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Total reflexive edge irregularity strength of staircase graphs and related graphs

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Abstract

Ryan proposed the concept of reflexive edge irregular labeling of graphs in 2017. The reflexive edge irregular total m-labeling on graph Ω with vertex set V_Ω and edge set E_Ω is defined as a mapping ϖ_e from the edge set E_Ω to the whole positive numbers $\{1, 2, ..., m_e\}$ and a mapping ϖ_v from the vertex set V_Ω to the even positive numbers $\{0, 2, ..., 2m_v\}$ such that $\varpi(x) = \varpi_v(x)$ if $x \in V_\Omega$ and $\varpi(xy) = \varpi_e(xy)$ if $xy \in E_\Omega$, where $m = \max\{m_e, 2m_v\}$. For each $xy \in E_\Omega$, we have a weight $wt(xy) = \varpi(x) + \varpi(y) + \varpi(xy)$. The lowest value m utilized in reflexive total m-labeling is referred to as the reflexive irregularity strength of the graph Ω. Symbolized by res(Ω). In this research, we computed the precise value of reflexive edge irregularity strength for staircase graphs, double staircase graphs, and mirror-staircase graphs.

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Keywords: *Edge irregular reflexive labeling, Staircase graphs, Double staircase graphs, Mirror staircase graphs.*

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1. Introduction

Assuming $\Omega(V_\Omega, E_\Omega)$ represents a basic, finite, undirected graph, where V_Ω denotes a non-empty collection of vertices and E_Ω signifies an unordered pair of two separate vertices $x, y \in V_\Omega$. Graphs are valuable models for a variety of connection and interactions in various disciplines, including biology, computer science, and chemistry, with a particular emphasis on chemical graph theory. Graphs have been employed by scientists to illustrate communication networks and arrange data [12, 13]. The labeling of a graph is a mapping that assigns graph components to positive or non-negative numbers. Whenever the domain consists solely of vertex set or encompasses all edges and vertices, the labeling is referred to as vertex labeling or total labeling. Haque [14] introduced the specific assignment of numerical values to the edges and vertices of a network using diverse total labeling methodologies. An extensive variety of scientific disciplines have implemented irregular graph labeling. For instance, the labels on the vertices refer to network routing difficulties that occur in computers. In [11] Chartrand et. al proposed irregular graph labeling, where the lowest biggest label of the graph Ω is marked as the graph strength $s(\Omega)$. Recently, irregular labeling classified into two categories: edge-irregular labeling (EIL) and vertex-irregular labeling (VIL). Ahmad, Al-Mushayt, and Bača [1] established that the vertex m-labeling $\varpi: V_\Omega \to \{1,2,\dots,m\}$, the relative weight $wt(xy) = \varpi(x) + \varpi(y)$ of edges $xy \in E_\Omega$. Vertex m-labeling is named EIL when all edges have various weights. The largest integer m reduced by every edge-irregular m-labeling, is referred to as the edge irregularity strength of Ω, es(Ω). By the same way, Imran, Baig, and Town [15] confirmed that for the edge m-labeling $\varpi: E_\Omega \to \{1,2,\dots,m\}$, the relative weight $wt(x) = \sum_{xy \in E_\Omega} \varpi(xy)$. Edge m-labeling is named VIL when all vertices have various weights. The vertex irregularity strength of Ω is represented by vs(Ω). Meanwhile Bača, Miller, and Ryan [3] confirmed a new labeling named a total m-labeling ϖ that label the vertices and edges of Ω by integers from the set $\{1,2,\dots,m\}$. A total labeling is classified as total vertex-irregular labeling (TVIL) when the weights of all vertices are differing from one another, and it is termed total edge-irregular labeling (TEIL) when the weights of the edges are also differing from each other, while the vertex weight and edge weight are given by $wt(x) = \varpi(x) + \sum_{xy \in E_\Omega} \varpi(xy)$ and $wt(xy) = \varpi(x) + \varpi(y) + \varpi(xy)$ respectively. Numerous results regarding TEIL and TVIL can be found in [4, 16-18, 20]. Tanna, Ryan, and Semaničová-Feňovčíková [21] motivated the concept of edge-irregular total m-labeling , where the vertices of Ω label by even positive numbers from the collection $\{0,2,\dots,2m_v\}$ and the edges label by positive numbers from the collection $\{1,2,\dots,m_e\}$, where $m = \max\{m_e, 2m_v\}$, which is referred to as reflexive edge-irregular labeling (REIL), whereas the lowest largest label of graph Ω is known as the reflexive edge irregularity strength of Ω represented by res(Ω). Baća et al. [5] put forward the subsequent conjecture and provided a proof for Lemma 1.1.

Conjecture 1.1: Consider Ω is a graph with maximum degree $\Delta(\Omega)$, then

$$res(\Omega) = max\left\{\left\lceil \frac{|E_\Omega|}{3} + c \right\rceil, \left\lfloor \frac{\Delta(\Omega)}{2} + 1 \right\rfloor\right\}$$

where $c = 1$ for $|E_\Omega| \equiv 2{,}3 \mod(6)$, otherwise, $c = 0$.

Lemma 1.1: *For each graph Ω,*

$$res(\Omega) \geq \begin{cases} \left\lceil \frac{|E_\Omega|}{3} \right\rceil & \text{if } |E_\Omega| \equiv 2,3 \pmod 6, \\ \left\lceil \frac{|E_\Omega|}{3} \right\rceil + 1 & \text{otherwise.} \end{cases}$$

is the lowest bound of the res(Ω), where $|E_\Omega|$ denotes the cardinality of the edge set E_Ω [22].

Tanna, Ryan, and Semaničová-Feňovčíková [21] computed the precise values of res(Ω), where Ω represents the prism, wheel, and friendship graphs. Also, Agustin, Utoyo, and Venkatachalam [2] computed the exact values of some graphs, namely ladder graph, triangular ladder graph, $P_n \odot P_2$ graph, $P_n \times C_3$ graph, and $P_n \odot C_3$ graph. In [6-10] the author estimated the precise values of the reflexive edge strength of the circulant graph, r-th power of the path graph, toroidal fullerene graph, and diamond graph. In this research, we obtained the exact values of res(Ω), where Ω is staircase, double-staircase and mirror staircase graphs.

