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Some results on the graph associated to centralizer of elements of a group

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Abstract

The aim of this paper is to give some results on the graph associated to a finite group G and subset S of $G - Z(G)$ as defined in [9] with vertex set consisting are all elements in $G - (Z(G) \cup S)$. Two distinct vertices x and y are adjacent if and only there exists an element $s \in S$ such that $[x, s] = [y, s] = 1$, where $[x, s] = x^{-1} s^{-1} xs$ is the commutator of elements x and s. The results stated here are mostly new or an improvement of results give in the above paper.

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1. Introduction

The graph theory is a fundamental subject in mathematics and computer science that studies relationships between objects using vertices (nodes) and edges (connections). It is crucial for modeling real-world problems such as social networks, transportation systems, and internet routing. Many topics are explored in graph theory, including graph coloring (with new coloring methods introduced in [14, 15]), matching theory [10], planar graphs [7], topological indices [12], spectral graph theory [6], random graphs [4], and more.

An alternative perspective on graph theory involves its applications to other branches of mathematics. In this paper, we focus on the intersection of **graph theory** and **algebra**. Graph theory provides a powerful tool for studying algebraic structures. Numerous articles associate graphs with groups, rings, modules, semigroups, and other algebraic structures, where graph properties are analyzed in terms of algebraic properties, or vice versa. For instance, commuting and non-commuting graph of finite groups [1, 8], coprime graph of finite groups [11], zero divisor graphs [2], and Jacobson graphs of finite rings [3]. One of the interesting graphs associated to centralizer of element in a finite group G is a graph with vertex set $G-(Z(G)\cup S)$, where S is a non-empty subset of $G-Z(G)$. Two distinct vertices x and y are adjacent if and only if there exists an element $s\in S$ such that $[x,s]=1=[y,s]$. This graph defined by Johari and Khashyarmanesh, in [9] and they stated some results of the graph for instance connectivity, girth and planarity when S is a singleton set. In this paper, we will improve some of these results for general size of subset S (not necessarily one). Moreover, we will state some new results on clique, independence and dominating numbers. Also, two new graphs similar to this graph are introduced at the end.

This paper is organized in three sections. The remaining of this section is devoted to recalling some basic concepts in graph theory. In section 2, we give some results on the connectivity and invariant graph parameters of the graph for arbitrary subset S of $G-Z(G)$. Planarity, outer planarity and defining the two new graphs are discussed in section 3.

A graph Γ is defined by a pair (V,E), where V is the set of vertices and E the set of edges. We consider only *simple* graphs, meaning no loops or multiple edges are allowed. A vertex $x\in V$ is called *isolated* if it has no neighbors, and a graph is *empty* if all its vertices are isolated (i.e., $E=\emptyset$). A *path* P_n of length $n-1$ is a sequence of distinct vertices $x_1,\dots,x_n$ such that consecutive vertices are adjacent. If any two vertices in Γ are connected by a path, Γ is *connected*; otherwise, it is *disconnected*. A *cycle* C_n is a connected graph with n vertices, each having exactly two neighbors. The smallest cycle, C_3, is a triangle. The *degree* of a vertex x, denoted $\deg(x)$, counts its neighbors, and the *girth* $\mathrm{gr}(\Gamma)$ is the length of the shortest cycle in Γ. The *complete graph* K_n has every pair of vertices adjacent. An *independent set* is a subset $X\subseteq V$ with no edges between its vertices, and the *independence number* $\alpha(\Gamma)$ is the size of the largest such set. A *dominating set* $D\subseteq V$ satisfies that every vertex outside D has at least one neighbor in D, and the *domination number* $\gamma(\Gamma)$ is the smallest size of such a set.

vertices so that adjacent vertices have distinct colors. The *complement* $\overline{\Gamma}$ shares V with Γ, but edges exist precisely where Γ does not. For $S \subseteq V(\Gamma)$, the *induced subgraph* $\Gamma[S]$ retains all edges of Γ between vertices in S. Throughout, G denotes a finite group, and all graphs are undirected and simple. Standard terminology follows [5] and [13].

