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Types of Zagreb indices for some operations on the regular graph

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Abstract

Topological indices are fundamental and important concepts in graph theory. They are numerical indicators used to describe the structural properties of graphs in a quantitative method. These indices are based on the topological structure of the graph, such as degrees of

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vertices and length of paths between vertices. In this article, our interest was in topological indices that depend on degrees between distinct two vertices (for adjacent and non-adjacent vertices) which is called the types of Zagreb indices. We found the types of Zagreb indices for both a regular graph and a regular graph after adding new vertices equal to the number of edges of the regular graph, such that each new vertex is adjacent to two adjacent vertices in the regular graph, this is called the cog-graph of the regular graph. We also found Zagreb topological indices for a regular graph after adding r end vertices for each vertex of the vertices of the regular graph, and this is what is called the $r\angle$thorn graph of the regular graph and a $r\angle$thorn graph for a cog-regular graph. In addition, some illustrative examples were taken that are consistent with the results we obtained, and finally the relationships between these types of evidence were discussed.

Subject Classification: *05C09, 05C38, 05C92.*

Keywords: *Zagreb indices, Zagreb coindices, Rregular graph, Cog-graph, r–thorn graph.*

1. Introduction

The mathematical structure of a simple graph $\Phi = (V, E)$ "consists of a finite, non-empty set of points V (the vertex set) and a finite set of unordered pairs of distinct elements of these points, which represent the edges". The order and the size of a graph Φ are the cardinalities of the set of vertices and the set of edges, respectively, represented by $p = p(\Phi)$ and $q = q(\Phi)$ respectively. In a graph Φ, the degree of a vertex v is the number of edges that are incident with it, and denoted its $deg_{\Phi}(v)$ or $degv$. If every vertex of Φ has the same degree with value n, then Φ is called $n-$ regular graph, [1,2]. See Figure 1 (a). Among the modern and effective concepts in the analysis of graph structures, topological indices have emerged as valuable numerical measures derived from the internal structure of a graph. The importance of topological indices lies in the study of the sum of energy scores, which is a spectral quantity extracted from the sum of scores matrix of the graph, [21] and it is also used to analyze many types of regular silica networks, relying on the iterative structure of these networks to derive accurate mathematical formulas for topological indices, [22]. Also, these indices are employed to describe and distinguish graphs mathematically. Topological indices vary depending on the parameters they rely on, and the most notable categories include: the first is distance-based indices, which depend on the lengths of paths between pairs of vertices, [3, 4], the second is the degree-based indices, which rely on the number of edges incident to each vertex, [5, 6], as for the third, depends on both of them, [7, 8].

Among the topological indices, Zagreb indices hold a prominent place due to their widespread application in the analysis of chemical graphs, as they effectively reflect the level of branching or connectivity within a graph and Zagreb indices have the following formats:

Zagreb's first and second indices, [9] are defined by

$ZI_1(\Phi) = \sum_{uv\in E}(degu + degv)$ and $ZI_2(\Phi) = \sum_{uv\in E}(degu \times degv)$ respectively,

Zagreb's first and second coindices, [10] are defined by

$ZCI_1(\Phi) = \sum_{uv \notin E}(degu + degv)$ and $ZCI_2(\Phi) = \sum_{uv \notin E}(degu \times degv)$,

where $degu$ and $degv$ are the degrees of a graph Φ's nonadjacent vertices u and v.

The Schultz polynomial and its modified version have also gained considerable importance for their ability to encode topological relationships among vertices in the form of a polynomial function. This importance has become apparent in many recent studies that have found Schultz and modified Schultz polynomials for graphs that are in the form of a chain or ring sequence of special graphs resulting from vertex identification, [11]. Other researchers have also calculated these polynomials for special graphs in addition to some graphs operations. [12-14], where Schultz polynomial and modified Schultz polynomial are defined as follows:

$$Sc(\Phi; x) = \sum_{\{u,v\} \subseteq V(\Phi)}(degu + degv)x^{d(u,v)},$$

$$Sc^*(\Phi; x) = \sum_{\{u,v\} \subseteq V(\Phi)}(degu \times degv)x^{d(u,v)},$$

where $d(u, v)$ is a distance between distinct vertices u and v, (the distance is a shortest path which connected between two vertices u and v). Both the Schultz and modified Schultz polynomials exhibit notable algebraic and geometric properties, and they serve as a foundation for deriving further topological indices at $x = 1$. It is very easy to give the relations between the Schultz and modified Schultz polynomials when x equals one and the indices of Zagreb indices are the proposition following:

Proposition 1.1: Let Φ be a graph, then

1. $Sc(\Phi; 1) = ZI_1(\Phi) + ZCI_1(\Phi)$.

2. $Sc^*(\Phi; 1) = ZI_2(\Phi) + ZCI_2(\Phi)$.

In addition, we can obtain formulas for each of $Sc(\Phi; x)$ and $Sc^*(\Phi; x)$ at $x = 1$ as the following proposition:

Proposition 1.2: Let Φ be a graph, then

1. $Sc(\Phi; 1) = 2|E(\Phi)|(|V(\Phi)| - 1)$.

