

sup-Hesitant fuzzy ideals and bi-ideals of semigroups

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Abstract

As a generalization of the concept of an interval-valued fuzzy bi-ideal (respectively, subsemigroup, generalized bi-ideal, left ideal, right ideal) of a semigroup, the concept of a sup-hesitant fuzzy bi-ideal (respectively, subsemigroup, generalized bi-ideal, left ideal, right ideal) is introduced, investigated some properties and discussed the generalizations of these hesitant fuzzy sets. We characterize sup-hesitant fuzzy subsemigroups, sup-hesitant fuzzy left (right) ideals, sup-hesitant fuzzy (generalized) bi-ideals and sup-hesitant fuzzy (interior) ideals in terms of level sets, fuzzy sets, Pythagorean fuzzy sets, interval-valued fuzzy sets, cubic sets, hesitant fuzzy sets and hybrid sets. Moreover, subsemigroups, (generalized) bi-ideals, and left (right) ideals of semigroups are characterized by sup-types of hesitant fuzzy sets.

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1. Introduction

Uncertainty can be considered of different types such as incompleteness [19], indistinguishability [25], randomness [24] and fuzziness [38]. Applications of fuzzy set theory are numerous and well-established, including computer science, automata theory, artificial intelligence, graph theory, logic theory, finite state machine, control engineering, expert, robotics, group theory, semigroup theory, and many branches of pure and applied mathematics. However, when many sources of imprecise and vague information appear simultaneously, the concept of fuzzy sets presents limitations. To get beyond these limitations, numerous expansions and generalizations of the fuzzy set notion have been proposed or examined in the literature, including interval-valued fuzzy sets [39], intuitionistic fuzzy sets [2], fuzzy bags [35], Pythagorean fuzzy sets [36, 37], hybrid sets [1, 14], hesitant fuzzy sets [33, 32], cubic sets [13] and so forth.

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Rosenfeld's seminal paper [27] introduced the notions of fuzzy groups and fuzzy ideals (subgroupoids, left ideals, right ideals), which marked the beginning of the study of fuzzy algebraic structures. Kuroki [17, 18] applied fuzzy sets to semigroups. Later, several authors applied fuzzy sets and extensions and generalizations of fuzzy sets to semigroups. Mordeson, Malik, and Kuroki [20] provided a methodical explanation of semigroup theory to fuzzy semigroup theory, along with examples of how they were applied in fuzzy languages, coding theory, and fuzzy finite state machines. In their studies on anti-types of fuzzy sets in semigroups, Shabir and Nawaz [29] and Khan and Asif [16] characterized several classes of semigroups according to the properties of their anti-types of fuzzy sets. Julatha and Siripitukdet [11] studied characterizations of anti-fuzzy subsemigroups and anti-fuzzy (generalized) bi-ideals of a semigroup A via the certain subsets of $\mathcal{A}$, $[0, 1]$ and $\mathcal{A} \times [0, 1]$. On semigroups, Chinnadurai and Arulselvam [3] introduced Pythagorean fuzzy ideals and investigated their properties. Thillaigovindan and Chinnadurai investigated the theory of semigroups via interval-valued fuzzy sets [31] and Narayanan and Manikantan [23]. Talee, Abassi, and Khan [30] and Jun et al. [15] used hesitant fuzzy sets to develop semigroup theory. Umar, Hadi, and Kham [34], Jun and Khan [12], Muhiuddin [22], Hadi and Khan [6] studied the theory of semigroups via cubic sets. Anis et al. [1], Elavarasan, Porselvi, and Jun [5], and Elavarasan and Jun [4] studied the theory of semigroups via hybrid sets.

Investigating hesitant fuzzy sets in the context of the supremum of its images, Jittburus and Julatha [7] presented the concept of sup-hesitant fuzzy ideals of semigroups and examined its characteristics through interval-valued fuzzy sets, sets, fuzzy sets and hesitant fuzzy sets. A sup-hesitant fuzzy ideal is a generalization of the ideas of an interval-valued fuzzy ideal and a hesitant fuzzy ideal. A sup-hesitant fuzzy interior ideal, which is a generalization of the ideas of a sup-hesitant fuzzy ideal, a hesitant fuzzy interior ideal, and an interval-valued fuzzy interior ideal, was introduced by Phummee et al. [26]. Its properties were examined using sets, fuzzy sets, hesitant fuzzy sets, and interval-valued fuzzy sets. Jittburus, Julatha, and Iampan [8] introduced the idea of INF-hesitant fuzzy ideals in semigroups and its INF-hesitant fuzzy translations. Sahu and Gupta developed an incomplete hesitant fuzzy preference relation method [28].

As said earlier, it inspired us to investigate hesitant fuzzy sets in the meaning of the supremum of their images. On semigroups, we introduce concepts of sup-hesitant fuzzy (generalized) bi-ideals and sup-hesitant fuzzy subsemigroups and study also sup-hesitant fuzzy left (right) ideals. We also explore some of their intriguing characteristics and prove their generalizations. It is demonstrated that

the notion of sup-hesitant fuzzy left (right) ideals is a generalization of the notions of interval-valued fuzzy left (right) ideals and hesitant fuzzy left (right) ideals. The notion of sup-hesitant fuzzy subsemigroups (generalized bi-ideals, bi-ideals) is a generalization of interval-valued fuzzy subsemigroups (generalized bi-ideals, bi-ideals). We characterize sup-hesitant fuzzy subsemigroups, sup-hesitant fuzzy left (right) ideals, sup-hesitant fuzzy (generalized) bi-ideals, and sup-hesitant fuzzy (interior) ideals via fuzzy sets, level sets, Pythagorean fuzzy sets, cubic sets, interval-valued fuzzy sets, hybrid sets and hesitant fuzzy sets. Furthermore, in terms of sup-types of hesitant fuzzy sets, characterizations of subsemigroups, left ideals, right ideals, and generalized bi-ideals are given.

2. Preliminaries

First, we present some basic concepts and results used throughout this article in this section.

Unless otherwise specified, the following will refer to $\mathcal{A}$ as a semigroup, $\mathcal{B}$ as an initial universal set, and $\mathcal{Z}$ as a nonempty set. By a *right ideal* (RId) (respectively, a *left ideal* (LId), an *interior ideal* (IId)) of $\mathcal{A}$ we mean a nonempty subset $\mathcal{Z}$ of $\mathcal{A}$ such that $\mathcal{Z}\mathcal{A} \subseteq \mathcal{Z}$ (respectively, $\mathcal{A}\mathcal{Z} \subseteq \mathcal{Z}$, $\mathcal{A}\mathcal{Z}\mathcal{A} \subseteq \mathcal{Z}$). When we refer to a nonempty subset of $\mathcal{A}$ as a *ideal* (Id) of $\mathcal{A}$, we mean that it is both an LId and an RId. By a *subsemigroup* (SS) of $\mathcal{A}$ we mean a nonempty subset $\mathcal{Z}$ of $\mathcal{A}$ such that $\mathcal{Z}^2 \subseteq \mathcal{Z}$, by a *generalized bi-ideal* (GBId) of $\mathcal{A}$ we mean a nonempty subset $\mathcal{Z}$ of $\mathcal{A}$ such that $\mathcal{Z}\mathcal{A}\mathcal{Z} \subseteq \mathcal{Z}$, and by a *bi-ideal* (BId) of $\mathcal{A}$ we mean a nonempty subset of $\mathcal{A}$ is both an SS and a GBId of $\mathcal{A}$. SSs and GBIds of $\mathcal{A}$ are then generalizations of a BId of $\mathcal{A}$, a generalization of the notions of an LId and an RId of $\mathcal{A}$. Accordingly, the concepts of an RId, an LId and an IId of $\mathcal{A}$ generalise the concept of an Id of $\mathcal{A}$.

Let $[0, 1]$ be the unit interval, the definition of a *fuzzy subset* (FS) [38] of $\mathcal{Z}$ is a function $\varphi : \mathcal{Z} \rightarrow [0, 1]$. For FSs φ and σ of $\mathcal{Z}$, define $\varphi \leq \sigma$ if $\varphi(z) \leq \sigma(z)$ for all $z \in \mathcal{Z}$. An FS φ of $\mathcal{A}$ is said to be a *fuzzy ideal* (FI) [20] of $\mathcal{A}$ if $\max\{\varphi(p), \varphi(w)\} \leq \varphi(pw)$ for all $p, w \in \mathcal{A}$. An FS φ of $\mathcal{A}$ is said to be a *fuzzy interior ideal* (FII) [20] of $\mathcal{A}$ if $\varphi(y) \leq \varphi(pyw)$ for all $p, w, y \in \mathcal{A}$.

We refer to an interval $[a^-, a^+]$, where $0 \leq a^- \leq a^+ \leq 1$, by an interval number $\check{a}$. Denoted by $\wp(\mathcal{B})$, the set of all subsets of $\mathcal{B}$ and $\mathcal{D}([0, 1])$, the set of all interval numbers. Obviously, we have $\mathcal{D}([0, 1]) \subseteq \wp([0, 1])$. For two elements $\check{a} = [a^-, a^+]$ and $\check{b} = [b^-, b^+]$ in $\mathcal{D}([0, 1])$, we define the following operations for two elements in $\mathcal{D}([0, 1])$: $\check{a} \preceq \check{b} \Leftrightarrow a^- \leq b^-$ and $a^+ \leq b^+$, $\check{a} = \check{b} \Leftrightarrow a^- = b^-$ and $a^+ = b^+$, $\check{a} \prec \check{b} \Leftrightarrow \check{a} \preceq \check{b}$ and $\check{a} \neq \check{b}$, $\text{rmax}(\text{rmax}\{\check{a}, \check{b}\} = [\max\{a^-, b^-\}, \max\{a^+, b^+\}])$ and $\text{rmin}(\text{rmin}\{\check{a}, \check{b}\} = [\min\{a^-, b^-\}, \min\{a^+, b^+\}])$.

A function $\check{v} : \mathcal{Z} \rightarrow \mathcal{D}([0, 1])$, where $\check{v}(w) = [\check{v}^-(w), \check{v}^+(w)]$ for all $w \in \mathcal{Z}$, $\check{v}^-$ and $\check{v}^+$ are FSs of $\mathcal{Z}$ such that $\check{v}^- \leq \check{v}^+$, is said to be an

interval-valued fuzzy set (IvFS) [39] on $\mathcal{Z}$. Let $\check{v}$ be an IvFS on $\mathcal{A}$. If $\text{rmax}\{\check{v}(p), \check{v}(w)\} \preceq \check{v}(pw)$ for all $p, w \in \mathcal{A}$, then $\check{v}$ is called an *interval-valued fuzzy ideal* (IvFI) [23, 31] of $\mathcal{A}$. If $\check{v}(y) \preceq \check{v}(pyw)$ for all $p, w, y \in \mathcal{A}$, then $\check{v}$ is said to be an *interval-valued fuzzy interior ideal* (IvFII) [31] of $\mathcal{A}$. An IvFII of $\mathcal{A}$ is a generalization of an IvFI.

A function $\widehat{\delta} : \mathcal{Z} \rightarrow \wp([0, 1])$ is called a *hesitant fuzzy set* (HFS) [33, 32] on $\mathcal{Z}$. All IvFSs on $\mathcal{Z}$ are also HFSs on $\mathcal{Z}$. An HFS $\widehat{\delta}$ on $\mathcal{A}$ is said to be a *hesitant fuzzy ideal* (HFI) [15, 7] of $\mathcal{A}$ if $\widehat{\delta}(p) \cup \widehat{\delta}(w) \subseteq \widehat{\delta}(pw)$ for all $p, w \in \mathcal{A}$. An HFS $\widehat{\delta}$ on $\mathcal{A}$ is called a *hesitant fuzzy interior ideal* (HFII) [30, 26] of $\mathcal{A}$ if $\widehat{\delta}(y) \subseteq \widehat{\delta}(pyw)$ for all $p, w, y \in \mathcal{A}$.

Let $\widehat{\delta}$ be a HFS on $\mathcal{Z}$, n an element of $[0, 1]$ and $\blacktriangledown$ an element of $\wp([0, 1])$. Define $\text{SUP}\blacktriangledown$ and $\mathcal{S}[\widehat{\delta}; \blacktriangledown]$ [7, 26] as follows: $\text{SUP}\blacktriangledown = \begin{cases} \sup \blacktriangledown & \text{if } \blacktriangledown \neq \emptyset, \\ 0 & \text{otherwise,} \end{cases}$ and $\mathcal{S}[\widehat{\delta}; \blacktriangledown] = \{z \in \mathcal{Z} \mid \text{SUP}\widehat{\delta}(z) \geq \text{SUP}\blacktriangledown\}$. Then $\text{SUP}\check{v}(z) = \sup \check{v}(z) = \check{v}^+(z)$ for every IvFS $\check{v}$ on $\mathcal{Z}$ and for all $z \in \mathcal{Z}$. The HFS $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}$ [9, 10] on $\mathcal{Z}$ is defined by $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)} : \mathcal{Z} \rightarrow \wp([0, 1])$, $z \mapsto \{n \in \blacktriangledown \mid \text{SUP}\widehat{\delta}(z) \geq n\}$. We denote $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, [0, 1])}$ by $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$, and then both an HFS and an IvFS on $\mathcal{Z}$. The subsets $L_{\text{SUP}}^{(\widehat{\delta}; n)}$ and $U_{\text{SUP}}^{(\widehat{\delta}; n)}$ [10, 21] of $\mathcal{Z}$ are defined by $L_{\text{SUP}}^{(\widehat{\delta}; n)} = \{z \in \mathcal{Z} \mid \text{SUP}\widehat{\delta}(z) \leq n\}$ and $U_{\text{SUP}}^{(\widehat{\delta}; n)} = \{z \in \mathcal{Z} \mid \text{SUP}\widehat{\delta}(z) \geq n\}$ are said to be a *sup-lower n -level subset* and a *sup-upper n -level subset* of $\widehat{\delta}$, respectively. The FS $\text{f}^{\widehat{\delta}}$ of $\mathcal{Z}$ is defined by $\text{f}^{\widehat{\delta}}(z) = \text{SUP}\widehat{\delta}(z)$ for all $z \in \mathcal{Z}$. An HFS $\widehat{\tau}$ on $\mathcal{Z}$ is called a *supremum complement* of $\widehat{\delta}$ on $\mathcal{Z}$ if $\text{SUP}\widehat{\tau}(z) = 1 - \text{SUP}\widehat{\delta}(z)$ for all $z \in \mathcal{Z}$. Let $\mathcal{SC}(\widehat{\delta})$ be the set of all supremum complements of $\widehat{\delta}$. Define the HFS $\widehat{\delta}^*$ by $\widehat{\delta}^*(z) = \{1 - \text{SUP}\widehat{\delta}(z)\}$ for all $z \in \mathcal{Z}$. Hence, we get the following results:

- (1) $\widehat{\delta}^* \in \mathcal{SC}(\widehat{\delta})$.
- (2) $\text{f}^{\widehat{\tau}}(z) = 1 - \text{SUP}\widehat{\delta}(z)$ for all $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$ and $z \in \mathcal{Z}$.
- (3) $\text{SUP}(\widehat{\delta}^*)(z) = \text{SUP}\widehat{\delta}(z) = \text{f}^{\widehat{\delta}}(z)$ for all $z \in \mathcal{Z}$.
- (4) $\text{SUP}\widehat{\delta}(z) = 1 - (1 - \text{SUP}\widehat{\delta}(z)) = 1 - \text{SUP}\widehat{\tau}(z)$ for all $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$ and $z \in \mathcal{Z}$.

