, was proposed. In this approach, an independent dominating set of vertices in a graph G was introduced and applied as a novel idea to construct a perfect secret sharing scheme such that the vertices of the graph represent the participants and the dominating set of vertices in G represents the minimal qualified set.

Related to that previous works on the hierarchical secret sharing, we propose in this article a simpler construction of a secret sharing scheme base on a tree.

2. The proposed construction algorithm

The propose scheme is based on the hierarchical concept of companies illustrated through its organization chart.The proposed construction algorithm include two phases, which are achieved by the dealer who can, for instance, be represented by the board of directors at a company.

2.1 *The initialization phase*

The hierarchical concept of any company is illustrated through its organization chart, which is represented by a tree $T = (V, E)$ such that:

- The height of T, correspond to the number of hierarchical levels at the company, denoted h, and each hierarchical level is denoted N_j, for $j = 1, \ldots, h$.
- The set of vertices V corresponding to the company's employees represents the set of participants P. As each participant i belong to a specified level j, we denote by P_{ij} such participant.
- The set of edges E corresponds to the hierarchical relations between participants (employees).

Figure 1 given below, illustrate an organization chart of a company with 9 employees and 3 hierarchical levels.

In the initialization phase, the dealer proceeds to the construction of the access structure Γ containing all the qualified subsets. A subset X of P is considered as qualified if and only if:

(i) No participant will have the veto right for reconstructing the secret alone, especially the first manager. This condition is formulated by:

$$\sum_{P_{ij} \in X} j \geq h + 1.$$

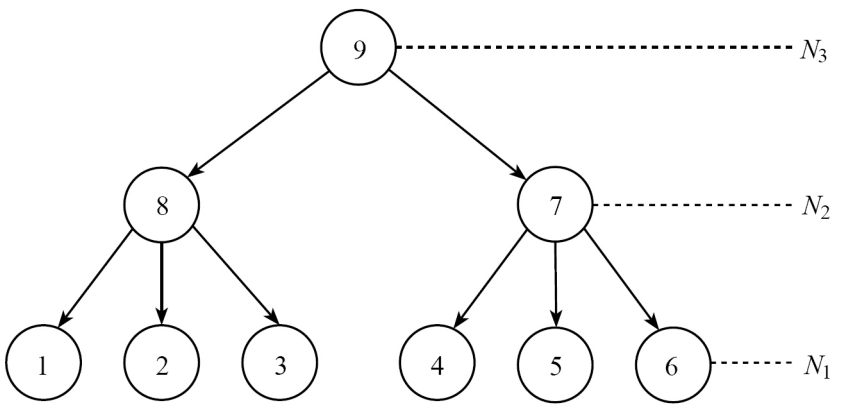

Figure 1

Company organization chart *T* with 9 employees.

(ii) The elements of X cannot all be at the same hierarchical level, in order to reduce the risk of corruption. This condition is expressed by:

$$\left|X \cap N_j\right| \leq \left\lceil \frac{h+1}{j} \right\rceil - 1, \text{ for } j = 1, \ldots, h.$$

The access structure Γ is then:

$$\Gamma = \left\{ X \subset P : \sum_{P_{ij} \in X} j \geq h+1, \text{ and } \left|X \cap N_j\right| \leq \left\lceil \frac{h+1}{j} \right\rceil - 1, \text{ for } j = 1, \ldots, h \right\}.$$

Finally, the minimum access structure Γ_0 is then:

$$\Gamma_0 = \left\{ X \in \Gamma : \forall X' \, (X' \subsetneq X \Rightarrow X' \notin \Gamma) \right\}.$$

2.2 *The decomposition phase*

In this phase, the dealer:

- choose a prime power number q;
- select the secret to share $K = (k_1, \ldots, k_h)$ that he encodes in the finite field $GF(q)$;
- generate randomly one value a_0 in $GF(q)$;
- construct the polynomial $f(x)$ of degree h:

$$f(x) = a_0 + k_1 x + \cdots + k_h x^h;$$

- Calculate and distribute the shares to all participants. The share given to each participant P_{ij}, denoted S_{ij}, consists on two parts. The first one is publicly revealed and correspond to there login i and hierarchical level j. The second part is sent in private and consists on j values of ordered pairs:

$$\left(x_{i1}, f(x_{i1})\right), \ldots, \left(x_{ij}, f(x_{ij})\right),$$

so that the number of participants who can pool their shares to reconstruct the secret depends on their importance.