2. Main results

A staircase graph with k-level is obtained by merging the grids of $P_2 \times P_2$, $P_2 \times P_3$, $P_2 \times P_4$ ... $P_2 \times P_k$ as shown in Figure 1.

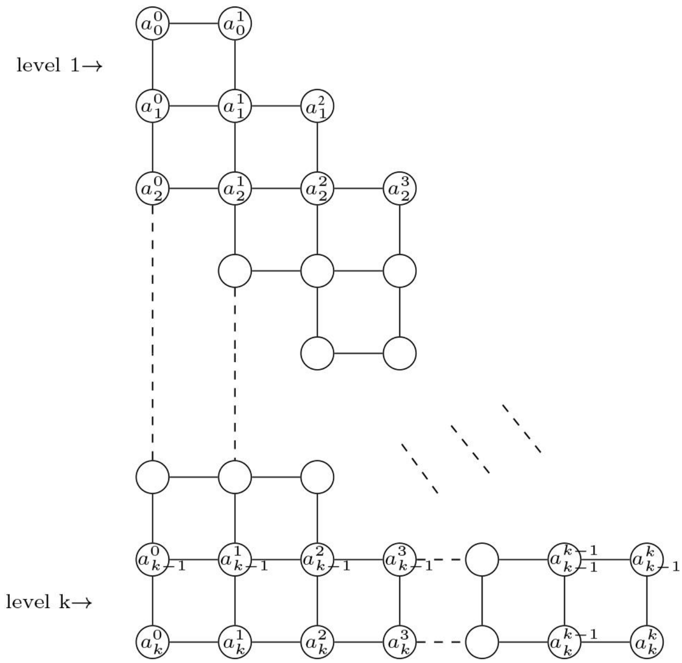

Figure 1

Thus, the vertex set and edge set of the SC_k, where $n \geq 1$ are $V(SC_k) = \{a_s^r | r = 0{,}1 \text{ and } s = 0{,}1, \ldots, k\} \cup \{a_s^r | r = 2, \ldots, k \text{ and } s = r-1, \ldots, k\}$ and $E(SC_k)$ comprises edges as follows.

edges	r	s
$a_s^r a_s^{r+1}$	0	$0, 1, 2, \ldots, k$
$a_s^r a_s^{r+1}$	$1, \ldots, k-1$	$r, \ldots, k$
$a_s^r a_{s+1}^r$	$0, 1$	$0, 1, \ldots, k-1$
$a_s^r a_{s+1}^r$	$2, \ldots, k$	$r-1, \ldots, k-1$

By performing a straightforward calculation, it is evident that $|V(SC_k)| = \frac{k^2}{2} + 2k + 1$ and $|E(SC_k)| = k(k+3)$. The subsequent theorem provides an estimation of the precise value of res for the staircase graph SC_k, $k \geq 1$ [19].

Theorem 2.1: *For any positive number* $k \geq 1$, *the following holds*

$$res(SC_k) = \left\lceil \frac{k(k+3)}{3} \right\rceil = m$$

Proof: Given that the cardinality of $E(SC_k)$ is $k(k+3)$, we can establish a lower bound for res(SC_k) by applying Lemma 1.1.

$$res(SC_k) = m \geq \left\lceil \frac{k(k+3)}{3} \right\rceil$$

Thus, demonstrating the equality of res(SC_k) suffices under the assumption that REIL with $m = \left\lceil \frac{k(k+3)}{3} \right\rceil$ exists. The optimal REIL for SC_k, where $k \geq 1$, is expressed as.

x	$\varpi(x)$	r and s
a_s^r	0	$r = 0{,}1,\ s = 0$
		$r = 0{,}1,\ s = 1$
	2	$r = 2,\ s = 1$
	$6\left(\left\lceil \frac{s-4}{6} \right\rceil^2 + \left\lfloor \frac{s-4}{6} \right\rfloor^2\right) + 8\left\lceil \frac{s-4}{6} \right\rceil + {}$ $+14\left\lfloor \frac{s-4}{6} \right\rfloor + 10$	$s = 2, \ldots, k,\ s \equiv 1 (\text{mod } 3)$ $r = 0{,}1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor + 1$
	$6\left(\left\lceil \frac{s-2}{6} \right\rceil^2 + \left\lfloor \frac{s-2}{6} \right\rfloor^2\right) + 4\left\lceil \frac{s-2}{6} \right\rceil + {}$ $+10\left\lfloor \frac{s-4}{6} \right\rfloor + 10$	$s = 2, \ldots, k,\ s \equiv 2 (\text{mod } 3)$ $r = 0{,}1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor + 1$
	$\frac{s^2}{3} + s$	$s = 2, \ldots, k,\ s \equiv 0\ (\text{mod } 3)$ $r = 0{,}1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor + 1$
$a_s^r a_s^{r+1}$	1	$r = 0,\ s = 0$
	4	$r = 0,\ s = 1$
	3	$r = 1,\ s = 1$