For any subset $\mathcal{Z}$ of B , the definitions of the characteristic hesitant fuzzy set (CHFS) $\widehat{\Pi}_{\mathcal{Z}}$ and the characteristic interval-valued fuzzy set (CIvFS) $\check{O}_{\mathcal{Z}}$ of $\mathcal{Z}$ on B are as follows:

$$\check{O}_{\mathcal{Z}} : B \rightarrow \mathcal{D}([0, 1]), b \mapsto \begin{cases} \check{1} & \text{if } b \in \mathcal{Z}, \\ \check{0} & \text{otherwise,} \end{cases}$$

and

$$\widehat{\Pi}_{\mathcal{Z}}: B \rightarrow \wp([0, 1]), b \mapsto \begin{cases} [0, 1] & \text{if } b \in \mathcal{Z}, \\ \emptyset & \text{otherwise,} \end{cases}$$

where $\check{1} = [1, 1]$ and $\check{0} = [0, 0]$.

For any subset $\mathcal{Z}$ of B and $\blacktriangledown, \blacktriangle \in \wp([0, 1])$ with $\text{SUP}\blacktriangle > \text{SUP}\blacktriangledown$, define a mapping $\widehat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}$ [9, 10] as follows:

$$\widehat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}: B \rightarrow \wp([0, 1]), b \mapsto \begin{cases} \blacktriangle & \text{if } b \in \mathcal{Z}, \\ \blacktriangledown & \text{otherwise.} \end{cases}$$

Then the HFS $\widehat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}$ on B is said to be the $\text{sup}(\blacktriangledown, \blacktriangle)$ -characteristic hesitant fuzzy set ($\text{sup}(\blacktriangledown, \blacktriangle)$ -CHFS) of $\mathcal{Z}$ on B . Observe that $\widehat{\omega}_{\mathcal{Z}}^{(\emptyset, [0, 1])} = \widehat{\Pi}_{\mathcal{Z}}$ and $\widehat{\omega}_{\mathcal{Z}}^{(\check{0}, \check{1})} = \check{O}_{\mathcal{Z}}$.

On semigroups, a sup-hesitant fuzzy ideal (sup-HFI) has been established by Jittburus and Julatha [7]. It is a generalization of the ideas of an HFI and an IvFI, and its properties were investigated in the following ways using HFSs, IvFSSs, FSSs and sets.

Definition 2.1 (7). Let $\widehat{\delta}$ be an HFS on $\mathcal{A}$. If $\mathcal{S}[\widehat{\delta}; \blacktriangle]$ is an Id of $\mathcal{A}$, then $\widehat{\delta}$ is said to be a *sup-hesitant fuzzy ideal of $\mathcal{A}$ related to $\blacktriangle$* ($\blacktriangle$ -sup-hesitant fuzzy ideal) of $\mathcal{A}$. If $\widehat{\delta}$ is a $\blacktriangle$ -sup-hesitant fuzzy ideal of $\mathcal{A}$ for each $\blacktriangle \in \wp([0, 1])$ with $\mathcal{S}[\widehat{\delta}; \blacktriangle] \neq \emptyset$, then $\widehat{\delta}$ is called a *sup-HFI of $\mathcal{A}$* .

Theorem 2.2 (7). We have that $\widehat{\delta}$ is a sup-HFI of $\mathcal{A}$ for each HFI $\widehat{\delta}$ of $\mathcal{A}$.

Theorem 2.3. [7] We have that $\check{\nu}$ is a sup-HFI of $\mathcal{A}$ for each IvFI $\check{\nu}$ of $\mathcal{A}$.

Theorem 2.4. [7] The following are equivalent for every HFS $\widehat{\delta}$ on $\mathcal{A}$.

- (1) $\widehat{\delta}$ is a sup-HFI of $\mathcal{A}$.
- (2) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is an IvFI of $\mathcal{A}$.
- (3) $\text{f}^{\widehat{\delta}}$ is an FI of $\mathcal{A}$.
- (4) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is an HFI of $\mathcal{A}$.
- (5) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is a sup-HFI of $\mathcal{A}$.
- (6) $\text{SUP}\widehat{\delta}(pw) \geq \max\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(w)\}$ for all $p, w \in \mathcal{A}$.
- (7) For all $n \in [0, 1]$, if $U_{\text{SUP}}^{(\widehat{\delta}; n)} \neq \emptyset$, then $U_{\text{SUP}}^{(\widehat{\delta}; n)}$ is an Id of $\mathcal{A}$.
- (8) For all $\blacktriangledown \in \wp([0, 1])$, we have that $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}$ is an HFI of $\mathcal{A}$.

Theorem 2.5. [7] *The following assertions are equivalent for any nonempty subset $\mathcal{Z}$ of $\mathcal{A}$.*

- (1) $\mathcal{Z}$ is an Id of $\mathcal{A}$.
- (2) $\widehat{\Pi}_{\mathcal{Z}}$ is a sup-HFI of $\mathcal{A}$.
- (3) $\check{\mathcal{O}}_{\mathcal{Z}}$ is a sup-HFI of $\mathcal{A}$.
- (4) If $\blacktriangledown, \blacktriangle \in \wp([0, 1])$ and $\text{SUP}\blacktriangledown < \text{SUP}\blacktriangle$, then the $\text{sup}(\blacktriangledown, \blacktriangle)$ -CHFS $\widehat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}$ is a sup-HFI of $\mathcal{A}$.

A sup-hesitant fuzzy interior ideal (sup-HFII) of a semigroup, presented by Phummee et al. [26], is a generalization of a sup-HFI, an IvFII, and an HFII, and its properties are examined using FSSs, IvFSSs, sets, and HFSs.

Definition 2.6. [26] Let $\widehat{\delta}$ be an HFS on $\mathcal{A}$. If $\mathcal{S}[\widehat{\delta}; \blacktriangle]$ is an IId of $\mathcal{A}$, then $\widehat{\delta}$ is said to be a *sup-hesitant fuzzy interior ideal of $\mathcal{A}$ related to $\blacktriangle$* ($\blacktriangle$ -sup-hesitant fuzzy interior ideal) of $\mathcal{A}$. If $\widehat{\delta}$ is a $\blacktriangle$ -sup-hesitant fuzzy interior ideal of $\mathcal{A}$ for each $\blacktriangle \in \wp([0, 1])$ with $\mathcal{S}[\widehat{\delta}; \blacktriangle] \neq \emptyset$, then $\widehat{\delta}$ is called a sup-HFII of $\mathcal{A}$.

Theorem 2.7. [26] *We have $\widehat{\delta}$ is a sup-HFII of $\mathcal{A}$ for every sup-HFI $\widehat{\delta}$ of $\mathcal{A}$.*

Theorem 2.8. [26] *We have $\widehat{\delta}$ is a sup-HFII of $\mathcal{A}$ for every HFII $\widehat{\delta}$ of $\mathcal{A}$.*

Theorem 2.9. [26] *We have $\check{\nu}$ is a sup-HFII of $\mathcal{A}$ for every IvFII $\check{\nu}$ of $\mathcal{A}$.*

Theorem 2.10. [26] *The following are equivalent for each HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFII of $\mathcal{A}$.
- (2) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is a sup-HFII of $\mathcal{A}$.
- (3) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is an IvFII of $\mathcal{A}$.
- (4) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is an HFII of $\mathcal{A}$.
- (5) $\mathbf{f}^{\widehat{\delta}}$ is an FII of $\mathcal{A}$.
- (6) $\text{SUP}\widehat{\delta}(pyw) \geq \text{SUP}\widehat{\delta}(y)$ for all $p, w, y \in \mathcal{A}$.
- (7) For all $n \in [0, 1]$, if $U_{\text{SUP}}^{(\widehat{\delta}; n)} \neq \emptyset$, then $U_{\text{SUP}}^{(\widehat{\delta}; n)}$ is an IId of $\mathcal{A}$.
- (8) $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}$ is an HFII of $\mathcal{A}$ for all $\blacktriangledown \in \wp([0, 1])$.

Theorem 2.11. [26] *Let $\widehat{\delta}$ be an HFS on $\mathcal{A}$. Then $\widehat{\delta}^*$ is a sup-HFII of $\mathcal{A}$ if and only if, for each $n \in [0, 1]$, if $L_{\text{SUP}}^{(\widehat{\delta}; n)} \neq \emptyset$, then $L_{\text{SUP}}^{(\widehat{\delta}; n)}$ is an IId of $\mathcal{A}$.*

Theorem 2.12. [26] *The following assertions are equivalent for any nonempty subset $\mathcal{Z}$ of $\mathcal{A}$.*

- (1) $\mathcal{Z}$ is an *IId* of $\mathcal{A}$.
- (2) $\check{O}_{\mathcal{Z}}$ is a sup-HFII of $\mathcal{A}$.
- (3) $\hat{\Pi}_{\mathcal{Z}}$ is a sup-HFII of $\mathcal{A}$.
- (4) If $\blacktriangledown, \blacktriangle \in \wp([0, 1])$ and $\text{SUP}\blacktriangledown < \text{SUP}\blacktriangle$, then the $\text{sup}(\blacktriangledown, \blacktriangle)$ -CHFS $\hat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}$ is a sup-HFII of $\mathcal{A}$.

3. sup-hesitant 1 fuzzy substructures

The notions of a sup-hesitant fuzzy subsemigroup (sup-HFSS), a sup-hesitant fuzzy left ideal (sup-HFLI), a sup-hesitant fuzzy right ideal (sup-HFRI), a sup-hesitant fuzzy generalized bi-ideal (sup-HFGBI), and a sup-hesitant fuzzy bi-ideal (sup-HFBI) of arbitrary semigroup will be presented. Some of their properties will be examined in this section.

Definition 3.1. An HFS $\hat{\delta}$ on $\mathcal{A}$ is called

- (1) a sup-HFRI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\text{SUP}\hat{\delta}(p) \leq \text{SUP}\hat{\delta}(pw))$, and
- (2) a sup-HFLI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\text{SUP}\hat{\delta}(w) \leq \text{SUP}\hat{\delta}(pw))$.

From Definition 3.1 and Theorem 2.4, we get that an HFS $\hat{\delta}$ of $\mathcal{A}$ is a sup-HFI of $\mathcal{A}$ if and only if $\hat{\delta}$ is both a sup-HFLI and a sup-HFRI of $\mathcal{A}$.

Example 3.2. Consider the semigroup $\mathcal{A} = \{(1, 1), (0, 1), (0, 0), (1, 0)\}$ with respect to multiplication defined by: $(a, b)(y, z) = (a, z)$ for all $a, b, y, z \in \{0, 1\}$.

- (1) An HFS $\hat{\delta}_1$ of $\mathcal{A}$ is defined by $\hat{\delta}_1((0, 0)) = \{0.2, 0.4, 0.6\}$, $\hat{\delta}_1((0, 1)) = \{0.3, 0.6\}$, $\hat{\delta}_1((1, 0)) = \{0.2, 0.4\}$ and $\hat{\delta}_1((1, 1)) = \{0.4\}$. Therefore, $\hat{\delta}_1$ is a sup-HFRI of $\mathcal{A}$. However, we see that $\hat{\delta}_1$ is not a sup-HFLI of $\mathcal{A}$ since

$$\text{SUP}\hat{\delta}_1((1, 0)(0, 0)) = 0.4 < 0.6 = \text{SUP}\hat{\delta}_1((0, 0)).$$

- (2) We define an HFS $\hat{\delta}_2$ of $\mathcal{A}$ by $\hat{\delta}_2((0, 0)) = \{0.3\}$, $\hat{\delta}_2((0, 1)) = \{0.9\}$, $\hat{\delta}_2((1, 0)) = \{0.2, 0.3\}$ and $\hat{\delta}_2((1, 1)) = \{0.1, 0.9\}$. Then $\hat{\delta}_2$ is a sup-HFLI but not a sup-HFRI of $\mathcal{A}$ since

$$\text{SUP}\hat{\delta}_2((1, 1)(1, 0)) = 0.3 < 0.9 = \text{SUP}\hat{\delta}_2((1, 1)).$$

Definition 3.3. An HFS $\widehat{\delta}$ on $\mathcal{A}$ is called

(1) a sup-HFSS of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\min\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(w)\} \leq \text{SUP}\widehat{\delta}(pw))$,

(2) a sup-HFGBI of $\mathcal{A}$ if

$$(\forall p, w, y \in \mathcal{A})(\min\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(y)\} \leq \text{SUP}\widehat{\delta}(pwy)),$$

(3) a sup-HFBI of $\mathcal{A}$ if it is both a sup-HFGBI and a sup-HFSS of $\mathcal{A}$.

Example 3.4. Let $\mathcal{A} = \{o, w, x\}$ be a semigroup with the following multiplication:

$\cdot$	o	w	x
o	o	o	o
w	o	w	w
x	o	w	x

Define an HFS $\widehat{\delta}$ of $\mathcal{A}$ by $\widehat{\delta}(o) = \emptyset, \widehat{\delta}(w) = [0.4, 0.6] \cup \{0.5, 0.7, 0.8\}, \widehat{\delta}(x) = (0.3, 0.9)$. Then $\widehat{\delta}$ is a sup-HFSS of $\mathcal{A}$ but not a sup-HFGBI of $\mathcal{A}$ because $\text{SUP}\widehat{\delta}(wox) = 0 < 0.8 = \min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\widehat{\delta}(x)\}$.

Example 3.5. Let $\mathcal{A} = \{\theta, w, x, y\}$ be a semigroup with the following multiplication:

$\cdot$	θ	w	x	y
θ	θ	θ	θ	θ
w	θ	θ	θ	θ
x	θ	θ	w	θ
y	θ	θ	w	w

Define an HFS $\widehat{\delta}$ of $\mathcal{A}$ by $\widehat{\delta}(\theta) = \check{1}, \widehat{\delta}(w) = \emptyset, \widehat{\delta}(x) = \{0.5, 0.6, 0.7\}, \widehat{\delta}(y) = \check{0}$. Then $\widehat{\delta}$ is a sup-HFGBI of $\mathcal{A}$ but is not a sup-HFSS of $\mathcal{A}$ because $\text{SUP}\widehat{\delta}(xx) = \text{SUP}\widehat{\delta}(w) = 0 < 0.7 = \min\{\text{SUP}\widehat{\delta}(x), \text{SUP}\widehat{\delta}(x)\}$.

Theorem 3.6. Every sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$ is a sup-HFBI of $\mathcal{A}$.

Proof. Let $\widehat{\delta}$ be a sup-HFLI of $\mathcal{A}$. Then $\text{SUP}\widehat{\delta}(pw) \geq \text{SUP}\widehat{\delta}(w) \geq \min\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(w)\}$ for all $p, w \in \mathcal{A}$, and $\text{SUP}\widehat{\delta}(pwy) \geq \text{SUP}\widehat{\delta}(y) \geq \min\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(y)\}$ for all $p, w, y \in \mathcal{A}$. consequently, $\widehat{\delta}$ is a sup-HFBI of $\mathcal{A}$.

As shown below, Theorem 3.6's converse is generally false.