The following algorithm resumes the proposed construction of secret sharing scheme.

Algorithm 1 Construction of secret sharing scheme

Require :

1. The set of company's participants $P = \{P_{ij}, i = 1,\dots,n;\ j = 1,\dots,h\}$;
2. A prime power q;
3. The polynomial $f(x) = a_0 + k_1 x + \cdots + k_h x^h$.

Ensure : The set of shares assigned to participants

$$S = \{S_{ij}, i = 1,\dots,n;\ j = 1,\dots,h\}.$$

1. For each participant P_{ij}, calculate $x_{im} = 1 + mih,\ m = 1,\dots,j$;
2. Calculate $S_{ij} = (i, j, (x_{i1}, f(x_{i1})), (x_{i2}, f(x_{i2})),\dots,(x_{ij}, f(x_{ij}))),\ i = 1,\dots,n;$ $j = 1,\dots,h$;
3. **Return:** $S = \{S_{ij}, i = 1,\dots,n;\ j = 1,\dots,h\}$.

According to Horner's method, Algorithm 1 can be achieved, in the worst case, in $O(n)$ time complexity.

3. The proposed reconstruction algorithm

Let $K = (k_1,\dots,k_h)$ be the secret shared over the finite set of participants P by application of Algorithm 1. According to the polynomial chosen by the dealer for calculating and distributing the shares, a group of participants X who want to collaborate with their shares in order to recover the secret K, should in first reconstruct the polynomial f, which can be done by interpolation, for that X should own at least $h+1$ values of ordered pairs, $(x_1, f(x_1)),\dots,(x_{h+1}, f(x_{h+1}))$. The proposed reconstruction is summarized in Algorithm 2.

Theorem 3.1 : *The constructed secret sharing scheme is perfect.*

Proof : Let X be a qualified subset of participants, then the conditions (*i*) and (*ii*), in the initialization phase 2.1 above, are satisfied. According to the decomposition phase 2.2, each P_{ij} belonging to X owns as much values of $(x, f(x))$ as his level j, $(x_{i1}, f(x_{i1})),\dots,(x_{ij}, f(x_{ij}))$. Thus, X owns at least $h+1$ values of $(x, f(x))$ and can recover $f(x)$, by using interpolation, and then the secret K. Therefore, any qualified subset can reconstruct the secret.

Now, let X be an unqualified subset of participants, then one of the conditions (*i*) and (*ii*), in the initialization phase 2.1, is not satisfied. If the condition (*i*) is not, X owns less than $h+1$ values of $(x, f(x))$, which don't allow the reconstruction of $f(x)$. In the other hand, as the elements of X cannot all be at the same hierarchical level, if the condition (ii) is not satisfied, the system denies access. Therefore, any unqualified subset has no information about the secret. □

Algorithm 2 Reconstruction of a secret K

Require:

1. A subset of participants $X \subset P$;
2. The set of hierarchical levels, $N_1, \ldots, N_h$;
3. The shares of participants belonging to X.

Ensure:

1. The secret $K = (k_1, \ldots, k_h)$ or
2. The system denies access.

If $X \in \Gamma$ **Then** Apply interpolation to reconstruct $f(x)$ and then the secret K

Else The system denies access and displays "The subset is not qualified".

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4. The efficiency of the proposed secret sharing scheme

To measure the efficiency of the proposed secret sharing scheme, we consider the information rate $\rho = \frac{\log_2(|K|)}{\log_2(|S|)}$, where S is the maximum share.

Theorem 4.1 : *The constructed secret sharing scheme is ideal.*

Proof : The secret $K = (k_1, \ldots, k_h)$ is an h-dimensional vector such that each k_i, $i = 1 \ldots, h$, is in $GF(q)$. The k_i's length is then equal to $\log_2(q)$. According to the decomposition phase 2.2, each share S_{ij} is represented by a vector of $j+2$ components, in which j components are private. The maximum share S is the one corresponding to the first manager of the company which is at the high level h, its length is then equal to $h\log_2(q)$. Hence, $\rho = 1$. □

5. Security analysis

The two main security requirements in a secret sharing scheme are confidentiality and authentication. Confidentiality is about ensuring that the information is only available to the qualified subsets, while the authentication is intended to ensure that each participant trying to collaborate in order to reconstruct the secret, is the one he claims to be.