2. Basic results on $\Gamma(G, S)$

First, let us remind definition of graph $\Gamma(G, S)$ given in [9] as the following.

Definition 2.1: Let G be a group with center $Z(G)$. Let S be a non-empty subset of $G - Z(G)$. Then the graph associate to group G and subset S , denoted by $\Gamma(G, S)$ is an undirected simple graph with vetex set $G - (Z(G) \cup S)$. Two distinct vertices x and y in $G - (Z(G) \cup S)$ are adjacent if and only if there exists an element $s \in S$ such that $[x, s] = 1 = [y, s]$. We may denote the vertex set $V(\Gamma(G, S))$ by V for shorten notation.

Now, we state the following proposition from [9] and our attempts to give an improvement for it.

Theorem 2.2: (Proposition 2.3 in [9]) *Let G be a group with center $Z(G)$, and let $S = \{g\}$ for some $g \in G \backslash Z(G)$. Then the graph $\Gamma(G, S)$ satisfies:*

(i) $|V(\Gamma(G, S))| \geq 3$;

(ii) $\Gamma(G, S)$ is disconnected and contains at least one isolated vertex;

(iii) For $x \in V(\Gamma(G, S))$:

- $\deg(x) = 0$ implies $x \in C_G(g)$;
- $\deg(x) \in \{0, |C_G(g)| - |Z(G)| - 2\}$.

Now, if S is not a singleton set and $|S| \geq 2$, then the situation is a little bit different and we may have different results. In the following lemma, we show that part (i) the above proposition is still true.

Lemma 2.3: *Let G be a non-abelian group, $S = \{a, b\}$ be a subset of $G - Z(G)$ and $V = V(\Gamma_{G,S})$ be a vertex set of the graph. Then $|V(\Gamma(G, S))| \geq 3$.*

Proof: On the contrary, we assume that $|V| \leq 2$. There are two cases:

Case 1: If $|V| = 1$ then we have $3 = |G| - |Z(G)|$. Or equivalently $\frac{|Z(G)|+3}{|Z(G)|} = \frac{|G|}{|Z(G)|}$. On the other hand, for the non-abelian group G, we have $\frac{|G|}{|Z(G)|} \geq 4$ which implies that $|G| = 4$. It is a contradiction.

Case 2: If $|V| = 2$ then we have $4 = |G| - |Z(G)|$ and so $\frac{|Z(G)|+4}{|Z(G)|} = \frac{|G|}{|Z(G)|}$. Again, $\frac{|G|}{|Z(G)|} \geq 4$ for the non-abelian group G. Consequently, $|G| = 5$ which is a contradiction.

In the case that $|S| = 3$, then we have a different lower bound for the number of vertices. The following lemma states that if $|S| = 3$, then $|V(\Gamma(G,S)| \geq 2$.

Lemma 2.4: *Let G be a non-abelian group and $S = \{a,b,c\}$ be a subset of $G - Z(G)$. Then $|V(\Gamma(G,S))| \geq 2$.*

Proof: We assume that $|V| \leq 1$. Then by the same method as lemma 2.3 we have two cases:

Case 1: If $|V| = 0$ then we have we have $3 = |G| - |Z(G)|$ and so $|G| = 4$ which is a contradiction.

Case 2: If $|V| = 1$ then $4 = |G| - |Z(G)|$ and therefor $|G| = 5$ which is a contradiction. Hence, $|V(\Gamma_{G,S})| \geq 2$ as required

If $|S| \geq 4$, different situations may occur. For example, if $G \cong S_6$ and $|S| = 4$, then the graph has only one isolated vertex and if $|S| = 5$, in this case, the graph has no vertices.