Example 3.7. Let $\mathcal{A} = \{\theta, x, y, z\}$ be a semigroup with the following multiplication:

$\cdot$	θ	x	y	z
θ	θ	θ	θ	θ
x	θ	x	y	θ
y	θ	θ	θ	θ
z	θ	z	θ	θ

Define an HFS $\widehat{\delta}$ of $\mathcal{A}$ by $\widehat{\delta}(\theta) = [0, 0.9]$, $\widehat{\delta}(x) = \{0.6, 0.7, 0.8\}$, $\widehat{\delta}(y) = (0.3, 0.6)$, $\widehat{\delta}(z) = \{0.6\}$. Hence, the following hold.

- (1) $\widehat{\delta}$ is a sup-HFBI of $\mathcal{A}$.
- (2) $\widehat{\delta}$ is not a sup-HFLI of $\mathcal{A}$ because $\text{SUP}\widehat{\delta}(zx) = 0.6 < 0.8 = \text{SUP}\widehat{\delta}(x)$.
- (3) $\widehat{\delta}$ is not a sup-HFRI of $\mathcal{A}$ because $\text{SUP}\widehat{\delta}(xy) = 0.6 < 0.8 = \text{SUP}\widehat{\delta}(x)$.

From Theorems 2.7 and 3.6, Definitions 3.1 and 3., and Examples 3.2, 3.4, 3.5 and 3.7, our conclusion is that a sup-HFSS (respectively, sup-HFGBI) is a general concept of a sup-HFBI, a sup-HFBI is a general concept of a sup-HFLI and a sup-HFRI, and a sup-HFLI (respectively, sup-HFRI, sup-HFII) is a general concept of a sup-HFI.

On semigroups, we provide characterizations of sup-HFBIs, sup-HFSSs and sup-HFGBIs in Theorem 3.8, and sup-HFLIs and sup-HFRIs in Theorem 3.9.

Theorem 3.8. *An HFS $\widehat{\delta}$ on $\mathcal{A}$ is a sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$ if and only if, for all $n \in [0, 1]$, if $U_{\text{SUP}}^{(\widehat{\delta};n)} \neq \emptyset$, then $U_{\text{SUP}}^{(\widehat{\delta};n)}$ is an SS (respectively, GBId, BId) of $\mathcal{A}$.*

Proof. Assume that $n \in [0, 1]$ and $\widehat{\delta}$ is a sup-HFSS of $\mathcal{A}$ such that $w, p \in U_{\text{SUP}}^{(\widehat{\delta};n)}$. Then $n \leq \min\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(w)\}$. By the assumption, we have $n \leq \text{SUP}\widehat{\delta}(pw)$ and so $pw \in U_{\text{SUP}}^{(\widehat{\delta};n)}$. Hence, $U_{\text{SUP}}^{(\widehat{\delta};n)}$ is an SS of $\mathcal{A}$ for all $n \in [0, 1]$ when $U_{\text{SUP}}^{(\widehat{\delta};n)} \neq \emptyset$.

Conversely, assume that for every $n \in [0, 1]$, if $U_{\text{SUP}}^{(\widehat{\delta};n)}$ is a nonempty subset of $\mathcal{A}$, then $U_{\text{SUP}}^{(\widehat{\delta};n)}$ is an SS of $\mathcal{A}$. Let $p, w \in \mathcal{A}$ and choose $n := \min\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(w)\}$. Then $\text{SUP}\widehat{\delta}(p) \geq n$ and $\text{SUP}\widehat{\delta}(w) \geq n$ which imply that $p, w \in U_{\text{SUP}}^{(\widehat{\delta};n)}$. By the assumption, we directly obtain that $U_{\text{SUP}}^{(\widehat{\delta};n)}$ is an SS of $\mathcal{A}$ and so $pw \in U_{\text{SUP}}^{(\widehat{\delta};n)}$. Thus $\min\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(w)\} = n \leq \text{SUP}\widehat{\delta}(pw)$. Hence, $\widehat{\delta}$ is a sup-HFSS of $\mathcal{A}$. $\square$

Theorem 3.9. *An HFS $\widehat{\delta}$ on $\mathcal{A}$ is a sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$ if and only if, for all $n \in [0, 1]$, if $U_{\text{SUP}}^{(\widehat{\delta};n)} \neq \emptyset$, then $U_{\text{SUP}}^{(\widehat{\delta};n)}$ is an LId (respectively, RId) of $\mathcal{A}$.*

Proof. Let $\widehat{\delta}$ be a sup-HFLI of $\mathcal{A}$, $n \in [0, 1]$, $p \in \mathcal{A}$ and $w \in U_{\text{SUP}}^{(\widehat{\delta};n)}$. Then $\text{SUP}\widehat{\delta}(pw) \geq \text{SUP}\widehat{\delta}(w) \geq n$ and so $pw \in U_{\text{SUP}}^{(\widehat{\delta};n)}$. Therefore, $U_{\text{SUP}}^{(\widehat{\delta};n)}$ is an LId of $\mathcal{A}$ for all $n \in [0, 1]$ when $U_{\text{SUP}}^{(\widehat{\delta};n)} \neq \emptyset$.

Conversely, assume that for each $n \in [0, 1]$, if $U_{\text{SUP}}^{(\widehat{\delta};n)} \neq \emptyset$, then $U_{\text{SUP}}^{(\widehat{\delta};n)}$ is an LId of $\mathcal{A}$. Let $p, w \in \mathcal{A}$. Then $w \in U_{\text{SUP}}^{(\widehat{\delta};\text{SUP}\widehat{\delta}(w))}$. By the assumption, we have $pw \in U_{\text{SUP}}^{(\widehat{\delta};\text{SUP}\widehat{\delta}(w))}$. Hence, $\text{SUP}\widehat{\delta}(w) \leq \text{SUP}\widehat{\delta}(pw)$. Consequently, $\widehat{\delta}$ is a sup-HFLI of $\mathcal{A}$. $\square$

In Theorems 3.10 and 3.11, we give characterizations of sup-HFLIs, sup-HFIs, sup-HFIIs, sup-HFSSs, sup-HFBIs, sup-HFGBIs and sup-HFRIs of $\mathcal{A}$ in terms of lower level sets.

Theorem 3.10. *The following assertions are equivalent for each HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFLI (respectively, sup-HFRI, sup-HFI, sup-HFII) of $\mathcal{A}$.
- (2) For all $n \in [0, 1]$, if $L_{\text{SUP}}^{(\widehat{\delta}^*;n)} \neq \emptyset$, then $L_{\text{SUP}}^{(\widehat{\delta}^*;n)}$ is an LId (respectively, RId, Id, IId) of $\mathcal{A}$.
- (3) For all $n \in [0, 1]$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, if $L_{\text{SUP}}^{(\widehat{\tau};n)} \neq \emptyset$, then $L_{\text{SUP}}^{(\widehat{\tau};n)}$ is an LId (respectively, RId, Id, IId) of $\mathcal{A}$.

Proof. (1) $\Rightarrow$ (3). Suppose that $\widehat{\delta}$ is a sup-HFLI of $\mathcal{A}$. Let $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, $n \in [0, 1]$, $p \in \mathcal{A}$ and $w \in L_{\text{SUP}}^{(\widehat{\tau};n)}$. Then $\text{SUP}\widehat{\tau}(w) \leq n$ and $\text{SUP}\widehat{\delta}(pw) \geq \text{SUP}\widehat{\delta}(w)$. Thus

$$\text{SUP}\widehat{\tau}(pw) = 1 - \text{SUP}\widehat{\delta}(pw) \leq 1 - \text{SUP}\widehat{\delta}(w) = \text{SUP}\widehat{\tau}(w) \leq n$$

which implies that $pw \in L_{\text{SUP}}^{(\widehat{\tau};n)}$. Hence, $L_{\text{SUP}}^{(\widehat{\tau};n)}$ is an LId of $\mathcal{A}$ for all $n \in [0, 1]$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$ when $L_{\text{SUP}}^{(\widehat{\tau};n)} \neq \emptyset$.

(3) $\Rightarrow$ (2). It is directly obtained from $\widehat{\delta}^* \in \mathcal{SC}(\widehat{\delta})$.

(2) $\Rightarrow$ (1). Assume that for each $n \in [0, 1]$, if $L_{\text{SUP}}^{(\widehat{\delta}^*;n)} \neq \emptyset$, then $L_{\text{SUP}}^{(\widehat{\delta}^*;n)}$ is an LId of $\mathcal{A}$. Let $p, w \in \mathcal{A}$. Then $w \in L_{\text{SUP}}^{(\widehat{\delta}^*; \text{SUP}\widehat{\delta}^*(w))}$. By the assumption, we obtain that $L_{\text{SUP}}^{(\widehat{\delta}^*; \text{SUP}\widehat{\delta}^*(w))}$ is an LId of $\mathcal{A}$ and so $pw \in L_{\text{SUP}}^{(\widehat{\delta}^*; \text{SUP}\widehat{\delta}^*(w))}$. Thus $\text{SUP}\widehat{\delta}^*(pw) \leq \text{SUP}\widehat{\delta}^*(w)$ and then $\text{SUP}\widehat{\delta}(w) = 1 - \text{SUP}\widehat{\delta}^*(w) \leq 1 - \text{SUP}\widehat{\delta}^*(pw) = \text{SUP}\widehat{\delta}(pw)$. Consequently, $\widehat{\delta}$ is a sup-HFLI of $\mathcal{A}$. $\square$

Theorem 3.11. *The following assertions are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$.
- (2) For every $n \in [0, 1]$, if $L_{\text{SUP}}^{(\widehat{\delta}^*; n)} \neq \emptyset$, then $L_{\text{SUP}}^{(\widehat{\delta}^*; n)}$ is an SS (respectively, GBId, BId) of $\mathcal{A}$.
- (3) For every $n \in [0, 1]$ and for every $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, if $L_{\text{SUP}}^{(\widehat{\tau}; n)} \neq \emptyset$, then $L_{\text{SUP}}^{(\widehat{\tau}; n)}$ is an SS (respectively, GBId, BId) of $\mathcal{A}$.

Proof. (1) $\Rightarrow$ (3). Suppose that $\widehat{\delta}$ is a sup-HFSS of $\mathcal{A}$. Let $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, $n \in [0, 1]$ and $w, y \in L_{\text{SUP}}^{(\widehat{\tau}; n)}$. Then $\text{SUP}\widehat{\tau}(w) \leq n$, $\text{SUP}\widehat{\tau}(y) \leq n$ and $\text{SUP}\widehat{\delta}(wy) \geq \min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\widehat{\delta}(y)\}$. Thus $n \geq \max\{\text{SUP}\widehat{\tau}(w), \text{SUP}\widehat{\tau}(y)\} = \max\{1 - \text{SUP}\widehat{\delta}(w), 1 - \text{SUP}\widehat{\delta}(y)\} = 1 - \min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\widehat{\delta}(y)\} \geq 1 - \text{SUP}\widehat{\delta}(wy) = \text{SUP}\widehat{\tau}(wy)$. Hence, $wy \in L_{\text{SUP}}^{(\widehat{\tau}; n)}$. Therefore, $L_{\text{SUP}}^{(\widehat{\tau}; n)}$ is an SS of $\mathcal{A}$ for all $n \in [0, 1]$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$ when $L_{\text{SUP}}^{(\widehat{\tau}; n)} \neq \emptyset$.

(3) $\Rightarrow$ (2). It is directly obtained from $\widehat{\delta}^* \in \mathcal{SC}(\widehat{\delta})$.

(2) $\Rightarrow$ (1). Assume that for each $n \in [0, 1]$, if $L_{\text{SUP}}^{(\widehat{\delta}^*; n)} \neq \emptyset$, then $L_{\text{SUP}}^{(\widehat{\delta}^*; n)}$ is an SS of $\mathcal{A}$. Let $w, y \in \mathcal{A}$ and choose $n := \max\{\text{SUP}\widehat{\delta}^*(w), \text{SUP}\widehat{\delta}^*(y)\}$. Then $w, y \in L_{\text{SUP}}^{(\widehat{\delta}^*; n)}$. By the assumption, $L_{\text{SUP}}^{(\widehat{\delta}^*; n)}$ is an SS of $\mathcal{A}$ which implies that $wy \in L_{\text{SUP}}^{(\widehat{\delta}^*; n)}$. Thus, $\text{SUP}\widehat{\delta}^*(wy) \leq n$ and so $\text{SUP}\widehat{\delta}(wy) = 1 - \text{SUP}\widehat{\delta}^*(wy) \geq 1 - n = 1 - \max\{\text{SUP}\widehat{\delta}^*(w), \text{SUP}\widehat{\delta}^*(y)\} = \min\{1 - \text{SUP}\widehat{\delta}^*(w), 1 - \text{SUP}\widehat{\delta}^*(y)\} = \min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\widehat{\delta}(y)\}$. Hence, $\widehat{\delta}$ is a sup-HFSS of $\mathcal{A}$. $\square$

We use the CInvFS, CHFS and $\text{sup}(\blacktriangledown, \blacktriangle)$ -CHFS to study SSs, GBIds and BIds seen in Theorems 3.12, and LIds and RIds seen in Theorems 3.13.

Theorem 3.12. *The following assertions are equivalent for any nonempty subset $\mathcal{Z}$ of $\mathcal{A}$.*

- (1) $\mathcal{Z}$ is an SS (respectively, GBId, BId) of $\mathcal{A}$.
- (2) The CInvFS $\check{\mathcal{O}}_{\mathcal{Z}}$ is a sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$.
- (3) The CHFS $\widehat{\Pi}_{\mathcal{Z}}$ is a sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$.
- (4) The $\text{sup}(\blacktriangledown, \blacktriangle)$ -CHFS $\widehat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}$ is a sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$ for all $\blacktriangledown, \blacktriangle \in \wp([0, 1])$ with $\text{SUP}\blacktriangledown < \text{SUP}\blacktriangle$.

Proof. (1) $\Rightarrow$ (4). Let $\mathcal{Z}$ be an SS of $\mathcal{A}$ and $\blacktriangledown, \blacktriangle \in \wp([0, 1])$ such that $\text{SUP}\blacktriangledown < \text{SUP}\blacktriangle$. Suppose that there exist $p, w \in \mathcal{A}$ such that $\text{SUP}\widehat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}(pw) < \min\{\text{SUP}\widehat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}(p), \text{SUP}\widehat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}(w)\}$. Then $\widehat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}(p) = \widehat{\omega}_{\mathcal{Z}}^{(\blacktriangledown, \blacktriangle)}(w) = \blacktriangle$ and

$\widehat{\omega}_{\mathcal{Z}}^{(\nabla, \blacktriangle)}(pw) = \nabla$. Thus, $p, w \in \mathcal{Z}$ and $pw \notin \mathcal{Z}$. Since $\mathcal{Z}$ is an SS of $\mathcal{A}$, we have $pw \in \mathcal{Z}$. It is a contradiction and hence $\widehat{\omega}_{\mathcal{Z}}^{(\nabla, \blacktriangle)}$ is a sup-HFSS of $\mathcal{A}$.

(4) $\Rightarrow$ (3) and (4) $\Rightarrow$ (2). They are clear.