	$6\left(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2\right)+5\left\lceil\frac{s-4}{6}\right\rceil+$ $+11\left\lfloor\frac{s-4}{6}\right\rfloor+r+5$	$s=2,\dots,k,\ s\equiv 1(\text{mod}3)$ $r=0,1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor$
	$6\left(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2\right)+\left\lceil\frac{s-2}{6}\right\rceil+$ $+7\left\lfloor\frac{s-2}{6}\right\rfloor+r+1$	$s=2,\dots,k,\ s\equiv 2(\text{mod}3)$ $r=0,1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor$
	$\frac{s^2}{3}+r+1$	$s=2,\dots,k,\ s\equiv 0(\text{mod}3)$ $r=0,1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor$
$a_s^r a_{s+1}^r$	2	$r=0,\ s=0$
	3	$r=1,\ s=0$
	1	$r=0,\ s=1$
	2	$r=1,\ s=1$
	1	$r=2,\ s=1$
	$6\left(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2\right)+6\left\lceil\frac{s-4}{6}\right\rceil+$ $+12\left\lfloor\frac{s-4}{6}\right\rfloor+r+6$	$s=2,\dots,k-1,\ s\equiv 1\ (\text{mod}\ 3)$ $r=0,1,\dots,s+1$
	$6\left(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2\right)+2\left\lceil\frac{s-2}{6}\right\rceil+$ $+8\left\lfloor\frac{s-2}{6}\right\rfloor+r+2$	$s=2,\dots,k-1,\ s\equiv 2\ (\text{mod}\ 3)$ $r=0,1,\dots,s+1$
	$6\left(\left\lceil\frac{s-3}{6}\right\rceil^2+\left\lfloor\frac{s-3}{6}\right\rfloor^2\right)+4\left\lceil\frac{s-3}{6}\right\rceil+$ $+10\left\lfloor\frac{s-3}{6}\right\rfloor+r+4$	$s=2,\dots,k-1,\ s\equiv 0\ (\text{mod}\ 3)$ $r=0,1,\dots,s+1$

Now, we shall demonstrate that each edge in SC_k possesses a distinct edge weight according to the labeling ϖ as follows.

x	$wt_{\varpi}(x)$	r and s
$a_s^r a_s^{r+1}$	1	$r=0,\ s=0$
	4	$r=0,\ s=1$
	5	$r=1,\ s=1$
	9	$r=0,\ s=2$
	10	$r=1,\ s=2$
	11	$r=1,\ s=2$
	$2s^2+2s+r+1$	$s=3,\dots,k,\ s\equiv 0\ (\text{mod}\ 3)$ $r=0,1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor+1$
	$18\left(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2\right)+21\left\lceil\frac{s-4}{6}\right\rceil+$ $+39\left\lfloor\frac{s-4}{6}\right\rfloor+r+25$	$s=3,\dots,k,\ s\equiv 1\ (\text{mod}\ 3)$ $r=0,1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor+1$
	$18\left(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2\right)+9\left\lceil\frac{s-2}{6}\right\rceil+$ $+27\left\lfloor\frac{s-2}{6}\right\rfloor+r+9$	$s=3,\dots,k,\ s\equiv 2\ (\text{mod}\ 3)$ $r=0,1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor+1$
$a_s^r a_{s+1}^r$	2	$r=0,\ s=0$
	3	$r=1,\ s=0$
	6	$r=0,\ s=1$

	7	$r = 2,\ s = 1$
	8	$r = 2,\ s = 1$
	$18\left(\left\lceil\frac{s-4}{6}\right\rceil^2 + \left\lfloor\frac{s-4}{6}\right\rfloor^2\right) + 24\left\lceil\frac{s-4}{6}\right\rceil +$ $+42\left\lfloor\frac{s-4}{6}\right\rfloor + r + 30$	$s = 3,\ldots,k-1,\ s \equiv 1 \pmod 3$ $r = 0,1,\ldots,s+1$
	$12\left(\left\lceil\frac{s-2}{6}\right\rceil^2 + \left\lfloor\frac{s-2}{6}\right\rfloor^2\right) + 6\left\lceil\frac{s-2}{6}\right\rceil +$ $+16\left\lfloor\frac{s-4}{6}\right\rfloor + \frac{s(s+3)}{3} + r + 6$	$s = 3,\ldots,k-1,\ s \equiv 2 \pmod 3$ $r = 0,1,\ldots,s+1$
	$12\left(\left\lceil\frac{s-3}{6}\right\rceil^2 + \left\lfloor\frac{s-3}{6}\right\rfloor^2\right) + 12\left\lceil\frac{s-3}{6}\right\rceil +$ $+24\left\lfloor\frac{s-3}{6}\right\rfloor + \frac{s(s+3)}{3} + r + 4$	$s = 3,\ldots,k-1,\ s \equiv 0 \pmod 3$ $r = 0,1,\ldots,s+1$

We can enumerate all edge weights using their indices. Subsequently, using this collection, we obtain a sequence of integers ranging from 1 to $k(k+3)$. It clarify that the biggest label is m and all weights are distinct. This also indicates that ϖ constitutes a reflexive edge-irregular total m-labeling with $m = \left\lceil\frac{k(k+3)}{3}\right\rceil$. Consequently, we demonstrate that res$(SC_k) = \left\lceil\frac{k(k+3)}{3}\right\rceil$.

We now examine res for the double staircase in the next theorem. We refer to the double staircase of level k as DSC_k (see Figure 2). We possess $V(SC_k) = \{a_s^r | r = 1,2,\ldots,k \text{ and } s = r-1,1,\ldots,k\} \cup \{b_s^r | r = 1,\ldots,k \text{ and } s = r-1,\ldots,k\}$ and $E(DSC_k)$, which is composed of edges as outlined in the list below.

edges	r	s
$a_s^r b_s^r$	1	$0,1,2,\ldots,k$
$a_s^{r+1} a_s^r$	$1,\ldots,k-1$	$1,\ldots,k$
$b_s^r b_s^{r+1}$	$1,\ldots,k-1$	$1,\ldots,k$
$a_s^r a_{s+1}^r$	$1,\ldots,k$	$r-1,\ldots,k$
$b_s^r b_{s+1}^r$	$1,\ldots,k$	$r-1,\ldots,k$

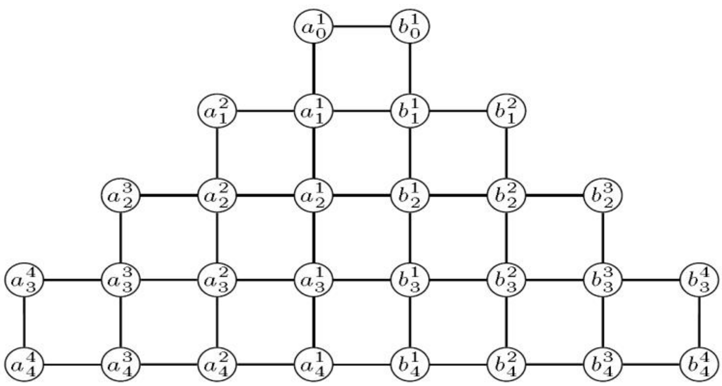

Figure 2
Double Staircase Graph DSC_4.