Therefore, the number of vertices of the graph or the lack of vertices of the graph depends on the number of elements in G, $Z(G)$ and S. Now, since $G \neq Z(G)$ we have $|G:Z(G)| = k$ there are elements $x_2, x_3, \ldots, x_k \in G - Z(G)$ such that $G = Z(G) \cup x_2Z(G) \cup \ldots \cup x_kZ(G)$ or equivalently $G - Z(G) = \bigcup_{i=2}^{k} x_iZ(G)$. Thus $G - (S \cup Z(G)) = \bigcup_{i=2}^{k} x_iZ(G) - S = \bigcup_{i=2}^{k} (x_iZ(G) - S)$.

We claim that for some j, $2 \leq j \leq k$, $x_iZ(G)ÚS$. On the contrary, if $x_iZ(G) \subseteq S$ for all i, $2 \leq i \leq k$, then we will have $G - Z(G) = \bigcup_{i=2}^{k} x_iZ(G) \subseteq S$ and consequently, $S = G - Z(G)$. Now, if S is a proper subset of $G - Z(G)$, then we can conclude that $V(\Gamma(G,S)) = G - (S \cup Z(G)) \neq \emptyset$ and we have atleast one vertex.

Therefore, we may always assume that S is a proper subset of $G - Z(G)$. Moreover, if G is abelian, then $G = Z(G)$ and so $G - Z(G) = \emptyset$. Thus $S = \emptyset$. Hence, we should also assume that G is not abelian to have at least a vertex in $\Gamma(G,S)$. For this reason, from now on we always assume without mentioning that G is not abelian and S is a proper subset of $G - Z(G)$.

Theorem 2.5: *Let G be a (non-abelian) group and $S = \{a_1, a_2, \ldots, a_k\}$ be a (proper) subset of $G - Z(G)$. Then $\Gamma(G,S)$ is an empty graph if and only if $|C_G(a_i) - (S \cup Z(G))| \leq 1$ for every i, $1 \leq i \leq k$.*

Proof: If $\Gamma(G,S)$ is an empty graph, then $|E(\Gamma(G,S))| = 0$. Also, $|V(\Gamma(G,S))| \neq 0$, as S is proper and G is non-abelian. Now, we claim that $|C_G(a_i) - (S \cup Z(G))| \leq 1$ for every i, $1 \leq i \leq k$. On the contrary, we assume that $|C_G(a_i) - (S \cup Z(G))| \geq 2$ for some $a_j \in S$, then still it has at least two vertices which are adjacent. So we will have at least one edge that is a contradiction. Conversely, if $|C_G(a_i) - (S \cup Z(G))| \leq 1$ for every i, $1 \leq i \leq k$. We have the following two cases:

Case 1: $|C_G(a_i) - (S \cup Z(G))| = 1$, for all $1 \leq i \leq k$. In this case we can see that there exists only one isolated vertex in $C_G(a_i) - (S \cup Z(G))$. Thus $V(\Gamma(G,S))$ consists of at least k isolated vertices.

Case 2: $|C_G(a_i) - (S \cup Z(G))| = 0$, for $1 \leq i \leq k$. In this case graph $\Gamma(G,S)$ has no vertex in $C_G(a_i) - (S \cup Z(G))$.

Theorem 2.6: *Suppose that G is a (non-abelian) group and $S = \{a_1, a_2, \ldots, a_k\}$ is a (proper) subset of $G - Z(G)$. If x is an isolated vertex and $I \cup J = \{1,2,\ldots,k\}$ and $I \cap J = \emptyset$ then one of the following statements hold:*

(i) $x \in \bigcap_{i \in I} C_G(a_i) - \bigcup_{j \in J} C_G(a_j)$ and $|C_G(a_i) - (Z(G) \cup S)| = 1$

(ii) $x \in G - \bigcup_1^k C_G(a_i)$

Proof:

(i) Assume that $x \in \bigcup_1^k C_G(a_i)$. Since x is a vertex, then $x \in C_G(a_i) - Z(G)$ for some $1 \leq i \leq k$. On the other hand, x is an isolated vertex and so there is no other vertex that adjacent to x. Thus $|C_G(a_i) - Z(G)| = 2$. Conversely, if for some $1 \leq i \leq k$ we have $x \in C_G(a_j) - Z(G)$ then clearly $x \in \bigcup_1^k C_G(a_i)$ as required.