(2) $\Rightarrow$ (1). Let $\check{O}_{\mathcal{Z}}$ be a sup-HFSS of $\mathcal{A}$ and $w, y \in \mathcal{Z}$. Then $\check{O}_{\mathcal{Z}}(w) = \check{1} = \check{O}_{\mathcal{Z}}(y)$. Thus, $\text{SUP}\check{O}_{\mathcal{Z}}(wy) \geq \min\{\text{SUP}\check{O}_{\mathcal{Z}}(w), \text{SUP}\check{O}_{\mathcal{Z}}(y)\} = \text{SUP}\check{1} = 1$. Hence, $\check{O}_{\mathcal{Z}}(wy) = \check{1}$ which implies that $wy \in \mathcal{Z}$. Therefore, $\mathcal{Z}$ is an SS of $\mathcal{A}$.

(3) $\Rightarrow$ (1). Proving that (2) implies (1) is analogous to this, we omit the proof. $\square$

Theorem 3.13. *Let $\mathcal{Z}$ be any nonempty subset of $\mathcal{A}$. Then, we have the following assertions that are equivalent.*

- (1) $\mathcal{Z}$ is an LId (respectively, RId) of $\mathcal{A}$.
- (2) The CivFS $\check{O}_{\mathcal{Z}}$ is a sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$.
- (3) The CHFS $\widehat{\Pi}_{\mathcal{Z}}$ is a sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$.
- (4) If $\nabla, \blacktriangle \in \wp([0, 1])$ and $\text{SUP}\nabla < \text{SUP}\blacktriangle$, then the $(\nabla, \blacktriangle)$ -CHFS $\widehat{\omega}_{\mathcal{Z}}^{(\nabla, \blacktriangle)}$ is a sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$.

Proof. (1) $\Rightarrow$ (4). Assume that $\mathcal{Z}$ is an LId of $\mathcal{A}$ and $\nabla, \blacktriangle \in \wp([0, 1])$ such that $\text{SUP}\nabla < \text{SUP}\blacktriangle$. To show that $\widehat{\omega}_{\mathcal{Z}}^{(\nabla, \blacktriangle)}$ is a sup-HFLI of $\mathcal{A}$, we suppose that $\widehat{\omega}_{\mathcal{Z}}^{(\nabla, \blacktriangle)}$ is not a sup-HFLI of $\mathcal{A}$. Hence $\text{SUP}\widehat{\omega}_{\mathcal{Z}}^{(\nabla, \blacktriangle)}(pw) < \text{SUP}\widehat{\omega}_{\mathcal{Z}}^{(\nabla, \blacktriangle)}(w)$ for some $p, w \in \mathcal{A}$. Then $\widehat{\omega}_{\mathcal{Z}}^{(\nabla, \blacktriangle)}(w) = \blacktriangle$ and $\widehat{\omega}_{\mathcal{Z}}^{(\nabla, \blacktriangle)}(pw) = \nabla$. Thus, $w \in \mathcal{Z}$ but $pw \notin \mathcal{Z}$. Since $\mathcal{Z}$ is an LId of $\mathcal{A}$, we get $pw \in \mathcal{Z}$ which is a contradiction. Consequently, we obtain that $\widehat{\omega}_{\mathcal{Z}}^{(\nabla, \blacktriangle)}$ is a sup-HFLI of $\mathcal{A}$.

(4) $\Rightarrow$ (3) and (4) $\Rightarrow$ (2). They are obvious.

(3) $\Rightarrow$ (1). Suppose that $\widehat{\Pi}_{\mathcal{Z}}$ is a sup-HFLI of $\mathcal{A}$. Let $z \in \mathcal{Z}$ and $p \in \mathcal{A}$. Then $\widehat{\Pi}_{\mathcal{Z}}(z) = [0, 1]$ and so $\text{SUP}\widehat{\Pi}_{\mathcal{Z}}(pz) \geq \text{SUP}\widehat{\Pi}_{\mathcal{Z}}(z) = \text{SUP}[0, 1] = 1$. Thus, $\widehat{\Pi}_{\mathcal{Z}}(pz) = [0, 1]$ which indicates that $pz \in \mathcal{Z}$. Hence, $\mathcal{Z}$ is an LId of $\mathcal{A}$.

(2) $\Rightarrow$ (1). Proving that (3) implies (1) is analogous to this, we omit the proof. $\square$

4. Fuzzy sets and Pythagorean fuzzy sets

We will study sup-HFSSs, sup-HFGBIs, sup-HFBIs, sup-HFLIs, sup-HFRIs, sup-HFIIIs, and sup-HFIs by FSs and PFSs in this section.

On an arbitrary semigroup, we recall the definitions of a fuzzy bi-ideal (FBI), a fuzzy subsemigroup (FSS) and a fuzzy generalized bi-ideal (FGBI) in the following:

Definition 4.1. [20] An FS φ of $\mathcal{A}$ is called

- (1) a FGBI of $\mathcal{A}$ if $(\forall p, w, y \in \mathcal{A})(\varphi(pyw) \geq \min\{\varphi(p), \varphi(w)\})$,
- (2) a FSS of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\varphi(pw) \geq \min\{\varphi(p), \varphi(w)\})$, and
- (3) a FBI of $\mathcal{A}$ if both an FGBI and an FSS of $\mathcal{A}$.

Definition 4.2. Let φ be an FS of $\mathcal{Z}$ and $n \in [0, 1]$. The FSs φ_n^+ and φ_n^- of $\mathcal{Z}$ defined by $\varphi_n^+ : \mathcal{Z} \rightarrow [0, 1], z \mapsto \min\{1, \varphi(z) + n\}$ and $\varphi_n^- : \mathcal{Z} \rightarrow [0, 1], z \mapsto \max\{0, \varphi(z) - n\}$.

For any FS φ of $\mathcal{Z}$ and $m, n \in [0, 1]$, we have the following results:

- (1) $\varphi_m^- \leq \varphi \leq \varphi_n^+$,
- (2) $\varphi_0^- = \varphi = \varphi_0^+$,
- (3) if $m \geq n$, then $\varphi_m^- \leq \varphi_n^-$, and
- (4) if $m \leq n$, then $\varphi_m^+ \leq \varphi_n^+$.

We describe sup-HFSSs, sup-HFBI, and sup-HFGBIs of arbitrary semi-group by FSs in the ensuing theorem.

Theorem 4.3. *The following statements are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFSS (respectively, sup-HFBI, sup-HFGBI) of $\mathcal{A}$.
- (2) $\widehat{f}$ is an FSS (respectively, FBI, FGBI) of $\mathcal{A}$.
- (3) For every $m \in [0, 1]$, we have that $(\widehat{f})_m^-$ is an FSS (respectively, FBI, FGBI) of $\mathcal{A}$.
- (4) For every $m \in [0, 1]$, $(\widehat{f})_m^+$ is an FSS (respectively, FBI, FGBI) of $\mathcal{A}$.

Proof. (1) $\Leftrightarrow$ (2) It follows from Definitions 3.3 and 4.1 and $\widehat{f}(p) = \text{SUP}\widehat{\delta}(p)$ for all $p \in \mathcal{A}$.

(3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (2) They are directly obtained by taking $n = 0$.

(1) $\Rightarrow$ (4) Assume that $\widehat{\delta}$ is a sup-HFSS of $\mathcal{A}$. Let $w, y \in \mathcal{A}$ and $n \in [0, 1]$. Then $\text{SUP}\widehat{\delta}(wy) \geq \min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\widehat{\delta}(y)\}$. Thus, $\text{SUP}\widehat{\delta}(wy) \geq \text{SUP}\widehat{\delta}(w)$ or $\text{SUP}\widehat{\delta}(wy) \geq \text{SUP}\widehat{\delta}(y)$. Suppose that $\text{SUP}\widehat{\delta}(wy) \geq \text{SUP}\widehat{\delta}(w)$. Then $\widehat{f}(wy) + n = \text{SUP}\widehat{\delta}(wy) + n \geq \text{SUP}\widehat{\delta}(w) + n = \widehat{f}(w) + n$ and so $(\widehat{f})_n^+(wy) = \min\{1, \widehat{f}(wy) + n\} \geq \min\{1, \widehat{f}(w) + n\} = (\widehat{f})_n^+(w) \geq \min\{(\widehat{f})_n^+(w), (\widehat{f})_n^+(y)\}$. Thus, $(\widehat{f})_n^+(wy) \geq \min\{(\widehat{f})_n^+(w), (\widehat{f})_n^+(y)\}$. On

the other hand, that $\text{SUP}\hat{\delta}(wy) \geq \text{SUP}\hat{\delta}(y)$, it can be proved that $(f^{\hat{\delta}})_n^+(wy) \geq \min\{(f^{\hat{\delta}})_n^+(w), (f^{\hat{\delta}})_n^+(y)\}$. Therefore, for all $n \in [0, 1]$, we get that $(f^{\hat{\delta}})_n^+$ is an FSS of $\mathcal{A}$.

(1) $\Rightarrow$ (3) Assume that $\hat{\delta}$ is a sup-HFSS of $\mathcal{A}$, $n \in [0, 1]$ and $w, y \in \mathcal{A}$. Then $\text{SUP}\hat{\delta}(wy) \geq \min\{\text{SUP}\hat{\delta}(w), \text{SUP}\hat{\delta}(y)\}$ which implies that $\text{SUP}\hat{\delta}(wy) \geq \text{SUP}\hat{\delta}(w)$ or $\text{SUP}\hat{\delta}(wy) \geq \text{SUP}\hat{\delta}(y)$. The next two cases $\text{SUP}\hat{\delta}(wy) \geq \text{SUP}\hat{\delta}(y)$ and $\text{SUP}\hat{\delta}(wy) \geq \text{SUP}\hat{\delta}(w)$ are examined.

Case 1, let $\text{SUP}\hat{\delta}(wy) \geq \text{SUP}\hat{\delta}(y)$. Then $f^{\hat{\delta}}(wy) - n = \text{SUP}\hat{\delta}(wy) - n \geq \text{SUP}\hat{\delta}(y) - n = f^{\hat{\delta}}(y) - n$. Thus, $(f^{\hat{\delta}})_n^-(wy) = \max\{0, f^{\hat{\delta}}(wy) - n\} \geq \max\{0, f^{\hat{\delta}}(y) - n\} = (f^{\hat{\delta}})_n^-(y) \geq \min\{(f^{\hat{\delta}})_n^-(w), (f^{\hat{\delta}})_n^-(y)\}$, which signifies that $(f^{\hat{\delta}})_n^-(wy) \geq \min\{(f^{\hat{\delta}})_n^-(w), (f^{\hat{\delta}})_n^-(y)\}$.

Case 2, suppose that $\text{SUP}\hat{\delta}(wy) \geq \text{SUP}\hat{\delta}(w)$. In the same way as proving Case 1, we have $(f^{\hat{\delta}})_n^-(wy) \geq \min\{(f^{\hat{\delta}})_n^-(w), (f^{\hat{\delta}})_n^-(y)\}$. Consequently, we get that $(f^{\hat{\delta}})_n^-$ is an FSS of $\mathcal{A}$ for all $n \in [0, 1]$. $\square$

On an arbitrary semigroup, we recall the definitions of a fuzzy left ideal (FLI) and a fuzzy right ideal (FRI) in the following:

Definition 4.4. [20] An FS φ of $\mathcal{A}$ is called

- (1) an FLI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\varphi(pw) \geq \varphi(w))$, and
- (2) an FRI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\varphi(pw) \geq \varphi(p))$.

The sup-HFIs, sup-HFIs, sup-HFLIs and sup-HFRIs of semigroups are characterized by FSs in the ensuing theorem.

Theorem 4.5. *The following statements are equivalent for an HFS $\hat{\delta}$ on $\mathcal{A}$.*

- (1) $\hat{\delta}$ is a sup-HFLI (respectively, sup-HFRI, sup-HFII, sup-HFI) of $\mathcal{A}$.
- (2) $f^{\hat{\delta}}$ is an FLI (respectively, FRI, FII, FI) of $\mathcal{A}$.
- (3) For any $m \in [0, 1]$, we have that $(f^{\hat{\delta}})_m^-$ is an FLI (respectively, FRI, FII, FI) of $\mathcal{A}$.
- (4) For each $m \in [0, 1]$, we get that $(f^{\hat{\delta}})_m^+$ is an FLI (respectively, FRI, FII, FI) of $\mathcal{A}$.

Proof. (1) $\Leftrightarrow$ (2) It follows from $f^{\hat{\delta}}(p) = \text{SUP}\hat{\delta}(p)$ for all $p \in \mathcal{A}$, Definitions 3.1 and 4.4, and Theorems 2.4 and 2.10.

(3) $\Rightarrow$ (2) and (4) $\Rightarrow$ (2). They are directly obtained by taking $n = 0$.

(1) $\Rightarrow$ (4). Let δ be a sup-HFLI of $\mathcal{A}$, $n \in [0, 1]$ and $p, y \in \mathcal{A}$. Then $\text{SUP}\hat{\delta}(py) \geq \text{SUP}\hat{\delta}(y)$ which indicates that $\hat{f}^{\delta}(py) + n \geq \hat{f}^{\delta}(y) + n$. Thus, $(\hat{f}^{\delta})_n^+(py) = \min\{1, \hat{f}^{\delta}(py) + n\} \geq \min\{1, \hat{f}^{\delta}(y) + n\} = (\hat{f}^{\delta})_n^+(y)$ and so $(\hat{f}^{\delta})_n^+(py) \geq (\hat{f}^{\delta})_n^+(y)$. Hence, $(\hat{f}^{\delta})_n^+$ is an FLI of $\mathcal{A}$ for all $n \in [0, 1]$.

(1) $\Rightarrow$ (3). Let $\hat{\delta}$ be a sup-HFLI of $\mathcal{A}$, $n \in [0, 1]$ and $p, y \in \mathcal{A}$. Then $\hat{f}^{\delta}(py) - n \geq \hat{f}^{\delta}(y) - n$. Thus, $(\hat{f}^{\delta})_n^-(py) = \max\{0, \hat{f}^{\delta}(py) - n\} \geq \max\{0, \hat{f}^{\delta}(y) - n\} = (\hat{f}^{\delta})_n^-(y)$. Therefore, $(\hat{f}^{\delta})_n^-$ is an FLI of $\mathcal{A}$ for all $n \in [0, 1]$. $\square$

On an arbitrary semigroup, we recall the definitions of an anti-fuzzy subsemigroup (AFSS), an anti-fuzzy generalized bi-ideal (AFGBI) and an anti-fuzzy generalized bi-ideal (AFBI) in the following:

Definition 4.6. [29] An FS φ of $\mathcal{A}$ is called

- (1) an AFSS of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\varphi(pw) \leq \max\{\varphi(p), \varphi(w)\})$,
- (2) an AFGBI of $\mathcal{A}$ if $(\forall p, w, y \in \mathcal{A})(\varphi(pyw) \leq \max\{\varphi(p), \varphi(w)\})$, and
- (3) an AFBI of $\mathcal{A}$ if it is both an AFGBI and an AFSS of $\mathcal{A}$.

In Theorem 4.7, we characterize sup-HFBIs, sup-HFGBIs and sup-HFSSs of $\mathcal{A}$ via anti-types of FSs.