In this paper, confidentiality has been demonstrated in Theorem 3.1 by proving that the proposed secret sharing scheme is perfect, while authentication is ensured by denying the access of all types of attacks. In fact, in such protocols, two types of attacks can arise: the insider and outsider attacks.

For the outsider attacks, where the attackers are not belonging to the system, the attacker aims to recover the secret by trying all possible combinations. As the secret K is an h-dimensional vector in which each component is in $GF(q)$, the number of possible combinations increases according to the number of hierarchical levels h. Thus, the brute force attack becomes a combinatorial explosion.

For the insider attacks, where the attackers are belonging to the system but consist on an unqualified subset of participants, as all parameters are public in the proposed scheme except the secret K, three types of insider attacks can arise:

- The first case consists on participant in level N_i who may pretend to be a participant of another lower level N_j, $j < i$, and use only a part of his share, in order to escape the condition (ii) described in the initialization phase 2.1. The following conditions (iii) and (iv) are then included in the proposed scheme and checked before proceeding to the reconstruction algorithm 2. In the case where these conditions are not satisfied, the system generates an authentication error and display an attack attempt message without executing the reconstruction algorithm 2.

  For each given share $S_{ij} = (i, j, (x_{i1}, f(x_{i1})), (x_{i2}, f(x_{i2})), \ldots, (x_{ij}, f(x_{ij}))),$

$$i = 1, \ldots, n;\ j = 1, \ldots, h:$$

(iii) The login i corresponds to a participant of the level j. This condition is formulated by:

$$\forall S_{ij},\ i = 1, \ldots, n \text{ and } j = 1, \ldots, h;\ P_{ij} \in N_j.$$

(iv) Each ordered pair $(x_{im}, f(x_{im}))$, $m = 1,\ldots, j,$ corresponds to the one sent by the dealer to the participant i belonging to the level j. This condition is expressed by:

$\forall S_{ij}, \forall x_{im}, x_{im} = 1 \pmod{ih}$ and $\left\lfloor \frac{x_{im}}{ih} \right\rfloor \leq j,$ for $i = 1,\ldots, n,\ j = 1,\ldots, h$ and $m = 1,\ldots, j;$ where $\lfloor . \rfloor$ denotes the floor function.

- The second case of insider attacks consists on participants in the same level $N_i,$ who are not allow to collaborate with their shares, according to condition (ii), in Section 2.1, trying to merge their shares to have only one and pretend to be a participant of another higher level $N_j,\ j > i.$ This case is treated as the first case described above.
- The last case of insider attacks consists on participant in level $N_i,$ who may pretend to be a participant of another higher level $N_j,\ j > i,$ and try to calculate another value of $f(x).$ This case is similar to the outsider attacks described above.

6. Didactic example

Let consider the case of a company whose organization chart is represented by the tree T given in Figure 1 above. According to the initialization phase 2.1:

- The number of hierarchical levels $h = 3.$
- The set of participants $P = \{P_{11}, P_{21}, P_{31}, P_{41}, P_{51}, P_{61}, P_{72}, P_{82}, P_{93}\}.$
- According to their hierarchical levels, participants are assigned as follow:

$$N_1 = \{P_{11}, P_{21}, P_{31}, P_{41}, P_{51}, P_{61}\},\ N_2 = \{P_{72}, P_{82}\} \text{ and } N_3 = \{P_{93}\}.$$

- The access structure Γ_0 containing all the minimal qualified subsets is given as follow:

$$\Gamma_0 = \{\{P_{93}, P_{11}\},\ \{P_{93}, P_{21}\}, \{P_{93}, P_{31}\}, \{P_{93}, P_{41}\}, \{P_{93}, P_{51}\}, \{P_{93}, P_{61}\}, \{P_{93}, P_{72}\},$$
$$\{P_{93}, P_{82}\}, \{P_{82}, P_{11}, P_{21}\}, \{P_{82}, P_{11}, P_{31}\}, \{P_{82}, P_{11}, P_{41}\}, \{P_{82}, P_{11}, P_{51}\}, \{P_{82}, P_{11}, P_{61}\},$$
$$\{P_{82}, P_{21}, P_{31}\}, \{P_{82}, P_{21}, P_{41}\}, \{P_{82}, P_{21}, P_{51}\}, \{P_{82}, P_{21}, P_{61}\}, \{P_{82}, P_{31}, P_{41}\},$$
$$\{P_{82}, P_{31}, P_{51}\}, \{P_{82}, P_{31}, P_{61}\}, \{P_{82}, P_{41}, P_{51}\}, \{P_{82}, P_{41}, P_{61}\}, \{P_{82}, P_{51}, P_{61}\},$$
$$\{P_{72}, P_{11}, P_{21}\}, \{P_{72}, P_{11}, P_{31}\}, \{P_{72}, P_{11}, P_{41}\}, \{P_{72}, P_{11}, P_{51}\}, \{P_{72}, P_{11}, P_{61}\},$$
$$\{P_{72}, P_{21}, P_{31}\}, \{P_{72}, P_{21}, P_{41}\}, \{P_{72}, P_{21}, P_{51}\}, \{P_{72}, P_{21}, P_{61}\}, \{P_{72}, P_{31}, P_{41}\},$$
$$\{P_{72}, P_{31}, P_{51}\}, \{P_{72}, P_{31}, P_{61}\}, \{P_{72}, P_{41}, P_{51}\}, \{P_{72}, P_{41}, P_{61}\}, \{P_{72}, P_{51}, P_{61}\}\}.$$

Suppose for instance that the key K is a 3-tuple of 32-bit integers and $q = 4294967311$ a prime number greater than $2^{32} - 1$. Based on the decomposition phase 2.2, let consider $k_1 = 4967295$, $k_2 = 94967$, $k_3 = 9496729$ and $a_0 = 429496$. The polynomial chosen by the dealer is then

$$f(x) = 429496 + 4967295x + 94967x^2 + 9496729x^3,$$

and the shares given to participants are:

$$S_{93} = (9, 3, (x_{91}, f(x_{91})), (x_{92}, f(x_{92})), (x_{93}, f(x_{93})))$$
$$= (9, 3, (28, 2527731964), (55, 31222823), (82, 1673628957));$$

$$S_{72} = (7, 2, (x_{71}, f(x_{71})), (x_{72}, f(x_{72})))$$
$$= (7, 2, (22, 2492596253), (43, 3826770342));$$

$$S_{82} = (8, 2, (x_{81}, f(x_{81})), (x_{82}, f(x_{82}))$$
$$= (8, 2, (25, 2541468297), (49, 1061011979));$$

$$S_{11} = (1, 1, (x_{11}, f(x_{11})))$$
$$= (1, 1, (4, 629608804));$$

$$S_{21} = (2, 1, (x_{21}, f(x_{21})))$$
$$= (2, 1, (7, 3297231991));$$

$$S_{31} = (3, 1, (x_{31}, f(x_{31})))$$
$$= (3, 1, (10, 966393524));$$

$$S_{41} = (4, 1, (x_{41}, f(x_{41})))$$
$$= (4, 1, (13, 3765498123));$$

$$S_{41} = (4,1,(x_{41}, f(x_{41}))) = (4,1,(13,3765498123));$$

$$S_{41} = (4,1,(x_{41}, f(x_{41}))) = (4,1,(13,3765498123));$$

$$S_{51} = (5,1,(x_{51}, f(x_{51}))) = (5,1,(16,348113953));$$

$$S_{61} = (6,1,(x_{61}, f(x_{61}))) = (6,1,(19,842645734)).$$

It's clear that each qualified subset belonging to Γ_0 can recover the secret K.

Let's take for instance the qualified subset $X = \{P_{82}, P_{11}, P_{21}\}$. According to the reconstruction Algorithm 3, the polynomial f can be reconstruct by applying interpolation.

The polynomial L defined bellow is the unique polynomial of degree at most h satisfying $L(x_i) = y_i = f(x_i)$:

$$L(x) = \sum_{j=0}^{h} f(x_j) l_j(x), \text{ where } l_j(x) = \prod_{\substack{i=0 \\ i \neq j}}^{h} \left(\frac{x - x_i}{x_j - x_i} \right).$$

For the considered qualified subset X, the h known values of $(x, f(x))$ are:

Table 1

$(x, f(x))$ values of qualified subset $X = \{P_{82}, P_{11}, P_{21}\}$.