(ii) Assume that $x \in G - \bigcup_1^k C_G(a_i)$ then $x \notin C_G(a_i)$ for every $1 \leq i \leq k$. So $[x, a_i] \neq 1$ for every $1 \leq i \leq k$. Conversely, if $[x, a_i] \neq 1$ for every $1 \leq i \leq k$, it is obvious that $x \notin C_G(a_i)$ for every $1 \leq i \leq k$ and so $x \notin \bigcup_1^k C_G(a_i)$. Hence $x \in G - \bigcup_1^k C_G(a_i)$.

Theorem 2.7: *For a given non-abelian group G and S as a proper subset of $G - Z(G)$, we have*

(i) if $S = \{x\}$ then $\Gamma(G,S) = K_{|V_1|} \cup \overline{K_{|V_2|}}$
where $|V_1| = |C_G(x)| - |Z(G)| - 1$ and $|V_2| = |V| - |V_1|$.

(ii) If $S = \{x, y\}$ such that $x \neq y$ then $\Gamma(G,S) = \left[K_{|V_3|} + (K_{|V_1|} \cup K_{|V_2|})\right] \cup \overline{K_t}$ where $|V_1| = |C_G(x) - (S \cup [C_G(x) \cap C_G(y)])|$,
$|V_2| = |C_G(y) - (S \cup [C_G(x) \cap C_G(y)])|$,
$|V_3| = |C_G(x) \cap C_G(y) - (S \cup Z(G))|$
and $t = |V| - (|V_1| + |V_2| + |V_3|)$.

Proof:

(i) $V = G - (\{x\} \cup Z(G)) = V_1 \cup V_2$, where $V_1 = C_G(x) - (Z(G) \cup \{x\})$ and $V_2 = V - V_1$. It is easy to see that every two vertices in V_1 are adjacent and so, the induced subgraph to to set V_1 is a complete graph $K_{|V_1|}$.

Moreover, every vertex in V_2 is an isolated vertex. Thus, the induced subgraph to set V_2 consist of $|V_2|$ isolated vertices. Hence, $\Gamma(G,S) = K_{|V_1|} \cup \overline{K_{|V_2|}}$.

(ii) We may give a partition for V as the following sets. $V = V_1 \cup V_2 \cup V_3 \cup V_4$, where $V_1 = C_G(x) - (S \cup [C_G(x) \cap C_G(y)])$, $V_2 = C_G(y) - (S \cup [C_G(x) \cap C_G(y)])$,

$$V_3 = C_G(x) \cap C_G(y) - (S \cup Z(G)) \text{ and } V_4 = V - (V_1 \cup V_2 \cup V_3).$$

Every two vertices in V_1, V_2 and V_3 are adjacent and every vertex in V_3 is adjacent to every vertex in V_1 and V_2. Moreover, all vertices in V_4 are isolated vertices. Thus, we have $\Gamma(G,S) = \left[K_{|V_3|} + (K_{|V_1|} \cup K_{|V_2|})\right] \cup \overline{K_{|V_4|}}$

The following example is an evidence for the above theorem.

Example 2.8: Let $G = S_4$. We know that $Z(G) = \{1\}$ and centralizer of elements $(12)(34)$ and $(13)(24)$ of G are the following sets :

$$C_G((12)(34)) = \{1, (12), (34), (12)(34), (13)(24), (14)(23), (1423), (1324)\}$$
$$C_G((13)(24)) = \{1, (13), (24), (13)(24), (12)(34), (14)(23), (1432), (1234)\}$$

Now, one can see that if $x = (13)(24)$ and $S = \{x\}$ then $\Gamma(G,S) = K_6 \cup \overline{K_{16}}$

If $S = \{x\}$ and $x = (13)(24)$ we have $\Gamma(G,S) = K_6 \cup \overline{K_{16}}$. Moreover, if $y = (13)(24)$ and $S = \{x, y\}$ then $\Gamma(G,S) = [K_1 + (K_4 \cup K_4)] \cup \overline{K_{12}}$ which is an evidence for 2.7 (see Figures 1 and 2).