Theorem 4.7. *The following statements are equivalent for an HFS $\hat{\delta}$ on $\mathcal{A}$.*

- (1) $\hat{\delta}$ is a sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$.
- (2) $\hat{f}^{\delta^*}$ is an AFSS (respectively, AFGBI, AFBI) of $\mathcal{A}$.
- (3) $\hat{f}^{\hat{\tau}}$ is an AFSS (respectively, AFGBI, AFBI) of $\mathcal{A}$ for all $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$.
- (4) $(\hat{f}^{\hat{\tau}})_n^-$ is an AFSS (respectively, AFGBI, AFBI) of $\mathcal{A}$ for all $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$ and for all $n \in [0, 1]$.
- (5) $(\hat{f}^{\hat{\tau}})_n^+$ is an AFSS (respectively, AFGBI, AFBI) of $\mathcal{A}$ for all $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$ and for all $n \in [0, 1]$.

Proof. (5) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3). They are directly obtained by taking $n = 0$.

(3) $\Rightarrow$ (2). Since $\hat{\delta}^* \in \mathcal{SC}(\hat{\delta})$, this proof holds.

(2) $\Rightarrow$ (1). Assume that $\hat{f}^{\delta^*}$ is an AFSS of $\mathcal{A}$. Then, for all $p, w \in \mathcal{A}$, we have $\text{SUP}\hat{\delta}(pw) = 1 - \text{SUP}\hat{\delta}^*(pw) = 1 - \hat{f}^{\delta^*}(pw) \geq 1 - \max\{\hat{f}^{\delta^*}(p), \hat{f}^{\delta^*}(w)\} =$

$1 - \max\{\text{SUP}\widehat{\delta}^*(p), \text{SUP}\widehat{\delta}^*(w)\} = \min\{1 - \text{SUP}\widehat{\delta}^*(p), 1 - \text{SUP}\widehat{\delta}^*(w)\} = \min\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(w)\}$. Hence, $\widehat{\delta}$ is a sup-HFSS of $\mathcal{A}$.

(1) $\Rightarrow$ (3). Let $\widehat{\delta}$ be a sup-HFSS of $\mathcal{A}$, $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$ and $p, w \in \mathcal{A}$. Then $\text{SUP}\widehat{\delta}(pw) \geq \min\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(w)\}$ and thus $f^{\widehat{\tau}}(pw) = \text{SUP}\widehat{\tau}(pw) = 1 - \text{SUP}\widehat{\delta}(pw) \leq 1 - \min\{\text{SUP}\widehat{\delta}(p), \text{SUP}\widehat{\delta}(w)\} = \max\{1 - \text{SUP}\widehat{\delta}(p), 1 - \text{SUP}\widehat{\delta}(w)\} = \max\{\text{SUP}\widehat{\tau}(p), \text{SUP}\widehat{\tau}(w)\} = \max\{f^{\widehat{\tau}}(p), f^{\widehat{\tau}}(w)\}$. Hence, $f^{\widehat{\tau}}$ is an AFSS of $\mathcal{A}$.

(3) $\Rightarrow$ (4). Assume that $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, $n \in [0, 1]$ and $f^{\widehat{\tau}}$ is an AFSS of $\mathcal{A}$. Then $(f^{\widehat{\tau}})_n^-(pw) = \max\{0, f^{\widehat{\tau}}(pw) - n\} \leq \max\{0, \max\{f^{\widehat{\tau}}(p), f^{\widehat{\tau}}(w)\} - n\} = \max\{0, \max\{f^{\widehat{\tau}}(p) - n, f^{\widehat{\tau}}(w) - n\}\} = \max\{\max\{0, f^{\widehat{\tau}}(p) - n\}, \max\{0, f^{\widehat{\tau}}(w) - n\}\} = \max\{(f^{\widehat{\tau}})_n^-(p), (f^{\widehat{\tau}})_n^-(w)\}$ for all $p, w \in \mathcal{A}$. Hence, $(f^{\widehat{\tau}})_n^-$ is an AFSS of $\mathcal{A}$ for all $n \in [0, 1]$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$.

(3) $\Rightarrow$ (5). Assume that $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, $n \in [0, 1]$ and $f^{\widehat{\tau}}$ is an AFSS of $\mathcal{A}$. Then $(f^{\widehat{\tau}})_n^+(pw) = \min\{1, f^{\widehat{\tau}}(pw) + n\} \leq \min\{1, \max\{f^{\widehat{\tau}}(p), f^{\widehat{\tau}}(w)\} + n\} = \min\{1, \max\{f^{\widehat{\tau}}(p) + n, f^{\widehat{\tau}}(w) + n\}\} = \max\{\min\{1, f^{\widehat{\tau}}(p) + n\}, \min\{1, f^{\widehat{\tau}}(w) + n\}\} = \max\{(f^{\widehat{\tau}})_n^+(p), (f^{\widehat{\tau}})_n^+(w)\}$ for all $p, w \in \mathcal{A}$. Thus, $(f^{\widehat{\tau}})_n^+$ is an AFSS of $\mathcal{A}$ for all $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$ and $n \in [0, 1]$. $\square$

On an arbitrary semigroup, we recall the definitions of an anti-fuzzy ideal (AFI), an anti-fuzzy left ideal (AFLI) and an anti-fuzzy right ideal (AFRI) in the following:

Definition 4.8. [29] An FS φ of $\mathcal{A}$ is called

- (1) an AFLI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\varphi(pw) \leq \varphi(p))$,
- (2) an AFRI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\varphi(pw) \leq \varphi(w))$, and
- (3) an AFI of $\mathcal{A}$ if it is both an AFRI and an AFLI of $\mathcal{A}$.

Definition 4.9. [16] An FS φ of $\mathcal{A}$ is called an *anti-fuzzy interior ideal* (AFII) of $\mathcal{A}$ if $(\forall p, w, y \in \mathcal{A})(\varphi(ypw) \leq \varphi(p))$.

In Theorem 4.10, we characterize sup-HFLIs, sup-HFRIs, sup-HFIIs and sup-HFIs of $\mathcal{A}$ via anti-types of FSs.

Theorem 4.10. *The following statements are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFLI (respectively, sup-HFII, sup-HFI, sup-HFRI) of $\mathcal{A}$.
- (2) $f^{\widehat{\delta}^*}$ is an AFLI (respectively, AFII, AFI, AFRI) of $\mathcal{A}$.
- (3) For any $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, we get that $f^{\widehat{\tau}}$ is an AFLI (respectively, AFII, AFI, AFRI) of $\mathcal{A}$.

(4) For each $n \in [0, 1]$ and $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$, we have that $(f^{\hat{\tau}})_n^-$ is an AFLI (respectively, AFII, AFI, AFRI) of $\mathcal{A}$.

(5) For each $n \in [0, 1]$ and $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$, we can observe that $(f^{\hat{\tau}})_n^+$ is an AFLI (respectively, AFII, AFI, AFRI) of $\mathcal{A}$.

Proof. (2) $\Rightarrow$ (1). Let $f^{\hat{\delta}^*}$ be an AFLI of $\mathcal{A}$. Then, for all $p, y \in \mathcal{A}$, we get $\text{SUP}\hat{\delta}(py) = 1 - \text{SUP}\hat{\delta}^*(py) = 1 - f^{\hat{\delta}^*}(py) \geq 1 - f^{\hat{\delta}^*}(y) = 1 - \text{SUP}\hat{\delta}^*(y) = \text{SUP}\hat{\delta}(y)$. Hence, $\hat{\delta}$ is a sup-HFLI of $\mathcal{A}$.

(1) $\Rightarrow$ (3). Let $\hat{\delta}$ be a sup-HFLI of $\mathcal{A}$. Then, for all $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$ and $p, y \in \mathcal{A}$, we have $f^{\hat{\tau}}(py) = \text{SUP}\hat{\tau}(py) = 1 - \text{SUP}\hat{\delta}(py) \leq 1 - \text{SUP}\hat{\delta}(y) = \text{SUP}\hat{\tau}(y) = f^{\hat{\tau}}(y)$. Thus, $f^{\hat{\tau}}$ is an AFLI of $\mathcal{A}$.

(3) $\Rightarrow$ (4). Assume that $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$ and $f^{\hat{\tau}}$ is an AFLI of $\mathcal{A}$. Then $(f^{\hat{\tau}})_n^-(py) = \max\{0, f^{\hat{\tau}}(py) - n\} \leq \max\{0, f^{\hat{\tau}}(y) - n\} = (f^{\hat{\tau}})_n^-(y)$ for all $p, y \in \mathcal{A}$ and $n \in [0, 1]$. Hence, $(f^{\hat{\tau}})_n^-$ is an AFLI of $\mathcal{A}$ for all $n \in [0, 1]$ and $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$.

(3) $\Rightarrow$ (5). Assume that $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$ and $f^{\hat{\tau}}$ is an AFLI of $\mathcal{A}$. Then $(f^{\hat{\tau}})_n^+(py) = \min\{1, f^{\hat{\tau}}(py) + n\} \leq \min\{1, f^{\hat{\tau}}(y) + n\} = (f^{\hat{\tau}})_n^+(y)$ for all $p, y \in \mathcal{A}$ and $n \in [0, 1]$. Thus, $(f^{\hat{\tau}})_n^+$ is an AFLI of $\mathcal{A}$ for all $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$ and $n \in [0, 1]$.

(5) $\Rightarrow$ (3) and (4) $\Rightarrow$ (3). They are directly obtained by taking $n = 0$.

(3) $\Rightarrow$ (2). Since $\hat{\delta}^* \in \mathcal{SC}(\hat{\delta})$, this proof holds. $\square$

Definition 4.11. [36, 37] The set $P = \{(w, \varphi(w), \sigma(w)) \mid w \in \mathcal{Z}\}$, where $\varphi : \mathcal{Z} \rightarrow [0, 1]$ denotes the degree of membership, $\sigma : \mathcal{Z} \rightarrow [0, 1]$ denotes the degree of nonmembership and $0 \leq (\varphi(w))^2 + (\sigma(w))^2 \leq 1$ for every $w \in \mathcal{Z}$, is called a *Pythagorean fuzzy set (PFS)* P in $\mathcal{Z}$. We use the symbol (φ, σ) of the PFS $\{(w, \varphi(w), \sigma(w)) \mid w \in \mathcal{Z}\}$.

For an FS φ of $\mathcal{Z}$, we define an FS $\frac{\varphi}{2}$ by $\frac{\varphi}{2}(z) = \frac{\varphi(z)}{2}$ for all $z \in \mathcal{Z}$. Then $(\frac{\varphi}{2}, \frac{\sigma}{2})$ is a PFS in $\mathcal{Z}$ for all FSs φ and σ of $\mathcal{Z}$.

Proposition 4.12. Let $\hat{\delta}$ be an HFS on $\mathcal{Z}$. Then the following statements are true:

(1) $(f^{\hat{\delta}}, f^{\hat{\tau}})$ is a PFS in $\mathcal{Z}$ for all $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$.

(2) $((f^{\hat{\delta}})_n^-, (f^{\hat{\tau}})_m^-), ((f^{\hat{\delta}})_n^+, (f^{\hat{\tau}})_m^-), ((f^{\hat{\delta}})_n^-, (f^{\hat{\tau}})_n^+)$ and $(\frac{(f^{\hat{\delta}})_m^\alpha}{2}, \frac{(f^{\hat{\tau}})_n^\beta}{2})$ are PFSs in $\mathcal{Z}$ for all $m, n \in [0, 1]$, $\alpha, \beta \in \{-, +\}$ and $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$.

(3) For all $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$, there exist $m, n \in [0, 1]$ and $\alpha, \beta \in \{-, +\}$ such that $((f^{\hat{\delta}})_m^\alpha, (f^{\hat{\tau}})_n^\beta)$ is a PFS in $\mathcal{Z}$.

On an arbitrary semigroup, we recall the definitions of a Pythagorean fuzzy subsemigroup (PFSS) and a Pythagorean fuzzy bi-ideal (PFBI) in the following:

Definition 4.13 (3). Let (φ, σ) be a PFS in $\mathcal{A}$.

- (1) (φ, σ) is called a PFSS of $\mathcal{A}$ if
 - (i) $(\forall p, w \in \mathcal{A})(\varphi(pw) \geq \min\{\varphi(p), \varphi(w)\})$, and
 - (ii) $(\forall p, w \in \mathcal{A})(\sigma(pw) \leq \max\{\sigma(p), \sigma(w)\})$.
- (2) (φ, σ) of $\mathcal{A}$ is called a PFBI of $\mathcal{A}$ if
 - (i) $(\forall p, w, y \in \mathcal{A})(\varphi(pwy) \geq \min\{\varphi(p), \varphi(y)\})$, and
 - (ii) $(\forall p, w, y \in \mathcal{A})(\sigma(pwy) \leq \max\{\sigma(p), \sigma(y)\})$.

Definition 4.14. A PFS (φ, σ) in $\mathcal{A}$ is called a *Pythagorean fuzzy generalized bi-ideal* (PFGBI) of $\mathcal{A}$ if

- (i) $(\forall p, w, y \in \mathcal{A})(\varphi(pwy) \geq \min\{\varphi(p), \varphi(y)\})$, and
- (ii) $(\forall p, w, y \in \mathcal{A})(\sigma(pwy) \leq \max\{\sigma(p), \sigma(y)\})$.

From Theorems 4.3 and 4.7, and Proposition 4.12, characterizations of sup-HFBIs, sup-HFSSs and sup-HFGBIs of $\mathcal{A}$ are given via PFSs seen in Theorem 4.15.

Theorem 4.15. *The following statements are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFSS (respectively, sup-HFBI, sup-HFGBI) of $\mathcal{A}$.
- (2) $(\widehat{f}^{\widehat{\delta}}, \widehat{f}^{\widehat{\delta}*})$ is a PFSS (respectively, PFBI, PFGBI) of $\mathcal{A}$.
- (3) For each $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, we see that $(\widehat{f}^{\widehat{\delta}}, \widehat{f}^{\widehat{\tau}})$ is a PFSS (respectively, PFBI, PFGBI) of $\mathcal{A}$.
- (4) For any $m, n \in [0, 1]$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, we have that $((\widehat{f}^{\widehat{\delta}})_m^-, (\widehat{f}^{\widehat{\tau}})_n^-)$ is a PFSS (respectively, PFBI, PFGBI) of $\mathcal{A}$.
- (5) For each $m \in [0, 1]$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, we get that $((\widehat{f}^{\widehat{\delta}})_m^+, (\widehat{f}^{\widehat{\tau}})_m^-)$ is a PFSS (respectively, PFBI, PFGBI) of $\mathcal{A}$.
- (6) For every $m \in [0, 1]$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, we observe that $((\widehat{f}^{\widehat{\delta}})_m^-, (\widehat{f}^{\widehat{\tau}})_m^+)$ is a PFSS (respectively, PFBI, PFGBI) of $\mathcal{A}$.

- (7) $(\frac{(f^{\widehat{\delta}})^\alpha_m}{2}, \frac{(f^{\widehat{\tau}})^\beta_n}{2})$ is a PFSS (respectively, PFBI, PFGBI) of $\mathcal{A}$ for each $\alpha, \beta \in \{-, +\}$, $m, n \in [0, 1]$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$.
- (8) If $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, $\alpha, \beta \in \{-, +\}$ and $m, n \in [0, 1]$ such that $((f^{\widehat{\delta}})^\alpha_m, (f^{\widehat{\tau}})^\beta_n)$ is a PFS in $\mathcal{A}$, then $((f^{\widehat{\delta}})^\alpha_m, (f^{\widehat{\tau}})^\beta_n)$ is a PFSS (respectively, PFBI, PFGBI) of $\mathcal{A}$.