2. $Sc^*(\Phi; 1) = 2|E(\Phi)|^2 - \frac{1}{2}ZI_1(\Phi)$.

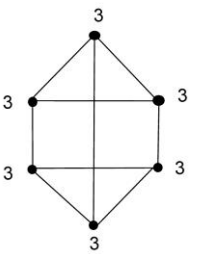

Figure 1 (a)
3- regular graph Φ

There are a number of researchers who have defined the cog for some special graphs on the edges that are on the outer face [15], but in this paper, we have given a more comprehensive definition of the cog for all edges in the graph whether they are located on the outer face or on the inner faces.

Definition 1.3: Let Φ be a connected simple graph with vertex set $\{v_1, v_2, \dots, v_p\}$ and edge set $\{e_1, e_2, \dots, e_q\}$. Its cog-graph, denoted Φ^c, is the simple connected graph obtained by introducing q new vertices $\{u_1, u_2, \dots, u_q\}$ each of vertex is degree 2. For each $i = \{1, 2, \dots, q\}$, the vertex u_i is adjacent precisely to the two original vertices that are end points of the edge e_i. Consequently, has $|V(\Phi^c)| = p + q$ and $|E(\Phi^c)| = 3q$. See Figure 1(b).

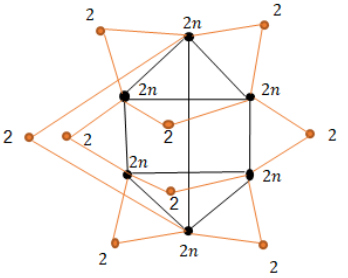

Figure 1(b)
The cog of 3- regular graph Φ^c

Definition 1.4: [16] Let Φ be a connected simple graph with p vertices and q edges. The $r-$ thorn graph of Φ, denoted Φ^t, is obtained by attaching r new pendant vertices to each original vertex of Φ, so that, each original vertex becomes adjacent to exactly r of these newly added vertices. Consequently, the overall number of vertices in Φ^t is $p(r+1)$ and the overall number of edges is $q + rp$, respectively. See Figure 1(c).

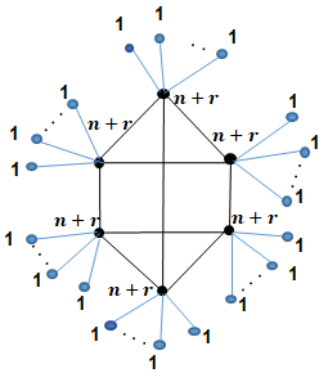

Figure 1 (c)
The r −thorn of 3-regular graph Φ^t

Definition 1.5: [17] The $r-$ thorn cog-graph Φ^{c^t} is a connected simple graph obtained by attaching r new pendant vertices to each cog-vertices of Φ^c, so that, each cog-vertex becomes adjacent to exactly r of these newly added vertices. Consequently, the overall number of vertices in Φ^{c^t} is $q(r+1)+p$ and the overall number of edges is $q(r+3)$, respectively. See Figure 1 (d).

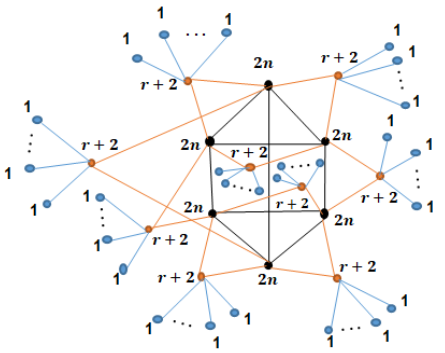

Figure 1(d)
The $r-$thorn of cog 3-regular graph Φ^{c^t}

This article aims to derive accurate analytical formulas for calculating these indices, then compare the resulting values and analyze the influence of regular geometric structure and score distribution on the indices. The results are expected to open new avenues for developing accurate quantitative models of molecular properties

Finally, we must mention that these topological indices and other from indices are used in many applications, some of which the researcher or reader can see in the references. [18-20]

2. Main results

This section presents several types of Zagreb indices associated with certain operations on regular graphs of degree n. This section is divided in to three subsections: the cog of $n-$ regular graph, the $r-$thorn of $n-$ regular graph, and the $r-$thorn of cog of $n-$regular graph.

2.1 *The Cog of $n-$ Regular Graph:*

Let $\aleph$ be $n-regular$ graph of order $\beth$, $\beth \in \mathbb{N}-\{1,2\}$, the cog of $n-regular$ graph $\aleph^c$ has order $\frac{1}{2}\beth(n+2)$ and size $\frac{3}{2}\beth n$. In the following theorems some topological indices will be obtained.

Theorem 2.1.1: *Let $\aleph$ be $n-regular$ graph of order $\beth$, $\beth \in \mathbb{N}-\{1,2\}$, the cog of $\aleph$ has:*

1. $Sc(\aleph^c;1) = \frac{3}{2}n\text{þ}(2\text{þ} + n\text{þ} - 2)$.
2. $Sc^*(\aleph^c;1) = \frac{1}{2}n\text{þ}(9n\text{þ} - 4n - 2)$.