$x_0 = x_{81} = 25$	$f(x_0) = 2541468297$
$x_1 = x_{82} = 49$	$f(x_1) = 1061011979$
$x_2 = x_{11} = 4$	$f(x_2) = 629608804$
$x_3 = x_{21} = 7$	$f(x_3) = 3297231991$

Lagrange polynomials are calculated as follow:

$$l_0(x) = \frac{(x-49)(x-4)(x-7)}{(25-49)(25-4)(25-7)} = \frac{1}{9072}(-x^3 + 60x^2 - 567x + 1372),$$

$$l_1(x) = \frac{(x-25)(x-4)(x-7)}{(49-25)(49-4)(49-7)} = \frac{1}{45360}(x^3 - 36x^2 + 303x - 700),$$

$$l_2(x) = \frac{(x-25)(x-49)(x-7)}{(4-25)(4-49)(4-7)} = \frac{1}{2835}(-x^3 + 81x^2 - 1743x + 8575),$$

$$l_3(x) = \frac{(x-25)(x-49)(x-4)}{(7-25)(7-49)(7-4)} = \frac{1}{2268}(x^3 - 78x^2 + 1521x - 4900).$$

Hence

$$\begin{aligned} L(x) &= 2541468297\, l_0(x) + 1061011979\, l_1(x) + 629608804\, l_2(x) \\ &\quad + 3297231991\, l_3(x) \pmod{q} \\ &= f(x). \end{aligned}$$

Therefore

Table 2

Reconstruction of the secret *K*.

	k_i**'s value**
k_1	4967295
k_2	94967
k_3	9496729

In case of insider attacks: as a first case of an insider attack, let's take the case in which the subset $\{P_{82}, P_{72}\}$, who is not qualified, try to reconstruct the secret by using the P_{82} share's as if it concerned those corresponding to participants P_{11} and P_{21}. For instance, instead of introducing the share S_{82} given above, P_{82} will introduce the following vectors S'_{11} and S'_{21} as shares of P_{11} and P_{21}, respectively:

$$S'_{11} = (1, 1, (x_{81}, f(x_{81}))) = (1, 1, (25, 2541468297)),$$

$$S'_{21} = (1,1,(x_{82}, f(x_{82}))) = (2,1,(49,1061011979)).$$

The condition (iv), in Section 5, is not satisfied in this case, since:

$$x_{81} = 1 \,(\text{mod } 3), \text{ but } \left\lfloor \frac{x_{81}}{3} \right\rfloor > 1,$$

$$x_{82} = 1 \,(\text{mod } 6), \text{ but } \left\lfloor \frac{x_{82}}{6} \right\rfloor > 1.$$

The system generates then an authentication error and display an attack attempt message.

As a second case of an insider attack, let's take the case in which the subset $\{P_{11}, P_{21}, P_{31}, P_{41}\}$, who is not qualified, according to condition (ii), in Section 2.1, try to reconstruct the secret by merging the shares of P_{31} and P_{41} and pretending to be the subset $\{P_{11}, P_{21}, P_{72}\}$ for instance.

In this case, instead of introducing the shares S_{31} and S_{41} given above, a merged share S'_{72} is introduced as if it was the one corresponding to the participant P_{72} : :

$$S'_{72} = (7,2,(x_{31}, f(x_{31})),(x_{41}, f(x_{41}))) = (7,2,(10,966393524),(13,3765498123)).$$

The condition (iv), in Section 5, is not satisfied in this case, since

$$\left\lfloor \frac{x_{31}}{21} \right\rfloor < 2, \text{ but } x_{31} = 10 \,(\text{mod } 21),$$

$$\left\lfloor \frac{x_{41}}{21} \right\rfloor < 2, \text{ but } x_{41} = 13 \,(\text{mod } 21).$$

The system generates then an authentication error and display an attack attempt message.

In case of outsider attacks: as all coefficients of f are taken in $GF(q)$, the attackers should try q^{h+1} possible combinations to reconstruct f. In our example, this requires 4294967311^4 possibilities, that exceeds 2^{128}.

7. Conclusion

In this paper, we were interested on the hierarchical concept of companies illustrated through its organization char. We propose a novel simple construction algorithm of a secret sharing scheme, where the access

structure is a tree and not uniform since the number of parts needed to reconstruct the secret depends on the importance of the participants within the company. For the reconstruction algorithm, the interpolation is used to reconstruct the polynomial and recover the secret K, as used in some previous works. We show that the proposed scheme is perfect and ideal. Furthermore, the security of the proposed scheme is analysed by discussing all possible kinds of attacks (insider and outsider) and proofing that confidentiality and authentication are ensured. Finally, we conclude by a detailed didactic example.

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