On an arbitrary semigroup, we recall the definitions of a Pythagorean fuzzy left ideal (PFLI), a Pythagorean fuzzy ideal (PFI), a Pythagorean fuzzy interior ideal (PFII) and a Pythagorean fuzzy right ideal (PFRI) in the following:

Definition 4.16 (3). Let (φ, σ) be a PFS in $\mathcal{A}$.

- (1) (φ, σ) is called a PFRI of $\mathcal{A}$ if
- (i) $(\forall p, w \in \mathcal{A})(\varphi(pw) \geq \varphi(p))$, and
 - (ii) $(\forall p, w \in \mathcal{A})(\sigma(pw) \leq \sigma(p))$.
- (2) (φ, σ) is called a PFLI of $\mathcal{A}$ if
- (i) $(\forall p, w \in \mathcal{A})(\varphi(pw) \geq \varphi(w))$, and
 - (ii) $(\forall p, w \in \mathcal{A})(\sigma(pw) \leq \sigma(w))$.
- (3) (φ, σ) is called a PFI of $\mathcal{A}$ if it is both a PFRI and a PFLI of $\mathcal{A}$.
- (4) (φ, σ) is called a PFII of $\mathcal{A}$ if
- (i) $(\forall p, w, y \in \mathcal{A})(\varphi(pwy) \geq \varphi(w))$, and
 - (ii) $(\forall p, w, y \in \mathcal{A})(\sigma(pwy) \leq \sigma(w))$.

From Theorems 4.5 and 4.10, and Proposition 4.12, characterizations of sup-HFLIs, sup-HFRIs, sup-HFIIs and sup-HFIs of $\mathcal{A}$ are investigated by PFSs seen in Theorem 4.17.

Theorem 4.17. *The following statements are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFLI (respectively, sup-HFII, sup-HFI, sup-HFRI) of $\mathcal{A}$.
- (2) $(f^{\widehat{\delta}}, f^{\widehat{\delta}*})$ is a PFLI (respectively, PFII, PFI, PFRI) of $\mathcal{A}$.
- (3) For each $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, we have that $(f^{\widehat{\delta}}, f^{\widehat{\tau}})$ is a PFLI (respectively, PFII, PFI, PFRI) of $\mathcal{A}$.

- (4) For every $m, n \in [0, 1]$ and $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$, we see that $((f^{\hat{\delta}})_m^-, (f^{\hat{\tau}})_n^-)$ is a PFLI (respectively, PFII, PFI, PFRI) of $\mathcal{A}$.
- (5) For each $n \in [0, 1]$ and $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$, we get that $((f^{\hat{\delta}})_n^+, (f^{\hat{\tau}})_n^-)$ is a PFLI (respectively, PFII, PFI, PFRI) of $\mathcal{A}$.
- (6) For each $n \in [0, 1]$ and $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$, we see that $((f^{\hat{\delta}})_n^-, (f^{\hat{\tau}})_n^+)$ is a PFLI (respectively, PFII, PFI, PFRI) of $\mathcal{A}$.
- (7) For each $\alpha, \beta \in \{-, +\}$, $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$ and $m, n \in [0, 1]$, we get that $(\frac{(f^{\hat{\delta}})_m^\alpha}{2}, \frac{(f^{\hat{\tau}})_n^\beta}{2})$ is a PFLI (respectively, PFII, PFI, PFRI) of $\mathcal{A}$.
- (8) If $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$, $\alpha, \beta \in \{-, +\}$ and $m, n \in [0, 1]$ such that $((f^{\hat{\delta}})_m^\alpha, (f^{\hat{\tau}})_n^\beta)$ is a PFS in $\mathcal{A}$, then $((f^{\hat{\delta}})_m^\alpha, (f^{\hat{\tau}})_n^\beta)$ is a PFLI (respectively, PFRI, PFII, PFI) of $\mathcal{A}$.

5. Hesitant fuzzy sets and hybrid sets

In this section, we will study sup-HFSSs, sup-HFGBIs, sup-HFBIs, sup-HFLIs, sup-HFRIs, sup-HFIIs, and sup-HFIs by HFSs and hybrid sets.

On an arbitrary semigroup, we recall the definitions of a hesitant fuzzy generalized bi-ideal (HFGBI), a hesitant fuzzy subsemigroup (HFSS) and a hesitant fuzzy bi-ideal (HFBI) in the following:

Definition 5.1. [15] An HFS $\hat{\delta}$ on $\mathcal{A}$ is called

- (1) an HFSS of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\hat{\delta}(p) \cap \hat{\delta}(w) \subseteq \hat{\delta}(pw))$,
- (2) an HFGBI of $\mathcal{A}$ if $(\forall p, w, y \in \mathcal{A})(\hat{\delta}(p) \cap \hat{\delta}(y) \subseteq \hat{\delta}(pwy))$, and
- (3) an HFBI of $\mathcal{A}$ if it is both an HFGBI and an HFSS of $\mathcal{A}$.

Proposition 5.2. Let $\hat{\delta}$ be an HFS of $\mathcal{Z}$. The following statements hold:

- (1) $(\forall z \in \mathcal{Z}, \forall \blacktriangledown \in \wp([0, 1]))(\text{SUP}\mathcal{H}_{\text{SUP}}^{(\hat{\delta}, \blacktriangledown)}(z) = \min\{\text{SUP}\hat{\delta}(z), \text{SUP}\blacktriangledown\})$.
- (2) $(\forall z \in \mathcal{Z})(\text{SUP}\mathcal{H}_{\text{SUP}}^{\hat{\delta}}(z) = \sup \mathcal{H}_{\text{SUP}}^{\hat{\delta}}(z) = \text{SUP}\hat{\delta}(z))$.
- (3) $(\forall y \in \mathcal{Z}, \forall \blacktriangledown \in \wp([0, 1]), \forall n \in [0, 1])(n \in \mathcal{H}_{\text{SUP}}^{(\hat{\delta}, \blacktriangledown)}(\underline{y}) \Leftrightarrow \min\{\text{SUP}\hat{\delta}(y), \text{SUP}\blacktriangledown\} \geq n \in \blacktriangledown)$.
- (4) $(\forall y \in \mathcal{Z}, \forall n \in [0, 1])(n \in \mathcal{H}_{\text{SUP}}^{\hat{\delta}}(y) \Leftrightarrow \text{SUP}\hat{\delta}(y) \geq n)$.

In Theorem 5.3, we characterize sup-HFBIs, sup-HFSSs and sup-HFGBIs of $\mathcal{A}$ in terms of HFSSs.

Theorem 5.3. *The following statements are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFSS (respectively, sup-HFBI, sup-HFGBI) of $\mathcal{A}$.
- (2) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is an HFSS (respectively, HFBI, HFGBI) of $\mathcal{A}$.
- (3) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is a sup-HFSS (respectively, sup-HFBI, sup-HFGBI) of $\mathcal{A}$.
- (4) For every $\blacktriangledown \in \wp([0, 1])$, we have that $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}$ is an HFSS (respectively, HFBI, HFGBI) of $\mathcal{A}$.
- (5) For every $\blacktriangledown \in \wp([0, 1])$, we see that $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}$ is a sup-HFSS (respectively, sup-HFBI, sup-HFGBI) of $\mathcal{A}$.

Proof. (1) $\Rightarrow$ (4). Assume that $\blacktriangledown \in \wp([0, 1])$ and $\widehat{\delta}$ is a sup-HFSS of $\mathcal{A}$. Let $w, y \in \mathcal{A}$ and $m \in \mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(w) \cap \mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(y)$. Thus, $m \in \blacktriangledown$, $\text{SUP}\widehat{\delta}(w) \geq m$ and $\text{SUP}\widehat{\delta}(y) \geq m$. By the assumption, we have $m \leq \min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\widehat{\delta}(y)\} \leq \text{SUP}\widehat{\delta}(wy)$. Then $m \in \mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(wy)$. Hence, we obtain $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(w) \cap \mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(y) \subseteq \mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(wy)$. Therefore, we have $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}$ is an HFSS of $\mathcal{A}$.

(4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3). They are directly obtained by taking $\blacktriangledown = [0, 1]$.

(2) $\Rightarrow$ (1) Assume that $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is an HFSS of $\mathcal{A}$. Let $w, y \in \mathcal{A}$. Then $\text{SUP}\widehat{\delta}(w) \in \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(w)$ and $\text{SUP}\widehat{\delta}(y) \in \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(y)$. Thus, $\min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\widehat{\delta}(y)\} \in \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(w) \cap \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(y)$. By the assumption, we obtain $\min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\widehat{\delta}(y)\} \in \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(wy)$. Hence, $\text{SUP}\widehat{\delta}(wy) \geq \min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\widehat{\delta}(y)\}$. Therefore, $\widehat{\delta}$ is a sup-HFSS of $\mathcal{A}$.

(1) $\Rightarrow$ (5). Let $\widehat{\delta}$ be a sup-HFSS of $\mathcal{A}$ and $\blacktriangledown \in \wp([0, 1])$. By Proposition 5.2, we get

$$\begin{aligned}
 \text{SUP}\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(wy) &= \min\{\text{SUP}\widehat{\delta}(wy), \text{SUP}\blacktriangledown\} \\
 &\geq \min\{\min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\widehat{\delta}(y)\}, \text{SUP}\blacktriangledown\} \\
 &= \min\{\min\{\text{SUP}\widehat{\delta}(w), \text{SUP}\blacktriangledown\}, \min\{\text{SUP}\widehat{\delta}(y), \text{SUP}\blacktriangledown\}\} \\
 &= \min\{\text{SUP}\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(w), \text{SUP}\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(y)\}
 \end{aligned}$$

for every $w, y \in \mathcal{A}$. Consequently, we obtain that $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}$ is a sup-HFSS of $\mathcal{A}$ for every $\blacktriangledown \in \wp([0, 1])$.

(3) $\Rightarrow$ (1). By using Proposition 5.2, this proof holds. $\square$

On an arbitrary semigroup, we recall the definitions of a hesitant fuzzy right ideal (HFRI) and a hesitant fuzzy left ideal (HFLI) in the following:

Definition 5.4. [15] An HFS $\widehat{\delta}$ on $\mathcal{A}$ is said to be

- (1) an HFRI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\widehat{\delta}(p) \subseteq \widehat{\delta}(pw))$, and
- (2) an HFLI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\widehat{\delta}(w) \subseteq \widehat{\delta}(pw))$.

Proposition 5.5. *Every HFLI (respectively, HFRI) of $\mathcal{A}$ is a sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$.*

Proof. Let $\widehat{\delta}$ be an HFLI of $\mathcal{A}$ and $p, w \in \mathcal{A}$. Then $\widehat{\delta}(w) \subseteq \widehat{\delta}(pw)$ and thus $\text{SUP}\widehat{\delta}(w) \subseteq \text{SUP}\widehat{\delta}(pw)$. Hence, $\widehat{\delta}$ is a sup-HFLI of $\mathcal{A}$. $\square$

Proposition 5.5's converse is generally false, as the illustration below illustrates.

Example 5.6. Let $\mathcal{A} = \{v, w, x\}$ be a semigroup defined in Example 3.4. We define an HFS $\widehat{\delta}$ on $\mathcal{A}$ as follows: $\widehat{\delta}(v) = (0, 1)$, $\widehat{\delta}(w) = [0, 0.3, 0.8]$ and $\widehat{\delta}(x) = \{0.8\}$. Then $\widehat{\delta}$ is a sup-HFI of $\mathcal{A}$, but not HFLI nor HFRI of $\mathcal{A}$ because $\widehat{\delta}(w) \not\subseteq \widehat{\delta}(wv)$ and $\widehat{\delta}(w) \not\subseteq \widehat{\delta}(vw)$.

From Proposition 5.5 and Example 5.6, a sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$ is a generalization of an HFLI (respectively, HFRI) of $\mathcal{A}$.

In Theorem 5.7, we characterize sup-HFLIs, sup-HFIs, sup-HFRIs and sup-HFRIIs of $\mathcal{A}$ by HFSs.

Theorem 5.7. *The following statements are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFLI (respectively, sup-HFI, sup-HFII, sup-HFRI) of $\mathcal{A}$.
- (2) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is an HFLI (respectively, HFI, HFII, HFRI) of $\mathcal{A}$.
- (3) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is a sup-HFLI (respectively, sup-HFI, sup-HFII, sup-HFRI) of $\mathcal{A}$.
- (4) For any $\blacktriangle \in \wp([0, 1])$, we see that $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangle)}$ is an HFLI (respectively, HFI, HFII, HFRI) of $\mathcal{A}$.
- (5) For each $\blacktriangle \in \wp([0, 1])$, we have that $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangle)}$ is a sup-HFLI (respectively, sup-HFI, sup-HFII, sup-HFRI) of $\mathcal{A}$.

Proof. (2) $\Rightarrow$ (3) and (4) $\Rightarrow$ (5). It follows from Proposition 5.5 and Theorems 2.2 and 2.8.

(4) $\Rightarrow$ (2) and (5) $\Rightarrow$ (3). We have the proofs by taking $\blacktriangledown = [0, 1]$.

(3) $\Rightarrow$ (1). By using Proposition 5.2, this proof holds.

(1) $\Rightarrow$ (4). Assume that $\blacktriangledown \in \wp([0, 1])$ and $\widehat{\delta}$ is a sup-HFLI of $\mathcal{A}$. Let $p, w \in \mathcal{A}$ and $m \in \mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(w)$. Then $\text{SUP}\widehat{\delta}(w) \geq m \in \blacktriangledown$. By the assumption, we have $\text{SUP}\widehat{\delta}(pw) \geq \text{SUP}\widehat{\delta}(w) \geq m \in \blacktriangledown$ and so $m \in \mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(pw)$. Thus, $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(w) \subseteq \mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}(pw)$. Hence, $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}$ is an HFLI of $\mathcal{A}$. $\square$

Corollary 5.8. *If $\widehat{\delta}$ is an HFLI (respectively, HFRI, HFI, HFII), then the following statements hold:*

- (1) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is an HFLI (respectively, HFRI, HFI, HFII) of $\mathcal{A}$.
- (2) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is a sup-HFLI (respectively, sup-HFRI, sup-HFI, sup-HFII) of $\mathcal{A}$.
- (3) For every $\blacktriangle \in \wp([0, 1])$, we get that $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangle)}$ is an HFLI (respectively, HFRI, HFI, HFII) of $\mathcal{A}$.
- (4) For every $\blacktriangle \in \wp([0, 1])$, we have that $\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangle)}$ is a sup-HFLI (respectively, sup-HFRI, sup-HFI, sup-HFII) of $\mathcal{A}$.

Proof. It follows from Proposition 5.5 and Theorems 2.2, 2.8 and 5.7. $\square$

Definition 5.9. [1] A hybrid set in $\mathcal{Z}$ over $\mathcal{B}$ is a function $(\widehat{\delta}, \sigma) : \mathcal{Z} \rightarrow \wp(\mathcal{B}) \times [0, 1]$, $z \mapsto (\widehat{\delta}(z), \sigma(z))$ where $\widehat{\delta} : \mathcal{Z} \rightarrow \wp(\mathcal{B})$ and $\sigma : \mathcal{Z} \rightarrow [0, 1]$.