Proof:

1. $$\begin{aligned} Sc(\aleph^c;1) &= \sum_{\{u,v\}\subseteq V(\aleph^c)}(degu + degv) \\ &= \sum_{\{u,v\}\subseteq V(\aleph)}(degu + degv) + \sum_{\substack{\{u\}\subseteq V(\aleph) \\ \{v\}\subseteq (V(\aleph^c)-V(\aleph))}}(degu + degv) \\ &+ \sum_{\{u,v\}\subseteq (V(\aleph^c)-V(\aleph))}(degu + degv) \\ &= \sum_{\{u,v\}\subseteq V(\aleph)}(2n + 2n) + \sum_{\substack{\{u\}\subseteq V(\aleph) \\ \{v\}\subseteq (V(\aleph^c)-V(\aleph))}}(2n + 2) \\ &+ \sum_{\{u,v\}\subseteq (V(\aleph^c)-V(\aleph))}(2 + 2) \\ &= 4n\binom{\text{þ}}{2} + 2(n+1)(\tfrac{1}{2}n\text{þ}^2) + 4\binom{\frac{1}{2}n\text{þ}}{2} \\ &= \tfrac{3}{2}n\text{þ}(2\text{þ} + n\text{þ} - 2). \end{aligned}$$

2. $$\begin{aligned} Sc^*(\aleph^c;1) &= \sum_{\{u,v\}\subseteq V(\aleph^c)}(degu \times degv) \\ &= \sum_{\{u,v\}\subseteq V(\aleph)}(degu \times degv) + \sum_{\substack{\{u\}\subseteq V(\aleph) \\ \{v\}\subseteq (V(\aleph^c)-V(\aleph))}}(degu \times degv) \\ &+ \sum_{\{u,v\}\subseteq (V(\aleph^c)-V(\aleph))}(degu \times degv) \\ &= \sum_{\{u,v\}\subseteq V(\aleph)} 4n^2 + \sum_{\substack{\{u\}\subseteq V(\aleph) \\ \{v\}\subseteq (V(\aleph^c)-V(\aleph))}} 4n + \sum_{\{u,v\}\subseteq (V(\aleph^c)-V(\aleph))}(4) \\ &= 4n^2\binom{\text{þ}}{2} + 4n(\tfrac{1}{2}n\text{þ}^2) + 4\binom{\frac{1}{2}n\text{þ}}{2} \\ &= \tfrac{1}{2}n\text{þ}(9n\text{þ} - 4n - 2). \end{aligned}$$

Theorem 2.1.2: *Let* $\aleph$ *be* $n-regular$ *graph of order* þ, $\text{þ} \in \mathbb{N} - \{1,2\}$, *the cog of* $\aleph$ *has:*

1. $ZI_1(\aleph^c) = 2n\text{þ}(2n+1)$.
2. $ZI_2(\aleph^c) = 2n^2\text{þ}(n+2)$.

Proof:

1. $$\begin{aligned} ZI_1(\aleph^c) &= \sum_{uv\in E(\aleph^c)}(degu + degv) \\ &= \sum_{uv\in E(\aleph)}(degu + degv) + \sum_{uv\in (E(\aleph^c)-E(\aleph))}(degu + degv) \\ &= \sum_{uv\in E(\aleph)}(2n + 2n) + \sum_{uv\in (E(\aleph^c)-E(\aleph))}(2n + 2) \\ &= 4n\left(\tfrac{1}{2}n\text{þ}\right) + 2(n+1)(n\text{þ}) \\ &= 2n\text{þ}(2n+1). \end{aligned}$$

2. $$\begin{aligned} ZI_2(\aleph^c) &= \sum_{uv\in E(\aleph^c)}(degu \times degv) \\ &= \sum_{uv\in E(\aleph)}(degu \times degv) + \sum_{uv\in (E(\aleph^c)-E(\aleph))}(degu \times degv) \\ &= \sum_{uv\in E(\aleph)} 4n^2 + \sum_{uv\in (E(\aleph^c)-E(\aleph))} 4n \end{aligned}$$

$$= 4n^2\left(\frac{1}{2}n\text{þ}\right) + 4n(n\text{þ})$$
$$= 2n^2\text{þ}(n+2).$$

From Proposition (1.1) and Theorems (2.1.1) and (2.1.2), we have:

Corollary 2.1.3: *Let* ℵ *be* $n-regular$ *graph of order* þ, $\text{þ} \in \mathbb{N} - \{1,2\}$, *the cog of* ℵ *has:*

1. $ZCI_1(\aleph^c) = \frac{1}{2}n\text{þ}(3n\text{þ} + 6\text{þ} - 8n - 10)$.

2. $ZCI_2(\aleph^c) = \frac{1}{2}n\text{þ}(9n\text{þ} - 4n^2 - 12n - 2)$.

Example 2.1.4: The complete graph $K_\text{þ}$ is $(\text{þ}-1)$ −regular graph, then

1. $Sc\left(K_\text{þ}^c; 1\right) = \frac{3}{2}\text{þ}(\text{þ}-1)^2(\text{þ}+2)$.