Theorem 2.2: *For any positive number* $k \geq 1$, *the following holds*

$$res(DSC_k) = m = \begin{cases} \left\lceil \frac{2k^2+3k-1}{3} \right\rceil + 1, & if\ k \equiv 3\ (mod\ 6), \\ \left\lceil \frac{2k^2+3k-1}{3} \right\rceil, & otherwise. \end{cases}$$

Proof: Because the size of DSC_k is $2k^2 + 3k - 1$, and based on Lemma 1.1 we can get a lower bound of res(DSC_k) as follows:

$$res(DSC_k) \geq \begin{cases} \left\lceil \frac{2k^2+3k-1}{3} \right\rceil + 1, & \text{if } k \equiv 3 \text{ (mod 6)}, \\ \left\lceil \frac{2k^2+3k-1}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

To reverse the inequality, it is sufficient to demonstrate the existence of a total m-labeling (REIL) ϖ of DSC_k as follows

x	$\varpi(x)$	r and s
$a_s^r = b_s^r$	0	$r = 1,\ s = 0$
	0	$r = 1,\ s = 1$
	2	$s = 2\ r = 1$
	$12\left(\left\lceil\frac{s-2}{6}\right\rceil^2 + \left\lfloor\frac{s-2}{6}\right\rfloor^2\right) + 6\left\lceil\frac{s-2}{6}\right\rceil + +16\left\lfloor\frac{s-2}{6}\right\rfloor + 4$	$s = 2,\dots,k,\ s \equiv 2 \text{ (mod 3)}$ $r = 0,1,\dots,s - \left\lfloor\frac{s}{k}\right\rfloor + 1$
	$12\left(\left\lceil\frac{s-3}{6}\right\rceil^2 + \left\lfloor\frac{s-3}{6}\right\rfloor^2\right) + 8\left\lceil\frac{s-3}{6}\right\rceil + +22\left\lfloor\frac{s-3}{6}\right\rfloor + 10$	$s = 2,\dots,k,\ s \equiv 0 \text{ (mod 3)}$ $r = 0,1,\dots,s - \left\lfloor\frac{s}{k}\right\rfloor + 1$
	$12\left(\left\lceil\frac{s-4}{6}\right\rceil^2 + \left\lfloor\frac{s-4}{6}\right\rfloor^2\right) + 14\left\lceil\frac{s-4}{6}\right\rceil + +24\left\lfloor\frac{s-4}{6}\right\rfloor + 10$	$s = 2,\dots,k,\ s \equiv 1 \text{ (mod 3)}$ $r = 0,1,\dots,s - \left\lfloor\frac{s}{k}\right\rfloor + 1$
$a_s^r b_s^r$	1	$r = 1,\ s = 0$
	4	$r = 1,\ s = 1$
	$12\left(\left\lceil\frac{s-2}{6}\right\rceil^2 - 3\left\lfloor\frac{s-2}{6}\right\rfloor^2\right) + 5\left\lceil\frac{s-2}{6}\right\rceil + +13\left\lfloor\frac{s-2}{6}\right\rfloor + 3$	$s = 2,\dots,k,\ s \equiv 2 \text{ (mod 3)}$ $r = 1$
	$12\left(\left\lceil\frac{s-3}{6}\right\rceil^2 + \left\lfloor\frac{s-3}{6}\right\rfloor^2\right) + 5\left\lceil\frac{s-3}{6}\right\rceil + +13\left\lfloor\frac{s-2}{6}\right\rfloor + 2$	$s = 2,\dots,k,\ s \equiv 0 \text{ (mod 3)}$ $r = 1$
	$12\left(\left\lceil\frac{s-4}{6}\right\rceil^2 + \left\lfloor\frac{s-4}{6}\right\rfloor^2\right) + 5\left\lceil\frac{s-4}{6}\right\rceil + +21\left\lfloor\frac{s-4}{6}\right\rfloor + 9$	$s = 2,\dots,k,\ s \equiv 1 \text{ (mod 3)}$ $r = 1$

x	$\varpi(x)$	r and s
$b_s^r b_s^{r+1}$	3	$s = 1,\ r = 1$
	$12(\left\lceil\frac{s-2}{6}\right\rceil^2 + \left\lfloor\frac{s-2}{6}\right\rfloor^2) - 3\left\lceil\frac{s-2}{6}\right\rceil + +13\left\lfloor\frac{s-2}{6}\right\rfloor + 2r + 2$	$s = 2,\dots,k,\ s \equiv 2 \text{ (mod 3)}$ $r = 0,1,\dots,s - \left\lfloor\frac{s}{k}\right\rfloor$