Definition 5.10 (1). A hybrid set $(\widehat{\delta}, \sigma)$ in $\mathcal{A}$ over $\mathcal{B}$ is called a *hybrid subsemigroup* (HSS) of $\mathcal{A}$ over $\mathcal{B}$ if

- (i) $(\forall p, w \in \mathcal{A})(\widehat{\delta}(p) \cap \widehat{\delta}(w) \subseteq \widehat{\delta}(pw))$, and
- (ii) $(\forall p, w \in \mathcal{A})(\sigma(pw) \leq \max\{\sigma(p), \sigma(w)\})$.

Definition 5.11 (5). A hybrid set $(\widehat{\delta}, \sigma)$ in $\mathcal{A}$ over $\mathcal{B}$ is called a *hybrid generalized bi-ideal* (HGBI) of $\mathcal{A}$ over $\mathcal{B}$ if

- (i) $(\forall p, w, y \in \mathcal{A})(\widehat{\delta}(p) \cap \widehat{\delta}(y) \subseteq \widehat{\delta}(pwy))$, and
- (ii) $(\forall p, w, y \in \mathcal{A})(\sigma(pwy) \leq \max\{\sigma(p), \sigma(y)\})$.

Definition 5.12 (5, 4). A hybrid set $(\widehat{\delta}, \sigma)$ in $\mathcal{A}$ over $\mathcal{B}$ is called a *hybrid bi-ideal* (HBI) of $\mathcal{A}$ over $\mathcal{B}$ if both an HSS and HGBI of $\mathcal{A}$ over $\mathcal{B}$.

By using Theorems 4.7 and 5.3, the sup-HFBIs, sup-HFSSs and sup-HFGBIs of arbitrary semigroups are characterized by hybrid sets in Theorem 5.13.

Theorem 5.13. *The following statements are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$.
- (2) $(\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}, \mathbf{f}^{\widehat{\delta}*})$ is an HSS (respectively, HGBI, HBI) of $\mathcal{A}$ over $[0, 1]$.
- (3) $(\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}, \mathbf{f}^{\widehat{\tau}})$ is an HSS (respectively, HGBI, HBI) of $\mathcal{A}$ over $[0, 1]$ for all $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$.
- (4) $(\mathcal{H}_{\text{SUP}}^{\widehat{\delta}, \blacktriangle}, \mathbf{f}^{\widehat{\tau}})$ is an HSS (respectively, HGBI, HBI) of $\mathcal{A}$ over $[0, 1]$ for every $\blacktriangle \in \wp([0, 1])$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$.
- (5) $(\mathcal{H}_{\text{SUP}}^{\widehat{\delta}, \blacktriangle}, (\mathbf{f}^{\widehat{\tau}})_n^-)$ is an HSS (respectively, HGBI, HBI) of $\mathcal{A}$ over $[0, 1]$ for all $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, $n \in [0, 1]$ and $\blacktriangle \in \wp([0, 1])$.
- (6) $(\mathcal{H}_{\text{SUP}}^{\widehat{\delta}, \blacktriangle}, (\mathbf{f}^{\widehat{\tau}})_n^+)$ is an HSS (respectively, HGBI, HBI) of $\mathcal{A}$ over $[0, 1]$ for all $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, $n \in [0, 1]$ and $\blacktriangle \in \wp([0, 1])$.

On an arbitrary semigroup, we recall the definitions of a hybrid right ideal (HRI), a hybrid left ideal (HLI) and a hybrid ideal (HI) in the following:

Definition 5.14 (1). A hybrid set $(\widehat{\delta}, \sigma)$ in $\mathcal{A}$ over $\mathcal{B}$ is called

- (1) an HRI of $\mathcal{A}$ over $\mathcal{B}$ if
 - (i) $(\forall p, w \in \mathcal{A})(\widehat{\delta}(p) \subseteq \widehat{\delta}(pw))$, and
 - (ii) $(\forall p, w \in \mathcal{A})(\sigma(pw) \leq \sigma(p))$,
- (2) an HLI of $\mathcal{A}$ over $\mathcal{B}$ if
 - (i) $(\forall p, w \in \mathcal{A})(\widehat{\delta}(w) \subseteq \widehat{\delta}(pw))$, and
 - (ii) $(\forall p, w \in \mathcal{A})(\sigma(pw) \leq \sigma(w))$,
- (3) an HI of $\mathcal{A}$ over $\mathcal{B}$ if it is both an HRI and an HLI of $\mathcal{A}$ over $\mathcal{B}$.

Definition 5.15. A hybrid set $(\widehat{\delta}, \sigma)$ in $\mathcal{A}$ over $\mathcal{B}$ is called a *hybrid interior ideal* (HII) of $\mathcal{A}$ over $\mathcal{B}$ if

- (i) $(\forall p, w, y \in \mathcal{A})(\widehat{\delta}(w) \subseteq \widehat{\delta}(pwy))$, and
- (ii) $(\forall p, w, y \in \mathcal{A})(\sigma(pwy) \leq \sigma(w))$.

Theorem 5.16. The following statements are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.

- (1) $\widehat{\delta}$ is an HFLI (respectively, HFRI, HFII, HFI) of $\mathcal{A}$.

- (2) $(\widehat{\delta}, f^{\widehat{\delta}^*})$ is an HLI (respectively, HRI, HII, HI) of $\mathcal{A}$ over $[0, 1]$.
- (3) For every $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, we get that $(\widehat{\delta}, f^{\widehat{\tau}})$ is an HLI (respectively, HRI, HII, HI) of $\mathcal{A}$ over $[0, 1]$.
- (4) For all $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$ and $n \in [0, 1]$, we have that $(\widehat{\delta}, (f^{\widehat{\tau}})_n^-)$ is an HLI (respectively, HRI, HII, HI) of $\mathcal{A}$ over $[0, 1]$.
- (5) For any $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$ and $n \in [0, 1]$, we see that $(\widehat{\delta}, (f^{\widehat{\tau}})_n^+)$ is an HLI (respectively, HRI, HII, HI) of $\mathcal{A}$ over $[0, 1]$.

Proof. It follows from Proposition 5.5 and Theorems 2.2, 2.8 and 4.10. $\square$

By using Theorems 4.10 and 5.7, we have Theorem 5.17 that characterizations of sup-HFLIs, sup-HFRIs, sup-HFIIs and sup-HFIs of $\mathcal{A}$ are given via hybrid sets.

Theorem 5.17. *The following assertions are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFLI (respectively, sup-HFRI, sup-HFII, sup-HFI) of $\mathcal{A}$.
- (2) $(\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}, f^{\widehat{\delta}^*})$ is an HLI (respectively, HRI, HII, HI) of $\mathcal{A}$ over $[0, 1]$.
- (3) $(\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}, f^{\widehat{\tau}})$ is an HLI (respectively, HRI, HII, HI) of $\mathcal{A}$ over $[0, 1]$ for all $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$.
- (4) For each $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$ and for each $\blacktriangledown \in \wp([0, 1])$, we have that $(\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}, f^{\widehat{\tau}})$ is an HLI (respectively, HRI, HII, HI) of $\mathcal{A}$ over $[0, 1]$.
- (5) $(\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}, (f^{\widehat{\tau}})_n^-)$ is an HLI (respectively, HRI, HII, HI) of $\mathcal{A}$ over $[0, 1]$ for all $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, $n \in [0, 1]$ and $\blacktriangledown \in \wp([0, 1])$.
- (6) $(\mathcal{H}_{\text{SUP}}^{(\widehat{\delta}, \blacktriangledown)}, (f^{\widehat{\tau}})_n^+)$ is an HLI (respectively, HRI, HII, HI) of $\mathcal{A}$ over $[0, 1]$ for all $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, $n \in [0, 1]$ and $\blacktriangledown \in \wp([0, 1])$.

6. Interval-valued fuzzy sets and cubic sets

Using IvFSs and cubic sets, we will examine sup-HFSSs, sup-HFGBIs, sup-HFBIIs, sup-HFLIs, sup-HFIs, sup-HFIIs, and sup-HFRIs in this section.

On arbitrary semigroup, we recall definitions of an interval-valued fuzzy subsemigroup (IvFSS), an interval-valued fuzzy generalized bi-ideal (IvFGBI) and an interval-valued fuzzy bi-ideal (IvFBI) in Definition 6.1.

Definition 6.1. [31] Let $\check{\nu}$ be an IvFS on $\mathcal{A}$.

- (1) $\check{v}$ is called an IvFSS of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\text{rmin}\{\check{v}(p), \check{v}(w)\} \preceq \check{v}(pw))$.
- (2) $\check{v}$ is called IvFGBI of $\mathcal{A}$ if $(\forall p, w, y \in \mathcal{A})(\text{rmin}\{\check{v}(p), \check{v}(y)\} \preceq \check{v}(pwy))$.
- (3) $\check{v}$ is called an IvFBI of $\mathcal{A}$ if it is both an IvFGBI and an IvFSS of $\mathcal{A}$.

Proposition 6.2. *Every IvFSS (respectively, IvFGBI, IvFBI) of $\mathcal{A}$ is a sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$.*

Proof. Let $\check{v}$ be an IvFSS of $\mathcal{A}$ and $w, y \in \mathcal{A}$. Thus, $\text{rmin}\{\check{v}(w), \check{v}(y)\} \preceq \check{v}(wy)$. Therefore,

$$\min\{\text{SUP}\check{v}(w), \text{SUP}\check{v}(y)\} = \min\{\check{v}^+(w), \check{v}^+(y)\} \leq \check{v}^+(wy) = \text{SUP}\check{v}(wy).$$

Hence, $\check{v}$ is a sup-HFSS of $\mathcal{A}$. $\square$

The following example demonstrates that Proposition 6.2's converse is generally false.

Example 6.3. Consider the semigroup $\mathcal{A} = \{w, x, y, z\}$ defined in Example 3.7. Define an HFS $\widehat{\delta}$ by $\widehat{\delta}(z) = \emptyset$, $\widehat{\delta}(x) = [0.4, 0.5]$, $\widehat{\delta}(w) = (0, 0.7)$, and $\widehat{\delta}(y) = \emptyset$. Thus, we obtain that $\widehat{\delta}$ is a sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$. However, $\widehat{\delta}$ is not an IvFSS (respectively, IvFGBI, IvFBI) of $\mathcal{A}$ since $\widehat{\delta}$ is not an IvFS.

A sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$ is a general concept of an IvFSS (respectively, IvFGBI, IvFBI) of $\mathcal{A}$, according to Proposition 6.2 and Example 6.3.

For FSs φ and σ of $\mathcal{Z}$ with $\varphi \leq \sigma$, define the IvFS $[\varphi, \sigma]$ on $\mathcal{Z}$ by $[\varphi, \sigma] : \mathcal{Z} \rightarrow \mathcal{D}([0, 1])$, $z \mapsto [\varphi(z), \sigma(z)]$. Let φ and σ be FSs of $\mathcal{Z}$ such that $\varphi \leq \sigma$ and $m, n \in [0, 1]$. The following statements are true:

We recall definitions of an interval-valued fuzzy right ideal (IvFRI) and an interval-valued fuzzy left ideal (IvFLI) of arbitrary semigroup in Definition 6.7.

Definition 6.7 (23). Let $\check{v}$ be an IvFS on $\mathcal{A}$.

- (1) $\check{v}$ is called an IvFLI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\check{v}(w) \preceq \check{v}(pw))$.
- (2) $\check{v}$ is called an IvFRI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\check{v}(p) \preceq \check{v}(pw))$.

Proposition 6.8. *Every IvFLI (respectively, IvFRI) of $\mathcal{A}$ is a sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$.*

Proof. Let $\check{v}$ be an IvFLI of $\mathcal{A}$ and $w, y \in \mathcal{A}$. Then $\check{v}(y) \preceq \check{v}(wy)$ and hence $\text{SUP}\check{v}(y) = \check{v}^+(y) \leq \check{v}^+(wy) = \text{SUP}\check{v}(wy)$. Therefore, $\check{v}$ is a sup-HFLI of $\mathcal{A}$. $\square$

The following case demonstrates that Proposition 6.8's converse is generally false.

Example 6.9. We know that $\widehat{\delta}_1$ is a sup-HFRI and $\widehat{\delta}_2$ is a sup-HFLI of $\mathcal{A}$ from Example 3.2. Since $\widehat{\delta}_1$ and $\widehat{\delta}_2$ are not IvFSs, we get that $\widehat{\delta}_1$ is not an IvFRI and $\widehat{\delta}_2$ is not an IvFLI of $\mathcal{A}$.

By Proposition 6.8 and Example 6.9, the concept of a sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$ is a general concept of an IvFLI (respectively, IvFRI) of $\mathcal{A}$.

Remark 6.10. An IvFS $\check{v}$ on $\mathcal{A}$ is an IvFLI (respectively, IvFRI, IvFII, IvFI) of $\mathcal{A}$ if and only if $\check{v}^-$ and $\check{v}^+$ are FLIs (respectively, FRIs, FIIs, FIs) of $\mathcal{A}$.

Remark 6.11. An FS φ of $\mathcal{A}$ is an FLI (respectively, FRI, FII, FI) of $\mathcal{A}$ if and only if $[\varphi, \varphi]$ is an IvFLI (respectively, IvFRI, IvFII, IvFI) of $\mathcal{A}$.

In Theorem 6.12, characterizations of sup-HFLIs, sup-HFRIs, sup-HFIIs and sup-HFIs of $\mathcal{A}$ via IvFSs.

Theorem 6.12. *The following assertions are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFLI (respectively, sup-HFI, sup-HFII, sup-HFRI) of $\mathcal{A}$.
- (2) For every $n \in [0, 1]$, we get that $[(f^{\widehat{\delta}})_n^-, f^{\widehat{\delta}}]$ is an IvFLI (respectively, IvFI, IvFII, IvFRI) of $\mathcal{A}$.
- (3) For every $n \in [0, 1]$, we get that $[f^{\widehat{\delta}}, (f^{\widehat{\delta}})_n^+]$ is an IvFLI (respectively, IvFI, IvFII, IvFRI) of $\mathcal{A}$.

We recall definitions of an interval-valued fuzzy right ideal (IvFRI) and an interval-valued fuzzy left ideal (IvFLI) of arbitrary semigroup in Definition 6.7.

Definition 6.7 (23). Let $\check{v}$ be an IvFS on $\mathcal{A}$.

- (1) $\check{v}$ is called an IvFLI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\check{v}(w) \lesssim \check{v}(pw))$.
- (2) $\check{v}$ is called an IvFRI of $\mathcal{A}$ if $(\forall p, w \in \mathcal{A})(\check{v}(p) \lesssim \check{v}(pw))$.

Proposition 6.8. *Every IvFLI (respectively, IvFRI) of $\mathcal{A}$ is a sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$.*

Proof. Let $\check{v}$ be an IvFLI of $\mathcal{A}$ and $w, y \in \mathcal{A}$. Then $\check{v}(y) \preceq \check{v}(wy)$ and hence $\text{SUP}\check{v}(y) = \check{v}^+(y) \leq \check{v}^+(wy) = \text{SUP}\check{v}(wy)$. Therefore, $\check{v}$ is a sup-HFLI of $\mathcal{A}$. $\square$

The following case demonstrates that Proposition 6.8's converse is generally false.

Example 6.9. We know that $\widehat{\delta}_1$ is a sup-HFRI and $\widehat{\delta}_2$ is a sup-HFLI of $\mathcal{A}$ from Example 3.2. Since $\widehat{\delta}_1$ and $\widehat{\delta}_2$ are not IvFSSs, we get that $\widehat{\delta}_1$ is not an IvFRI and $\widehat{\delta}_2$ is not an IvFLI of $\mathcal{A}$.