2. $Sc^*\left(K_\text{þ}^c; 1\right) = \frac{1}{2}\text{þ}(\text{þ}-1)(9\text{þ}^2 - 13\text{þ} + 2)$.

3. $ZI_1\left(K_\text{þ}^c\right) = 2\text{þ}(\text{þ}-1)(2\text{þ}-1)$.

4. $ZI_2\left(K_\text{þ}^c\right) = 2\text{þ}(\text{þ}-1)^2(\text{þ}+1)$.

5. $ZCI_1\left(K_\text{þ}^c\right) = \frac{1}{2}\text{þ}(\text{þ}-1)(3\text{þ}+1)(\text{þ}-2)$.

6. $ZCI_2\left(K_\text{þ}^c\right) = \frac{1}{2}\text{þ}(\text{þ}-1)(5\text{þ}-3)(\text{þ}-2)$.

2.2 *The* r *−Thorn of* n *− Regular Graph:*

Let ℵ be $n-$ regular graph of order þ, $\text{þ} \in \mathbb{N} - \{1\}$, the $r-$thorn of $n-regular$ graph $\aleph^t$ has order $\text{þ}(r+1)$ and size $\frac{1}{2}\text{þ}(n+2r)$. In the following theorems some topological indices will be obtained.

Theorem 2.2.1: *Let* ℵ *be* $n-regular$ *graph of order* þ, $\text{þ} \in \mathbb{N} - \{1,2\}$, *the* r *−thorn of* ℵ *has:*

1. $Sc(\aleph^t; 1) = \text{þ}(n+2r)(r\text{þ} + \text{þ} - 1)$
2. $Sc^*(\aleph^t; 1) = \frac{1}{2}\text{þ}(n^2\text{þ} - n^2 - 4n\text{þ}r - 2nr + 4\text{þ}r^2 - r^2 - r)$.

Proof:

1. $Sc(\aleph^t; 1) = \sum_{\{u,v\}\subseteq V(\aleph^t)}(degu + degv)$
$= \sum_{\{u,v\}\subseteq V(\aleph)}(degu + degv) + \sum_{\substack{\{u\}\subseteq V(\aleph) \\ \{v\}\subseteq\left(V(\aleph^t)-V(\aleph)\right)}}(degu + degv)$
$+ \sum_{\{u,v\}\subseteq(V(\aleph^t)-V(\aleph))}(degu + degv)$.
$= \sum_{\{u,v\}\subseteq V(\aleph)}(n+r+n+r) + \sum_{\substack{\{u\}\subseteq V(\aleph) \\ \{v\}\subseteq\left(V(\aleph^t)-V(\aleph)\right)}}(n+r+1)$
$+ \sum_{\{u,v\}\subseteq(V(\aleph^t)-V(\aleph))}(1+1)$
$= 2(n+r)\binom{\text{þ}}{2} + (n+r+1)(r\text{þ}^2) + 2\binom{r\text{þ}}{2}$

$$= \text{þ}(n + 2r)(r\text{þ} + \text{þ} - 1)$$

2. $Sc^*(\aleph^t; 1) = \sum_{\{u,v\}\subseteq V(\aleph^t)}(degu \times degv)$
$= \sum_{\{u,v\}\subseteq V(\aleph)}(degu \times degv)|$
$+ \sum_{\{u\}\subseteq V(\aleph)\ \{v\}\subseteq\left(V(\aleph^t)-V(\aleph)\right)}(degu \times degv)$
$+ \sum_{\{u,v\}\subseteq (V(\aleph^t)-V(\aleph))}(degu \times degv)$
$= \sum_{\{u,v\}\subseteq V(\aleph)}(n + r)^2 + \sum_{\{u\}\subseteq V(\aleph)\ \{v\}\subseteq\left(V(\aleph^t)-V(\aleph)\right)}(n + r)$
$+ \sum_{\{u,v\}\subseteq (V(\aleph^t)-V(\aleph))}(1)$
$= (n + r)^2 \binom{\text{þ}}{2} + (n + r)(r\text{þ}^2) + \binom{r\text{þ}}{2}$
$= \frac{1}{2}\text{þ}(n^2\text{þ} - n^2 + 4n\text{þ}r - 2nr + 4\text{þ}r^2 - r^2 - r)$.

Theorem 2.2.2: *Let* $\aleph$ *be* $n -$ *regular graph of order* þ, $\text{þ} \in \mathbb{N} - \{1,2\}$, *then the* $r -$*thorn of* $\aleph$ *has:*

1. $ZI_1(\aleph^t) = n\text{þ}(n + 2r) + r\text{þ}(r + 1)$.
2. $ZI_2(\aleph^t) = \frac{1}{2}\text{þ}(n + r)(n^2 + nr + 2r)$.