	$12\left(\left\lceil\frac{s-3}{6}\right\rceil^2+\left\lfloor\frac{s-3}{6}\right\rfloor^2\right)+$ $5\left\lceil\frac{s-3}{6}\right\rceil+13\left\lfloor\frac{s-3}{6}\right\rfloor+2r+1$	$s=2,\ldots,k,\ s\equiv 0\ (\mathrm{mod}\ 3)$ $r=0,1,\ldots,s-\left\lfloor\frac{s}{k}\right\rfloor$
	$12(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2)+5\left\lceil\frac{s-4}{6}\right\rceil+$ $+21\left\lfloor\frac{s-4}{6}\right\rfloor+2r+8$	$s=2,\ldots,k,\ s\equiv 1\ (\mathrm{mod}\ 3)$ $r=0,1,\ldots,s-\left\lfloor\frac{s}{k}\right\rfloor$
$a_s^{r+1}a_s^r$	4	$s=1,\ r=1$
	$12(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2)-3\left\lceil\frac{s-2}{6}\right\rceil+$ $+13\left\lfloor\frac{s-2}{6}\right\rfloor+2r+3$	$s=2,\ldots,k,\ s\equiv 2\ (\mathrm{mod}\ 3)$ $r=0,1,\ldots,s-\left\lfloor\frac{s}{k}\right\rfloor$
	$12(\left\lceil\frac{s-3}{6}\right\rceil^2+\left\lfloor\frac{s-3}{6}\right\rfloor^2)+5\left\lceil\frac{s-3}{6}\right\rceil+$ $+13\left\lfloor\frac{s-3}{6}\right\rfloor+2r+2$	$s=2,\ldots,k,\ s\equiv 0\ (\mathrm{mod}\ 3)$ $r=0,1,\ldots,s-\left\lfloor\frac{s}{k}\right\rfloor$
	$12(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2)+5\left\lceil\frac{s-4}{6}\right\rceil+$ $+21\left\lfloor\frac{s-4}{6}\right\rfloor+2r+9$	$s=2,\ldots,k,\ s\equiv 1\ (\mathrm{mod}\ 3)$ $r=0,1,\ldots,s-\left\lfloor\frac{s}{k}\right\rfloor$
$b_s^r b_{s+1}^r$	2	$s=0,\ r=1$
	3	$s=1,\ r=1$
	$12(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2)-3\left\lceil\frac{s-2}{6}\right\rceil+$ $+13\left\lfloor\frac{s-2}{6}\right\rfloor+2r$	$s=2,\ldots,k-1,\ s\equiv 2\ (\mathrm{mod}\ 3)$ $r=1,\ldots,s+1$
	$12(\left\lceil\frac{s-3}{6}\right\rceil^2+\left\lfloor\frac{s-3}{6}\right\rfloor^2)+5\left\lceil\frac{s-3}{6}\right\rceil+$ $+17\left\lfloor\frac{s-3}{6}\right\rfloor+2r+3$	$s=2,\ldots,k-1,\ s\equiv 0\ (\mathrm{mod}\ 3)$ $r=1,\ldots,s+1$
	$12(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2)+9\left\lceil\frac{s-4}{6}\right\rceil+$ $+21\left\lfloor\frac{s-4}{6}\right\rfloor+2r+8$	$s=2,\ldots,k-1,\ s\equiv 1\ (\mathrm{mod}\ 3)$ $r=1,\ldots,s+1$
$a_s^r a_{s+1}^r$	3	$s=0,\ r=1$
	4	$s=1,\ r=1$
	$12(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2)-3\left\lceil\frac{s-2}{6}\right\rceil+$ $+13\left\lfloor\frac{s-2}{6}\right\rfloor+2r+1$	$s=2,\ldots,k-1,\ s\equiv 2\ (\mathrm{mod}\ 3)$ $r=1,\ldots,s+1$
	$12(\left\lceil\frac{s-3}{6}\right\rceil^2+\left\lfloor\frac{s-3}{6}\right\rfloor^2)+5\left\lceil\frac{s-3}{6}\right\rceil+$ $+17\left\lfloor\frac{s-3}{6}\right\rfloor+2r+4$	$s=2,\ldots,k-1,\ s\equiv 0\ (\mathrm{mod}\ 3)$ $r=1,\ldots,s+1$
	$12(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2)+9\left\lceil\frac{s-4}{6}\right\rceil+$ $+21\left\lfloor\frac{s-4}{6}\right\rfloor+2r+9$	$s=2,\ldots,k-1,\ s\equiv 1\ (\mathrm{mod}\ 3)$ $r=0,1,\ldots,s+1$

Moreover, to demonstrate that ϖ constitutes a reflexive edge-irregular m-labeling, it is essential to evaluate the weight rise at all edges in DSC_k. We acquired the weight across every edge, as follows:

x	$wt_{\varpi}(x)$	r and s
$a_s^r b_s^r$	1	$r=1,\ s=0$
	4	$r=1,\ s=1$
	$36\left(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2\right)+9\left\lceil\frac{s-2}{6}\right\rceil+$ $+45\left\lfloor\frac{s-2}{6}\right\rfloor+11$	$s=2,\dots,k-1,\ s\equiv 1 \pmod 3$ $r=1$
	$36\left(\left\lceil\frac{s-3}{6}\right\rceil^2+\left\lfloor\frac{s-3}{6}\right\rfloor^2\right)+21\left\lceil\frac{s-3}{6}\right\rceil+$ $+57\left\lfloor\frac{s-3}{6}\right\rfloor+22$	$s=2,\dots,k-1,\ s\equiv 0 \pmod 3$ $r=1$
	$36\left(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2\right)+33\left\lceil\frac{s-2}{6}\right\rceil+$ $+69\left\lfloor\frac{s-4}{6}\right\rfloor+37$	$s=2,\dots,k-1,\ s\equiv 0 \pmod 3$ $r=1$