By Proposition 6.8 and Example 6.9, the concept of a sup-HFLI (respectively, sup-HFRI) of $\mathcal{A}$ is a general concept of an IvFLI (respectively, IvFRI) of $\mathcal{A}$.

Remark 6.10. An IvFS $\check{v}$ on $\mathcal{A}$ is an IvFLI (respectively, IvFRI, IvFII, IvFI) of $\mathcal{A}$ if and only if $\check{v}^-$ and $\check{v}^+$ are FLIs (respectively, FRIs, FIIs, FIs) of $\mathcal{A}$.

Remark 6.11. An FS φ of $\mathcal{A}$ is an FLI (respectively, FRI, FII, FI) of $\mathcal{A}$ if and only if $[\varphi, \varphi]$ is an IvFLI (respectively, IvFRI, IvFII, IvFI) of $\mathcal{A}$.

In Theorem 6.12, characterizations of sup-HFLIs, sup-HFRIs, sup-HFIIs and sup-HFIs of $\mathcal{A}$ via IvFSSs.

Theorem 6.12. *The following assertions are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFLI (respectively, sup-HFI, sup-HFII, sup-HFRI) of $\mathcal{A}$.
- (2) For every $n \in [0, 1]$, we get that $[(f^{\widehat{\delta}})_n^-, f^{\widehat{\delta}}]$ is an IvFLI (respectively, IvFI, IvFII, IvFRI) of $\mathcal{A}$.
- (3) For every $n \in [0, 1]$, we get that $[f^{\widehat{\delta}}, (f^{\widehat{\delta}})_n^+]$ is an IvFLI (respectively, IvFI, IvFII, IvFRI) of $\mathcal{A}$.
- (4) For every $m, n \in [0, 1]$, we get that $[(f^{\widehat{\delta}})_m^-, (f^{\widehat{\delta}})_n^+]$ is an IvFLI (respectively, IvFI, IvFII, IvFRI) of $\mathcal{A}$.
- (5) For every $m, n \in [0, 1]$ with $m \geq n$, we get that $[(f^{\widehat{\delta}})_m^-, (f^{\widehat{\delta}})_n^-]$ is an IvFLI (respectively, IvFI, IvFII, IvFRI) of $\mathcal{A}$.
- (6) For every $m, n \in [0, 1]$ with $m \leq n$, we have that $[(f^{\widehat{\delta}})_m^+, (f^{\widehat{\delta}})_n^+]$ is an IvFLI (respectively, IvFI, IvFII, IvFRI) of $\mathcal{A}$.
- (7) $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is an IvFLI (respectively, IvFI, IvFII, IvFRI) of $\mathcal{A}$.

Proof. By using Theorem 4.5 and Remarks 6.10 and 6.11, the conditions (1) – (6) are equivalent. Since Theorems 2.3 and 2.9, Proposition 6.8 and $\sup \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(w) = \text{SUP} \widehat{\delta}(w)$ for all $w \in \mathcal{A}$, the proof that (7) implies (1) holds. To show that (1) implies (7), let $\widehat{\delta}$ be a sup-HFLI of $\mathcal{A}$ and $w, y \in \mathcal{A}$. Then

$\sup \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(wy) \geq \sup \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(y)$. Since $\inf \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(wy) = 0 = \inf \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(y)$, we obtain that $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(y) \preceq \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}(wy)$. Hence, $\mathcal{H}_{\text{SUP}}^{\widehat{\delta}}$ is an IvFLI of $\mathcal{A}$. $\square$

Definition 6.13 (13). A *cubic set* in $\mathcal{Z}$ is a function $\langle \check{v}, \varphi \rangle : \mathcal{Z} \rightarrow \mathcal{D}([0, 1]) \times [0, 1]$, $z \mapsto (\check{v}(z), \varphi(z))$, where $\check{v} : \mathcal{Z} \rightarrow \mathcal{D}([0, 1])$ and $\varphi : \mathcal{Z} \rightarrow [0, 1]$.

It should be noted that every cubic set in $\mathcal{Z}$ over $[0, 1]$ is a hybrid set in $\mathcal{Z}$. On an arbitrary semigroup, we recall definitions of a cubic subsemigroup (CSS), a cubic generalized bi-ideal (CGBI) and a cubic bi-ideal (CBI) in the following three definitions:

Definition 6.14 (12). A cubic set $\langle \check{v}, \varphi \rangle$ in $\mathcal{A}$ is said to be a CSS of $\mathcal{A}$ if

- (i) $(\forall p, w \in \mathcal{A})(\text{rmin}\{\check{v}(p), \check{v}(w)\} \preceq \check{v}(pw))$, and
- (ii) $(\forall p, w \in \mathcal{A})(\varphi(pw) \leq \max\{\varphi(p), \varphi(w)\})$.

Definition 6.15 (6). A cubic set $\langle \check{v}, \varphi \rangle$ in $\mathcal{A}$ is said to be a CGBI of $\mathcal{A}$ if

- (i) $(\forall p, w, y \in \mathcal{A})(\text{rmin}\{\check{v}(w), \check{v}(y)\} \preceq \check{v}(wpy))$, and
- (ii) $(\forall p, w, y \in \mathcal{A})(\varphi(wpy) \leq \max\{\varphi(w), \varphi(y)\})$.

Definition 6.16 (34). A cubic set in $\mathcal{A}$ is said to be a CBI of $\mathcal{A}$ if it is both a CSS and a CGBI of $\mathcal{A}$.

Remark 6.17. A cubic set $\langle \check{v}, \varphi \rangle$ in $\mathcal{A}$ is a CSS (respectively, CGBI, CBI) of $\mathcal{A}$ if and only if $\check{v}$ is an IvFSS (respectively, IvFGBI, IvFBI) and φ is an AFSS (respectively, AFGBI, AFBI) of $\mathcal{A}$.

Theorem 6.18. Let $\check{v}$ be an IvFS on $\mathcal{A}$. The following assertions are equivalent.

- (1) $\check{v}$ is an IvFSS (respectively, IvFGBI, IvFBI) of $\mathcal{A}$.
- (2) $\langle \check{v}, f^{\check{v}^*} \rangle$ is a CSS (respectively, CGBI, CBI) of $\mathcal{A}$.
- (3) $\langle \check{v}, f^{\widehat{\tau}} \rangle$ is a CSS (respectively, CGBI, CBI) of $\mathcal{A}$ for all $\widehat{\tau} \in \mathcal{SC}(\check{v})$.
- (4) $\langle \check{v}, (f^{\widehat{\tau}})_n^- \rangle$ is a CSS (respectively, CGBI, CBI) of $\mathcal{A}$ for all $\widehat{\tau} \in \mathcal{SC}(\check{v})$ and $n \in [0, 1]$.
- (5) $\langle \check{v}, (f^{\widehat{\tau}})_n^+ \rangle$ is a CSS (respectively, CGBI, CBI) of $\mathcal{A}$ for all $\widehat{\tau} \in \mathcal{SC}(\check{v})$ and $n \in [0, 1]$.

Proof. This theorem is obtained by applying Theorem 4.7 and Remark 6.17. $\square$

Theorem 6.19. *The following assertions are equivalent for an HFS $\widehat{\delta}$ on $\mathcal{A}$.*

- (1) $\widehat{\delta}$ is a sup-HFSS (respectively, sup-HFGBI, sup-HFBI) of $\mathcal{A}$.
- (2) $\langle \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}, \mathbf{f}^{\widehat{\delta}*} \rangle$ is a CSS (respectively, CGBI, CBI) of $\mathcal{A}$.
- (3) For every $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, we have that $\langle \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}, \mathbf{f}^{\widehat{\tau}} \rangle$ is a CSS (respectively, CGBI, CBI) of $\mathcal{A}$.
- (4) For every $m \in [0, 1]$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, we have $\langle \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}, (\mathbf{f}^{\widehat{\tau}})_m^- \rangle$ is a CSS (respectively, CGBI, CBI) of $\mathcal{A}$.
- (5) For every $m \in [0, 1]$ and $\widehat{\tau} \in \mathcal{SC}(\widehat{\delta})$, we have that $\langle \mathcal{H}_{\text{SUP}}^{\widehat{\delta}}, (\mathbf{f}^{\widehat{\tau}})_m^+ \rangle$ is a CSS (respectively, CGBI, CBI) of $\mathcal{A}$.

Proof. This theorem is obtained by applying Theorems 4.7 and 6.6. $\square$

Definition 6.20 (12). Let $\langle \check{v}, \varphi \rangle$ be a cubic set in $\mathcal{A}$.

- (1) $\langle \check{v}, \varphi \rangle$ is said to be a *cubic left ideal* (CLI) of $\mathcal{A}$ if
 - (i) $(\forall p, w \in \mathcal{A})(\check{v}(w) \lesssim \check{v}(pw))$, and
 - (ii) $(\forall p, w \in \mathcal{A})(\varphi(pw) \leq \varphi(w))$.
- (2) $\langle \check{v}, \varphi \rangle$ is said to be a *cubic right ideal* (CRI) of $\mathcal{A}$ if
 - (i) $(\forall p, w \in \mathcal{A})(\check{v}(p) \lesssim \check{v}(pw))$, and
 - (ii) $(\forall p, w \in \mathcal{A})(\varphi(pw) \leq \varphi(p))$.
- (3) $\langle \check{v}, \varphi \rangle$ is said to be a *cubic ideal* (CI) of $\mathcal{A}$ if both a CLI and a CRI of $\mathcal{A}$.

Definition 6.21 (22). A cubic set $\langle \check{v}, \varphi \rangle$ in $\mathcal{A}$ is said to be a *cubic interior ideal* (CII) of $\mathcal{A}$ if

- (i) $(\forall p, w, y \in \mathcal{A})(\check{v}(w) \lesssim \check{v}(pwy))$, and
- (ii) $(\forall p, w, y \in \mathcal{A})(\varphi(pwy) \leq \varphi(w))$.

Remark 6.22. A cubic set $\langle \check{v}, \varphi \rangle$ in $\mathcal{A}$ is a CLI (respectively, CRI, CII, CI) of $\mathcal{A}$ if and only if $\check{v}$ is an IvFLI (respectively, IvFRI, IvFII, IvFI) and φ is an AFLI (respectively, AFRI, AFII, AFI) of $\mathcal{A}$.

Theorem 6.23. *Let $\check{v}$ be an IvFS on $\mathcal{A}$. The following assertions are equivalent.*

- (1) $\check{v}$ is an IvFLI (respectively, IvFRI, IvFII, IvFI) of $\mathcal{A}$.
- (2) $\langle \check{v}, f^{\check{v}*} \rangle$ is a CLI (respectively, CRI, CII, CI) of $\mathcal{A}$.
- (3) $\langle \check{v}, f^{\hat{\tau}} \rangle$ is a CLI (respectively, CRI, CII, CI) of $\mathcal{A}$ for all $\hat{\tau} \in \mathcal{SC}(\check{v})$.
- (4) $\langle \check{v}, (f^{\hat{\tau}})_n^- \rangle$ is a CLI (respectively, CRI, CII, CI) of $\mathcal{A}$ for all $\hat{\tau} \in \mathcal{SC}(\check{v})$ and $n \in [0, 1]$.
- (5) $\langle \check{v}, (f^{\hat{\tau}})_n^+ \rangle$ is a CLI (respectively, CRI, CII, CI) of $\mathcal{A}$ for all $\hat{\tau} \in \mathcal{SC}(\check{v})$ and $n \in [0, 1]$.

Proof. This theorem is obtained by applying Theorem 4.10 and Remark 6.22. $\square$

By using Theorems 4.10 and 6.12, we have Theorem 6.24 that characterizations of sup-HFLIs, sup-HFRIs, sup-HFIIs and sup-HFIs of $\mathcal{A}$ are given via cubic sets.

Theorem 6.24. *The following assertions are equivalent for an HFS $\hat{\delta}$ on $\mathcal{A}$.*

- (1) $\hat{\delta}$ is a sup-HFLI (respectively, sup-HFRI, sup-HFII, sup-HFI) of $\mathcal{A}$.
- (2) $\langle \mathcal{H}_{\text{SUP}}^{\hat{\delta}}, f^{\hat{\delta}*} \rangle$ is a CLI (respectively, CRI, CII, CI) of $\mathcal{A}$.
- (3) $\langle \mathcal{H}_{\text{SUP}}^{\hat{\delta}}, f^{\hat{\tau}} \rangle$ is a CLI (respectively, CRI, CII, CI) of $\mathcal{A}$ for all $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$.
- (4) $\langle \mathcal{H}_{\text{SUP}}^{\hat{\delta}}, (f^{\hat{\tau}})_n^- \rangle$ is a CLI (respectively, CRI, CII, CI) of $\mathcal{A}$ for all $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$ and $n \in [0, 1]$.
- (5) $\langle \mathcal{H}_{\text{SUP}}^{\hat{\delta}}, (f^{\hat{\tau}})_n^+ \rangle$ is a CLI (respectively, CRI, CII, CI) of $\mathcal{A}$ for all $\hat{\tau} \in \mathcal{SC}(\hat{\delta})$ and $n \in [0, 1]$.

7. Conclusions and Future 1 Works

In the present paper, we have introduced a sup-HFSS (respectively, sup-HFGBI, sup-HFBI, sup-HFLI, sup-HFRI), which is a general concept of an IvFSS (respectively, IvFGBI, IvFBI, IvFLI, IvFRI), on semigroups, investigated some of their interesting properties and proved their generalizations. Hence, we obtain the generalization diagram of HFSs in the meaning of the supremum of its images in semigroups, as shown in Figure 1. As important study results, characterizations of sup-HFSSs, sup-HFGBIs, sup-HFBIs, sup-HFLIs, sup-HFRIs, sup-HFIIs, and sup-HFIs have been discussed by FSSs, level sets, PFSSs, IvFSSs, cubic sets, HFSs, and hybrid sets. Further,

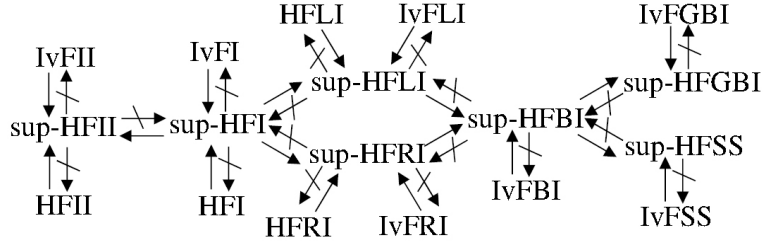


Figure 1

sup-type of HFSs in semigroups

we have characterized SSs, GBIds, BIds, LIds, and RIds in terms of the CIvFS, CHFS, and $\text{sup}(\nabla, \blacktriangle)$ -CHFS.

In our future study of semigroups and related structures, the following objectives were considered:

- to obtain further HFSs results regarding the significance of the supremum of its images,
- to characterize semigroups in terms of sup-types of HFSs,
- use HFSs in the meaning of the supremum of its images to define new types of neutrosophic sets in semigroups and related structures.
- to define anti-types of sup-HFSSs, sup-HFGBIs, sup-HFBIs, sup-HFLIs, sup-HFRIs, sup-HFIIs and sup-HFIs in semigroups.

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