Proof:

1. $ZI_1(\aleph^t) = \sum_{uv\in E(\aleph^t)}(degu + degv)$
$= \sum_{uv\in E(\aleph)}(degu + degv) + \sum_{uv\in (E(\aleph^t)-E(\aleph))}(degu + degv)$
$= \sum_{uv\in E(\aleph)}(n + r + n + r) + \sum_{uv\in (E(\aleph^t)-E(\aleph))}(n + r + 1)$
$= 2(n + r)(\frac{1}{2}n\text{þ}) + (n + r + 1)(r\text{þ})$
$= n\text{þ}(n + 2r) + r\text{þ}(r + 1)$.

2. $ZI_2(\aleph^t) = \sum_{uv\in E(\aleph^t)}(degu \times degv)$
$= \sum_{uv\in E(\aleph)}(degu \times degv) + \sum_{uv\in (E(\aleph^t)-E(\aleph))}(degu \times degv)$
$= \sum_{uv\in E(\aleph)}(n + r)^2 + \sum_{uv\in (E(\aleph^t)-E(\aleph))}(n + r)$
$= (n + r)^2(\frac{1}{2}n\text{þ}) + (n + r)(r\text{þ})$
$= \frac{1}{2}\text{þ}(n + r)(n^2 + nr + 2r)$.

From Proposition (1.1) and Theorems (2.2.1) and (2.2.2), we have:

Corollary 2.2.3: *Let* $\aleph$ *be* $n -$ *regular graph of order* þ, $\text{þ} \in \mathbb{N} - \{1,2\}$, *then the* $r -$*thorn of* $\aleph$ *has:*

1. $ZCI_1(\aleph^t) = \text{þ}(n + 2r)(\text{þ}r + \text{þ} - n - 1) - r\text{þ}(r + 1)$.
2. $ZCI_2(\aleph^t) = \frac{1}{2}\text{þ}\,(r(4\text{þ}r - 3r - 1) + nr(4\text{þ} - r - 4) + n^2(\text{þ} - 2r - 1) - n^3)$.

Example 2.2.4: *The complete bipartite graph* $K_{þ,þ}$ *is* $þ-$*regular graph, then from Theorems (2.2.1) and (2.2.2.) and from Corollary (2.2.3) we have:*

1. $Sc\left(K_{þ,þ}^{t};1\right)=\frac{1}{2}þ(þ+4r)(þ(r+1)-1).$
2. $Sc^{*}\left(K_{þ,þ}^{t};1\right)=\frac{1}{8}þ(þ^{3}+þ^{2}(8r-1)+4þr(4r-1)-4r(r+1)).$
3. $ZI_{1}\left(K_{þ,þ}^{t}\right)=\frac{1}{4}þ^{2}(þ+4r)+rþ(r+1).$
4. $ZI_{2}\left(K_{þ,þ}^{t}\right)=\frac{1}{16}þ(þ+2r)(þ^{2}+2rþ+8r).$
5. $ZCI_{1}\left(K_{þ,þ}^{t}\right)=\frac{1}{4}þ(þ^{2}(2r+1)+2þ(4r^{2}+2r-1)-4r(r+3)).$
6. $ZCI_{2}\left(K_{þ,þ}^{t}\right)=\frac{1}{16}\ p\ (p^{3}+2p^{2}(6\ r-1)+4pr(7r-4)-8r(3r+1))$

2.3 *The* $r-$*Thorn of Cog of* $n-$*Regular Graph:*

Let $ℵ$ be $n-regular$ graph of order þ, $þ\in\mathbb{N}-\{1,2\}$, then the $r-$thorn of cog of $n-regular$ graph $ℵ^{ct}$ has order $þ+\frac{1}{2}\ þn(r+1)$ and size $\frac{1}{2}nþ(r+3)$. In the following theorems some topological indices will be obtained.

Theorem 2.3.1: *Let* $ℵ$ *be* $n-regular$ *graph of order* þ, $þ\in\mathbb{N}-\{1,2\}$, *the* $r-$*thorn of cog of* $n-regular$ *graph* $ℵ$ *has:*

1. $Sc\left(ℵ^{ct};1\right)=\frac{1}{2}n\ þ\ (nþr^{2}+4nþr+3nþ+2þr+6þ-2r-6).$
2. $Sc^{*}\left(ℵ^{ct};1\right)=\frac{1}{4}nþ(2nþr^{2}+12nþr+18nþ-8n-r^{2}-5r-4)$.

Proof:

1. $$\begin{aligned}Sc\left(ℵ^{ct};1\right)&=\sum_{\{u,v\}\subseteq V\left(ℵ^{ct}\right)}(degu+degv)\\&=\sum_{\{u,v\}\subseteq V(ℵ)}(degu+degv)\\&+\sum_{\substack{\{u\}\subseteq V(ℵ)\\ \{v\}\subseteq\left(V(ℵ^{c})-V(ℵ)\right)}}(degu+degv)\\&+\sum_{\substack{\{u\}\subseteq V(ℵ)\\ \{v\}\subseteq\left(V\left(ℵ^{ct}\right)-V(ℵ^{c})\right)}}(degu+degv)\\&+\sum_{\{u,v\}\subseteq\left(V(ℵ^{c})-V(ℵ)\right)}(degu+degv)\\&+\sum_{\substack{\{u\}\subseteq\{V(ℵ^{c})-V(ℵ)\}\\ \{v\}\subseteq\left(V\left(ℵ^{ct}\right)-V(ℵ^{c})\right)}}(degu+degv)\\&+\sum_{\{u,v\}\subseteq\left(V\left(ℵ^{ct}\right)-V(ℵ^{c})\right)}(degu+degv).\\&=\sum_{\{u,v\}\subseteq V(ℵ)}(2n+2n)+\sum_{\substack{\{u\}\subseteq V(ℵ)\\ \{v\}\subseteq\left(V(ℵ^{c})-V(ℵ)\right)}}(2n+r+2)\\&+\sum_{\substack{\{u\}\subseteq V(ℵ)\\ \{v\}\subseteq\left(V\left(ℵ^{ct}\right)-V(ℵ^{c})\right)}}(2n+1)\\&+\sum_{\{u,v\}\subseteq\left(V(ℵ^{c})-V(ℵ)\right)}(r+2+r+2)\end{aligned}$$

$$+\sum_{\{u\}\subseteq\{V(\aleph^c)-V(\aleph)\}}\ \sum_{\{v\}\subseteq\left(V\left(\aleph^{ct}\right)-V(\aleph^c)\right)}(r+2+1)$$
$$+\sum_{\{u,v\}\subseteq\left(V\left(\aleph^{ct}\right)-V(\aleph^c)\right)}(1+1).$$
$$=4n\binom{þ}{2}+(2n+r+2)\left(\frac{1}{2}nþ^2\right)+(2n+1)\left(\frac{1}{2}nrþ^2\right)$$
$$+2(r+2)\binom{\frac{1}{2}nþ}{2}+(r+3)\left(\frac{1}{4}n^2þ^2r\right)+2\binom{\frac{1}{2}nrþ}{2}$$
$$=\frac{1}{2}n\,þ\,(nþr^2+4nþr+3nþ+2þr+6þ-2r-6)$$

2. $Sc^*\left(\aleph^{ct};1\right)=\sum_{\{u,v\}\subseteq V\left(\aleph^{ct}\right)}(degu\times degv)$
$$=\sum_{\{u,v\}\subseteq V(\aleph)}(degu\times degv)$$
$$+\sum_{\{u\}\subseteq V(\aleph)}\ \sum_{\{v\}\subseteq\left(V(\aleph^c)-V(\aleph)\right)}(degu\times degv)$$
$$+\sum_{\{u\}\subseteq V(\aleph)}\ \sum_{\{v\}\subseteq\left(V\left(\aleph^{ct}\right)-V(\aleph^c)\right)}(degu\times degv)$$
$$+\sum_{\{u,v\}\subseteq\left(V(\aleph^c)-V(\aleph)\right)}(degu\times degv)$$
$$+\sum_{\{u\}\subseteq\{V(\aleph^c)-V(\aleph)\}}\ \sum_{\{v\}\subseteq\left(V\left(\aleph^{ct}\right)-V(\aleph^c)\right)}(degu\times degv)$$
$$+\sum_{\{u,v\}\subseteq\left(V\left(\aleph^{ct}\right)-V(\aleph^c)\right)}(degu\times degv).$$
$$=\sum_{\{u,v\}\subseteq V(\aleph)}4n^2+\sum_{\{u\}\subseteq V(\aleph)}\ \sum_{\{v\}\subseteq\left(V(\aleph^c)-V(\aleph)\right)}2n(r+2)$$
$$+\sum_{\{u\}\subseteq V(\aleph)}\ \sum_{\{v\}\subseteq\left(V\left(\aleph^{ct}\right)-V(\aleph^c)\right)}2n+\sum_{\{u,v\}\subseteq\left(V(\aleph^c)-V(\aleph)\right)}(r+2)^2$$
$$+\sum_{\{u\}\subseteq\{V(\aleph^c)-V(\aleph)\}}\ \sum_{\{v\}\subseteq\left(V\left(\aleph^{ct}\right)-V(\aleph^c)\right)}(r+2)+\sum_{\{u,v\}\subseteq\left(V\left(\aleph^{ct}\right)-V(\aleph^c)\right)}(1)$$
$$=4n^2\binom{þ}{2}+2n(r+2)\left(\frac{1}{2}nþ^2\right)+2n\left(\frac{1}{2}nrþ^2\right)$$
$$+(r+2)^2\binom{\frac{1}{2}nþ}{2}+(r+2)\left(\frac{1}{4}n^2þ^2r\right)+\binom{\frac{1}{2}nrþ}{2}$$
$$=\frac{1}{4}nþ(2nþr^2+12nþr+18nþ-8n-r^2-5r-4)\,.$$

Theorem 2.3.2: *Let* $\aleph$ *be* $n-regular$ *graph of order* þ, $þ\in\mathbb{N}-\{1,2\}$, *the* r *−thorn of cog of* $n-regular$ *graph* $\aleph$ *has:*