x	$wt_{\varpi}(x)$	r and s
$b_s^r b_s^{r+1}$	5	$r=1,\ s=1$
	$36\left(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2\right)+9\left\lceil\frac{s-2}{6}\right\rceil+$ $+35\left\lfloor\frac{s-2}{6}\right\rfloor+r+10$	$s=2,\dots,k,\ s\equiv 2 \pmod 3$ $r=1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor$
	$36\left(\left\lceil\frac{s-3}{6}\right\rceil^2+\left\lfloor\frac{s-3}{6}\right\rfloor^2\right)+21\left\lceil\frac{s-3}{6}\right\rceil+$ $+57\left\lfloor\frac{s-3}{6}\right\rfloor+2r+21$	$s=2,\dots,k,\ s\equiv 0 \pmod 3$ $r=1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor$
	$36\left(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2\right)+49\left\lceil\frac{s-4}{6}\right\rceil+$ $+69\left\lfloor\frac{s-4}{6}\right\rfloor+2r+36$	$s=2,\dots,k,\ s\equiv 1 \pmod 3$ $r=1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor$
$a_s^{r+1} a_s^r$	6	$r=1,\ s=1$
	$36\left(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2\right)+9\left\lceil\frac{s-2}{6}\right\rceil+$ $+35\left\lfloor\frac{s-2}{6}\right\rfloor+2r+11$	$s=2,\dots,k,\ s\equiv 2 \pmod 3$ $r=1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor$
	$36\left(\left\lceil\frac{s-3}{6}\right\rceil^2+\left\lfloor\frac{s-3}{6}\right\rfloor^2\right)+21\left\lceil\frac{s-3}{6}\right\rceil+$ $+57\left\lfloor\frac{s-3}{6}\right\rfloor+2r+22$	$s=2,\dots,k,\ s\equiv 0 \pmod 3$ $r=1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor$
	$36\left(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2\right)+49\left\lceil\frac{s-4}{6}\right\rceil+$ $+69\left\lfloor\frac{s-4}{6}\right\rfloor+2r+37$	$s=2,\dots,k,\ s\equiv 0 \pmod 3$ $r=1,\dots,s-\left\lfloor\frac{s}{k}\right\rfloor$
$b_s^r b_{s+1}^r$	2	$r=1,\ s=0$
	7	$r=1,\ s=1$
	9	$r=2,\ s=1$
	$36\left(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2\right)+15\left\lceil\frac{s-2}{6}\right\rceil+$ $+51\left\lfloor\frac{s-2}{6}\right\rfloor+2r+14$	$s=2,\dots,k-1,\ s\equiv 2 \pmod 3$ $r=1,\dots,s+1$

	$36\left(\left\lceil\frac{s-3}{6}\right\rceil^2+\left\lfloor\frac{s-3}{6}\right\rfloor^2\right)+27\left\lceil\frac{s-3}{6}\right\rceil+$ $+63\left\lfloor\frac{s-3}{6}\right\rfloor+2r+27$	$s=2,\dots,k-1,\ s\equiv 0 \pmod 3)$ $r=1,\dots,s+1$
	$36\left(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2\right)+39\left\lceil\frac{s-4}{6}\right\rceil+$ $+75\left\lfloor\frac{s-4}{6}\right\rfloor+2r+37$	$s=2,\dots,k-1,\ s\equiv 0 \pmod 3)$ $r=1,\dots,s+1$
$a_s^r a_{s+1}^r$	3	$r=1,\ s=0$
	8	$r=1,\ s=1$
	10	$r=2,\ s=1$
	$36\left(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2\right)+15\left\lceil\frac{s-2}{6}\right\rceil+$ $+51\left\lfloor\frac{s-2}{6}\right\rfloor+2r+15$	$s=2,\dots,k-1,\ s\equiv 2 \pmod 3)$ $r=1,\dots,s+1$
	$36\left(\left\lceil\frac{s-3}{6}\right\rceil^2+\left\lfloor\frac{s-3}{6}\right\rfloor^2\right)+27\left\lceil\frac{s-3}{6}\right\rceil+$ $+63\left\lfloor\frac{s-3}{6}\right\rfloor+2r+28$	$s=2,\dots,k-1,\ s\equiv 0 \pmod 3)$ $r=1,\dots,s+1$
	$36\left(\left\lceil\frac{s-4}{6}\right\rceil^2+\left\lfloor\frac{s-4}{6}\right\rfloor^2\right)+39\left\lceil\frac{s-4}{6}\right\rceil+$ $+75\left\lfloor\frac{s-4}{6}\right\rfloor+2r+38$	$s=2,\dots,k-1,\ s\equiv 1 \pmod 3)$ $r=1,\dots,s+1$

The weights of the edges can be determined using their indices. Subsequently, using this collection, we get a sequence of integers ranging from 1 to $2k^2+3k-1$. This indicates that each distinct edge has a unique weight. This also indicates that ϖ constitutes a reflexive edge-irregular total m-labeling, where $m=\left\lceil\frac{2k^2+3k-1}{3}\right\rceil+1$ for $k\equiv 3 \pmod 6$, and $m=\left\lceil\frac{2k^2+3k-1}{3}\right\rceil$ otherwise. Consequently, we demonstrate that the following:

$$res(DSC_k)=\begin{cases}\left\lceil\frac{2k^2+3k-1}{3}\right\rceil+1, & \text{if } k\equiv 3 \pmod 6,\\ \left\lceil\frac{2k^2+3k-1}{3}\right\rceil, & otherwise.\end{cases}$$

Example 2.1: Figure 3 show the reflexive 15-labeling of DSC_4.

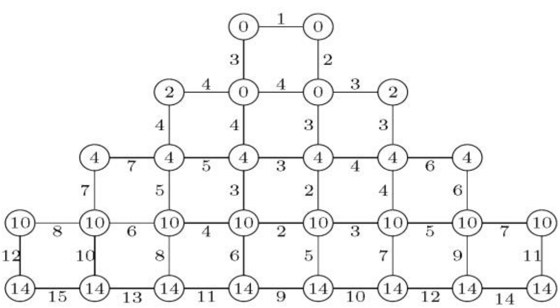

Figure 3
Reflexive 15-labeling of DSC_4.

Next, we find res for another related graph to the staircase graph called the mirror staircase graph of level $k \geq 1$, which we refer to as MSC_k see Figure 4.