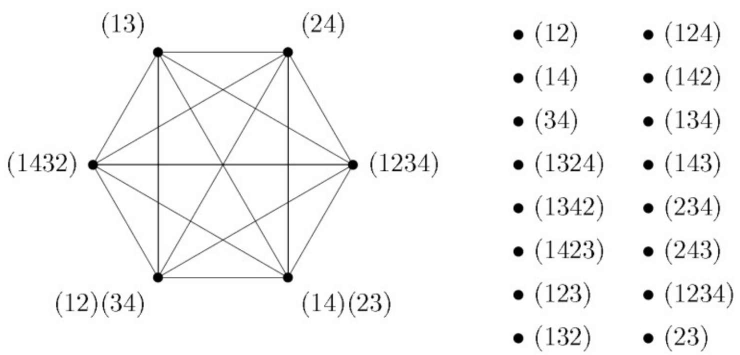

Figure 1
$\boldsymbol{x = (13)(24)}$

It can also be checked for cases where S has more than two members.
The following theorem is a generalization of Proposition 2.3. part (iii) in [9].

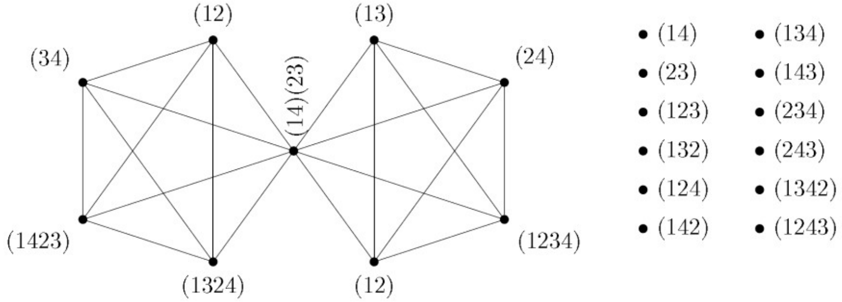

Figure 2
$\boldsymbol{x = (12)(34)}$ **and** $\boldsymbol{y = (13)(24)}$

Theorem 2.9: *Let G be a non-abelian group and $S = \{a_1, a_2, \dots, a_k\}$ be a proper subset of $G - Z(G)$ and $x \in V$.*

(i) If $x \in C_G(a_i) - \left(\bigcup_{j=1, i\neq j}^{k} C_G(a_j)\right)$ for all $1 \leq i \leq k$ then $degx = |C_G(a_i)| - |Z(G)| - 2$.

(ii) If $S' \subseteq S$ and $x \in C_G(S') - \bigcup_{a\in S-S'} C_G(a)$ then $degx = |\bigcup_{b\in S'} C_G(b)| - |Z(G)| - |S'| - 1$.

(iii) If $x \notin C_G(a_1) \cup C_G(a_2) \cup \dots \cup C_G(a_k)$ then $degx = 0$.

Proof:

(i) Assume that $x \in C_G(a_i) - \left(\bigcup_{j=1, i\neq j}^{k} C_G(a_j)\right)$ for all $1 \leq i \leq k$. Then $C_G(a_i)$ contains at least x, a_i and $Z(G)$.
Thus x is adjacent to all elements in $C_G(a_i) - (Z(G) \cup \{x, a_i\})$ and so $degx = |C_G(a_i)| - |Z(G)| - 2$.

(ii) It is very similar to the proof of part (i).

(iii) Let $x \notin C_G(a_1) \cup C_G(a_2) \cup \dots \cup C_G(a_k)$ then $x \notin C_G(a_i)$ for all $1 \leq i \leq k$ and the proof follows.

The following theorem is a generalization of Lemma 2.6. in [9].