1. $ZI_1\left(\aleph^{ct}\right)=\frac{1}{2}nþ(8n+r^2+5r+4)\,.$
2. $ZI_2\left(\aleph^{ct}\right)=\frac{1}{2}nþ(4n^2+4nr+8n+r^2+2r).$

Proof:

1. $$\begin{aligned} ZI_1\left(\aleph^{c^t}\right) &= \sum_{uv\in E\left(\aleph^{c^t}\right)}(degu + degv) \\ &= \sum_{uv\in E(\aleph)}(degu + degv) + \sum_{uv\in\left(E(\aleph^c)-E(\aleph)\right)}(degu + degv) \\ &+ \sum_{uv\in\left(E\left(\aleph^{c^t}\right)-E(\aleph^c)\right)}(degu + degv) \\ &= \sum_{uv\in E(\aleph)}(2n + 2n) + \sum_{uv\in(E(\aleph^c)-E(\aleph))}(2n + r + 2) \\ &+ \sum_{uv\in\left(E\left(\aleph^{c^t}\right)-E(\aleph^c)\right)}(r + 2 + 1) \\ &= 4n\left(\tfrac{1}{2}n\text{þ}\right) + (2n + r + 2)(n\text{þ}) + (r + 3)\left(\tfrac{1}{2}n\text{þ}r\right) \\ &= \tfrac{1}{2}n\text{þ}(8n + r^2 + 5r + 4)\,. \end{aligned}$$
2. $$\begin{aligned} ZI_2\left(\aleph^{c^t}\right) &= \sum_{uv\in E\left(\aleph^{c^t}\right)}(degu \times degv) \\ &= \sum_{uv\in E(\aleph)}(degu \times degv) + \sum_{uv\in\left(E(\aleph^c)-E(\aleph)\right)}(degu \times degv) \\ &+ \sum_{uv\in\left(E\left(\aleph^{c^t}\right)-E(\aleph^c)\right)}(degu \times degv) \\ &= \sum_{uv\in E(\aleph)} 4n^2 + \sum_{uv\in(E(\aleph^c)-E(\aleph))} 2n(r + 2) \\ &+ \sum_{uv\in\left(E\left(\aleph^{c^t}\right)-E(\aleph^c)\right)}(r + 2) \\ &= 2n^3\text{þ} + 2n^2\text{þ}(r + 2) + \tfrac{1}{2}n\text{þ}r(r + 2) \\ &= \tfrac{1}{2}n\text{þ}(4n^2 + 4nr + 8n + r^2 + 2r). \end{aligned}$$

From Proposition (1.1) and Theorems (2.3.1) and (2.3.2), we have:

Corollary 2.3.3: *Let* $\aleph$ *be* $n-$ *regular graph of order* þ, $\text{þ} \in \mathbb{N} - \{1,2\}$, *then the* r *−thorn of cog of* $n-$ *regular graph* $\aleph$ *has:*

1. $ZCI_1\left(\aleph^{c^t}\right) = \tfrac{1}{2}n\text{þ}\,(n\text{þ}r^2 - r^2 + 4n\text{þ}r + 3n\text{þ} + 2\text{þ}r + 6\text{þ} - 7r - 8n - 10)$.
2. $ZCI_2\left(\aleph^{c^t}\right) = \tfrac{1}{4}n\text{þ}(2n\text{þ}r^2 - 3r^2 - 8n^2 + 12n\text{þ}r + 18n\text{þ} - 8nr - 24n - 9r - 4)$.

Example 2.3.4: The cycle graph $C_\text{þ}$ is $2-$regular graph, then from Theorems (2.3.1) and (2.3.2.) and from Corollary (2.3.3) we have :

1. $Sc\left(C_\text{þ}^{c^t}; 1\right) = 2\text{þ}\,(\text{þ}r^2 + 5\text{þ}r + 6\text{þ} - r - 3)$.
2. $Sc^*\left(C_\text{þ}^{c^t}; 1\right) = \tfrac{1}{2}\text{þ}\,(4\text{þ}r^2 + 24\text{þ}r + 36\text{þ} - r^2 - 5r - 20)$.
3. $ZI_1\left(C_\text{þ}^{c^t}\right) = \text{þ}\,(r^2 + 5r + 20)$.
4. $ZI_2\left(C_\text{þ}^{c^t}\right) = \text{þ}\,(r^2 + 10r + 32)$.
5. $ZCI_1\left(C_\text{þ}^{c^t}\right) = \text{þ}\,(2\text{þ}r^2 - r^2 + 10\text{þ}r + 12\text{þ} - 7r - 26)$.

6. $ZCI_2\left(C_{\text{þ}}^{c^t}\right) = \frac{1}{2}\text{þ}\,(4\text{þ}r^2 - 3r^2 + 24\text{þ}r + 36\text{þ} - 25r - 84).$

3. Conclusions

We can conclude from Figure 2, 3, and 4, representations of special classes of regular graphs: specifically cycle, complete, and complete bipartite graphs together with namely cog- regular graphs, thorny regular graphs, and thorny of cog - regular graphs (constructed via specific graph operations), it is evident that the first and second Zagreb indices, along with their corresponding coindices, demonstrate a uniform behavioral pattern across all such configurations. This consistency is preserved irrespective of whether the structure considered is the base regular graph, its cog, its thorn, or its thorn of cog regular graph. Moreover, the analysis appears that a broad range of topological indices, regardless of their individual definitions or the degree of regularity of the basic graph, tend to follow a similar behavioral trajectory. Such findings suggest that the investigation of a single, strategically chosen class of regular graphs supported by the application of defined graph operations can serve as a representative framework for inferring the behavior of these indices through the related graph families.