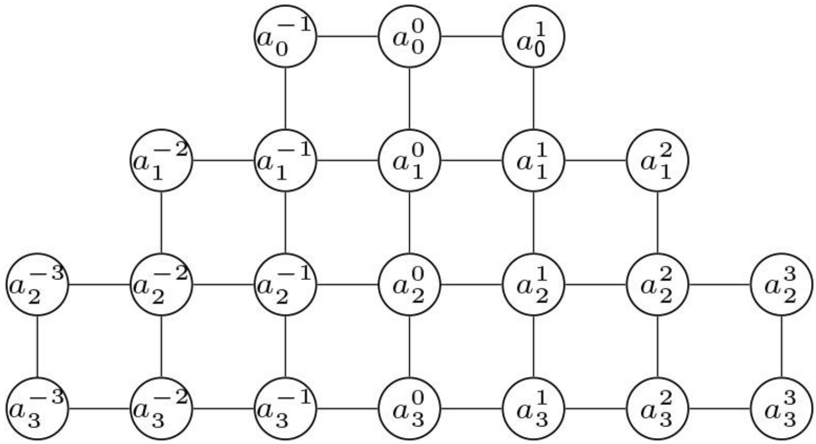

Figure 4
Mirror Staircase Graph MSC_3.

We possess $V(MSC_k) = \{a_s^r | r = -1,0,1, s = 0,1,\ldots,k\} \cup \{a_s^r | r = 2,\ldots, k, s = r-1,\ldots,k\} \cup \{a_s^r | r = -2,\ldots,-k, s = -r-1,\ldots,k\}$ and $E(MSC_k)$ comprises the edges enumerated below.

edges	r	s
$a_s^r a_s^{r+1}$	$-1, 0$	$0, 1, \ldots, k$
$a_s^r a_s^{r+1}$	$1, \ldots, k-1$	$r, \ldots, k$
$a_s^r a_s^{r+1}$	$-2, \ldots, -k$	$-r-1, \ldots, k$
$a_s^r a_{s+1}^r$	$-1, 0, 1$	$0, \ldots, k-1$
$a_s^r a_{s+1}^r$	$2, \ldots, k$	$r-1, \ldots, k-1$
$a_s^r a_{s+1}^r$	$-2, \ldots, -k$	$-r-1, \ldots, k-1$

Theorem 2.3: *For any positive number $k \geq 1$, the following holds*

$$res(MSC_k) = m = \begin{cases} \left\lceil \frac{2k^2+5k}{3} \right\rceil + 1, & \text{if } k \equiv 3, 5 \pmod 6, \\ \left\lceil \frac{2k^2+5k}{3} \right\rceil, & otherwise. \end{cases}$$

Proof: It is evident that the MSC_k has $2k^2 + 5k$ edges. Thus from Lemma 1.1, we get

$$es(MSC_k) = m \geq \begin{cases} \left\lceil \frac{2k^2+5k}{3} \right\rceil + 1, & \text{if } k \equiv 3, 5 \pmod 6, \\ \left\lceil \frac{2k^2+5k}{3} \right\rceil, & \text{otherwise.} \end{cases}$$

Consequently, we demonstrated that m serves as an upper bound for $\text{res}(MSC_k)$, where $k \geq 1$. By indicating the existence of a total m-labeling (REIL) ϖ of MSC_k as follows

x	$\varpi(x)$	r and s
a_s^r	$12\left(\left\lceil \frac{s}{6} \right\rceil^2 + \left\lfloor \frac{s}{6} \right\rfloor^2\right) + 10\left\lfloor \frac{s}{6} \right\rfloor$	$s = 0, 1, \ldots, k,\ s \equiv 0 \pmod 3$ $r = -s + \left\lfloor \frac{s}{k} \right\rfloor - 1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor + 1$
	$12\left(\left\lceil \frac{s-2}{6} \right\rceil^2 + \left\lfloor \frac{s-2}{6} \right\rfloor^2\right) + 8\left\lceil \frac{s-2}{6} \right\rceil +$ $+18\left\lfloor \frac{s-2}{6} \right\rfloor + 6$	$s = 0,1, \ldots, k,\ s \equiv 2 \pmod 3$ $r = -s + \left\lfloor \frac{s}{k} \right\rfloor - 1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor + 1$
	$12\left(\left\lceil \frac{s-1}{6} \right\rceil^2 + \left\lfloor \frac{s-1}{6} \right\rfloor^2\right) + 4\left\lceil \frac{s-1}{6} \right\rceil +$ $+14\left\lfloor \frac{s-1}{6} \right\rfloor + 2$	$s = 0, 1, \ldots, k,\ s \equiv 1 \pmod 3$ $r = -s + \left\lfloor \frac{s}{k} \right\rfloor - 1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor + 1$
$a_s^r a_s^{r+1}$	$12\left(\left\lceil \frac{s}{6} \right\rceil^2 + \left\lfloor \frac{s}{6} \right\rfloor^2\right) - 9\left\lceil \frac{s}{6} \right\rceil +$ $+7\left\lfloor \frac{s}{6} \right\rfloor + r + s - \left\lfloor \frac{s}{k} \right\rfloor + 2$	$s = 0,1, \ldots, k,\ s \equiv 0 \pmod 3$ $r = -s + \left\lfloor \frac{s}{k} \right\rfloor - 1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor$
	$12\left(\left\lceil \frac{s-1}{6} \right\rceil^2 + \left\lfloor \frac{s-1}{6} \right\rfloor^2\right) - 5\left\lceil \frac{s-1}{6} \right\rceil +$ $+11\left\lfloor \frac{s-1}{6} \right\rfloor + r + s - \left\lfloor \frac{s}{k} \right\rfloor + 3$	$s = 0, 1, \ldots, k,\ s \equiv 1 \pmod 3$ $r = -s + \left\lfloor \frac{s}{k} \right\rfloor - 1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor$
	$12\left(\left\lceil \frac{s-2}{6} \right\rceil^2 + \left\lfloor \frac{s-2}{6} \right\rfloor^2\right) - \left\lceil \frac{s-2}{6} \right\rceil +$ $+15\left\lfloor \frac{s-2}{6} \right\rfloor + r + s - \left\lfloor \frac{s}{k} \right\rfloor + 4$	$s = 0, 1, \ldots, k,\ s \equiv 2 \pmod 3$ $r = -s + \left\lfloor \frac{s}{k} \right\rfloor - 1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor$
$a_s^r a_{s+1}^r$	$12\left(\left\lceil \frac{s}{6} \right\rceil^2 + \left\lfloor \frac{s}{6} \right\rfloor^2\right) - 7\left\lceil \frac{s}{6} \right\rceil +$ $+9\left\lfloor \frac{s}{6} \right\rfloor + r + s + 2$	$s = 0, 1, \ldots, k-1,\ s \equiv 0 \pmod 3$ $r = -s - 1, \ldots, s + 1$
	$12\left(\left\lceil \frac{s-1}{6} \right\rceil^2 + \left\lfloor \frac{s-1}{6} \right\rfloor^2\right) - 3\left\lceil \frac{s-1}{6} \right\rceil +$ $+13\left\lfloor \frac{s-1}{6} \right\rfloor + r + s + 3$	$s = 0, 1, \ldots, k-1,\ s \equiv 1 \pmod 3$ $r = -s - 1, \ldots, s + 1$
	$12\left(\left\lceil \frac{s-2}{6} \right\rceil^2 + \left\lfloor \frac{s-2}{6} \right\rfloor^2\right) + 3\left\lceil \frac{s-2}{6} \right\rceil +$ $+15\left\lfloor \frac{s-2}{6} \right\rfloor + r + s + 4$	$s = 0, 1, \ldots, k-1,\ s \equiv 2 \pmod 3$ $r = -s - 1, \ldots, s + 1$