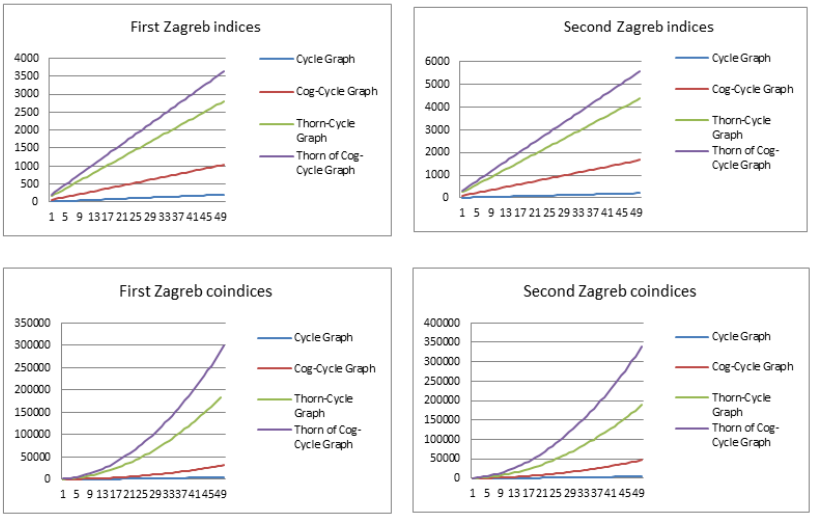

Figure 2
The Zagreb indices for cycle, cog-cycle, thorn-cycle and thorn of cog-cycle

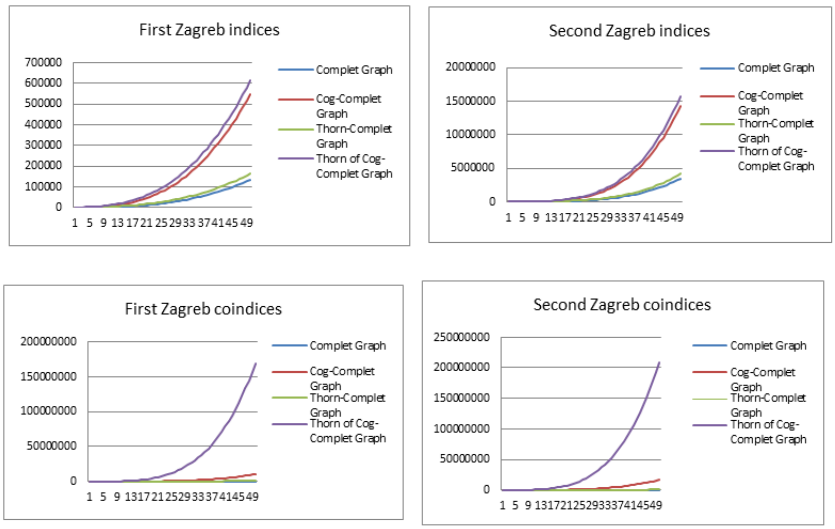

Figure 3

The Zagreb indices for complete, cog-complete, thorn – complete and thorn of cog-complete

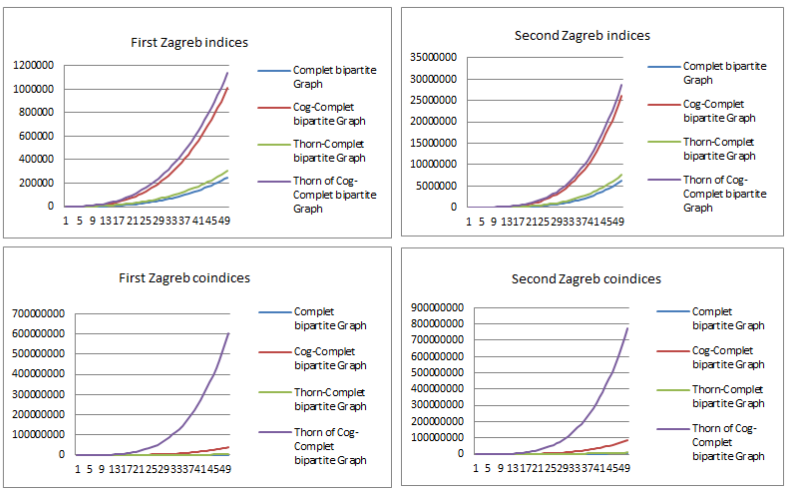

Figure 4

The Zagreb indices for complete bipartite, cog-complete bipartite, thron-complete bipartite and thorn of cog-complete bipartite

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