According to the aforementioned labeling, it is evident that the labels allocated each of the edges and vertices do not exceed m. Now, we can indicate the edge weights associated with the given labeling ϖ as follows.

x	$wt_{\varpi}(x)$	r and s
$a_s^r a_s^{r+1}$	$36\left(\left\lceil \frac{s}{6} \right\rceil^2 + \left\lfloor \frac{s}{6} \right\rfloor^2\right) - 9\left\lceil \frac{s}{6} \right\rceil +$ $+27\left\lfloor \frac{s}{6} \right\rfloor + r + s - \left\lfloor \frac{s}{k} \right\rfloor + 2$	$s = 0, 1, \ldots, k,\ s \equiv 0 \pmod 3$ $r = -s + \left\lfloor \frac{s}{k} \right\rfloor - 1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor$
	$36\left(\left\lceil \frac{s-1}{6} \right\rceil^2 + \left\lfloor \frac{s-1}{6} \right\rfloor^2\right) + 3\left\lceil \frac{s-1}{6} \right\rceil +$ $+39\left\lfloor \frac{s-1}{6} \right\rfloor + r + s - \left\lfloor \frac{s}{k} \right\rfloor + 7$	$s = 0, 1, \ldots, k,\ s \equiv 1 \pmod 3$ $r = -s + \left\lfloor \frac{s}{k} \right\rfloor - 1, \ldots, s - \left\lfloor \frac{s}{k} \right\rfloor$

	$36\left(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2\right)+15\left\lceil\frac{s-2}{6}\right\rceil+$ $+51\left\lfloor\frac{s-2}{6}\right\rfloor+r+s-\left\lfloor\frac{s}{k}\right\rfloor+16$	$s=0,1,\ldots,k,\ s\equiv 2 \pmod 3$ $r=-s+\left\lfloor\frac{s}{k}\right\rfloor-1,\ldots,s-\left\lfloor\frac{s}{k}\right\rfloor$
$a_s^r a_{s+1}^r$	$36\left(\left\lceil\frac{s}{6}\right\rceil^2+\left\lfloor\frac{s}{6}\right\rfloor^2\right)-3\left\lceil\frac{s}{6}\right\rceil+$ $+33\left\lfloor\frac{s}{6}\right\rfloor+r+s+4$	$s=0,1,\ldots,k-1,\ s\equiv 0 \pmod 3$ $r=-s-1,\ldots,s+1$
	$36\left(\left\lceil\frac{s-1}{6}\right\rceil^2+\left\lfloor\frac{s-1}{6}\right\rfloor^2\right)+9\left\lceil\frac{s-1}{6}\right\rceil+$ $+45\left\lfloor\frac{s-1}{6}\right\rfloor+r+s+11$	$s=0,1,\ldots,k-1,\ s\equiv 1 \pmod 3$ $r=-s-1,\ldots,s+1$
	$36\left(\left\lceil\frac{s-2}{6}\right\rceil^2+\left\lfloor\frac{s-2}{6}\right\rfloor^2\right)+21\left\lceil\frac{s}{6}\right\rceil+$ $+57\left\lfloor\frac{s-2}{6}\right\rfloor+r+s+22$	$s=0,1,\ldots,k-1,\ s\equiv 2 \pmod 3$ $r=-s-1,\ldots,s+1$

From the list above, we can ascertain that the edge weights form a sequence of successive integers from 1 to $2k^2+5k$. This indicates that the total m-labeling ϖ is a REIL of MSC_k, $k \geq 1$.

3. Conclusions

In the preceding section, we have showed that the res of the staircase graph SC_k, where $k \geq 1$, is given by $\mathrm{res}(SC_k)=\left\lceil\frac{k(k+3)}{3}\right\rceil$. Furthermore, we ascertained the exact quantity of res for two graphs: the double staircase graph DSC_k and the mirror staircase graph MSC_k, where $k \geq 1$, both derived from the staircase graph. For DSC_k graph we get $\mathrm{res}(DSC_k)=\left\lceil\frac{2k^2+3k-1}{3}\right\rceil+1$ for $k \equiv 3 \pmod 6$ and $\mathrm{res}(DSC_k)=\left\lceil\frac{2k^2+3k-1}{3}\right\rceil$, otherwise. For MSC_k graph we get $\mathrm{res}(MSC_k)=\left\lceil\frac{2k^2+5k}{3}\right\rceil+1$ for $k \equiv 3,5 \pmod 6$ and $\mathrm{res}(MSC_k)=\left\lceil\frac{2k^2+5k}{3}\right\rceil$, otherwise.

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