Theorem 2.10: *Assume that G is a non-abelian group and $S = \{a_1, a_2, \dots, a_k\}$ is a proper subset of $G - Z(G)$. The following propositions are equivalent :*

(i) $gr(\Gamma(G, S)) = 3$

(ii) *For some $1 \leq i \leq k$ we have $|C_G(a_i) - (S \cup Z(G))| \geq 3$*

(iii) *There are vertices x_1, x_2, x_3 such that for some $a_i \in S$ with $1 \leq i \leq k$ we have $a_i \in C_G(\langle x_1, x_2, x_3 \rangle)$*

Proof: Assume that $gr(\Gamma(G, S)) = 3$, then there exist three vertices x_1, x_2 and x_3 such that x_1, x_2, $x_3 \in C_G(a_i)$ for some $1 \leq i \leq k$. Thus $C_G(a_i) - (S \cup$

$Z(G)$) contains this three vertices and we get (ii). To prove (ii) to (iii), it is obvious that $[x_1, a_i] = [x_2, a_i] = [x_3, a_i] = 1$ which implies that $a_i \in C_G(\langle x_1, x_2, x_3 \rangle)$. (iii) to (i) is obvious.

The following theorem is a generalization of Corollary 2.7. in [9].

Theorem 2.11: *If G is a non-abelian group and $S = \{a_1, a_2, \ldots., a_k\}$ is a proper subset of $G - Z(G)$, then $\Gamma(G, S)$ is an acyclic graph if both of the following conditions hold:*

(i) $S \subseteq \{x \in G | x \notin Z(G) \text{and} |C_G(x) - Z(G)| \leq 3\}$

(ii) if $|C_G(a_i) - (Z(G) \cup \{a_i\})| = |C_G(a_j) - (Z(G) \cup \{a_j\})| = |C_G(a_l) - (Z(G) \cup \{a_l\})| = 2$ and $|(C_G(a_i) \cap C_G(a_j)) - (S \cup Z(G))| = |(C_G(a_j) \cap C_G(a_l)) - (S \cup Z(G))| = 1$ then $|(C_G(a_i) \cap C_G(a_l)) - (S \cup Z(G))| = 0$ where i, j, l are distinct and $1 \leq i, j, l \leq k$.

Proof: Since the number of elements of $C_G(x) - Z(G)$, is less than or equal to 3 then after removing x from $C_G(x) - Z(G)$, we have $|C_G(x) - (Z(G) \cup \{x\})| = 1$ or 2. If $|C_G(x) - (Z(G) \cup \{x\})| = 1$ then clearly the graph is acyclic.

Otherwise, for every i, j, l that are distinct and $1 \leq i, j, l \leq k$ if we have $x \in (C_G(a_i) \cap C_G(a_j)) - (S \cup Z(G))$ and $y \in (C_G(a_j) \cap C_G(a_l)) - (S \cup Z(G))$ then by condition (ii) there is not any element in $(C_G(a_i) \cap C_G(a_l)) - (S \cup Z(G))$. Hence, $\Gamma(G, S)$ is acyclic graph.

The following theorem is a generalization of Theorem 2.5. in [9].

Theorem 2.12: *Let G be a non-abelian group and $S = \{a_1, a_2, \ldots., a_k\}$ be a proper subset of $G - Z(G)$. Then $\Gamma(G, S)[C_G(a_i) - (S \cup Z(G))]$ is isomorphism to $K_{|C_G(a_i) - (S \cup Z(G))|}$, for some $a_i \in S$ if and only if $|C_G(a_i)| > |C_G(a_i) \cap S| + |Z(G)|$.*

Proof; Assume that $|C_G(a_i)| > |C_G(a_i) \cap S| + |Z(G)|$ for some $a_i \in S$, then $|C_G(a_i) - (S \cup Z(G))| \geq 1$. If $|C_G(a_i) - (S \cup Z(G))| = 1$ it is obvious that $\Gamma(G, S)[C_G(a_i) - (S \cup Z(G))]$ is isomorphism to K_1, and if $|C_G(a_i) - (S \cup Z(G))| = 2$ we have $\Gamma(G, S)[C_G(a_i) - (S \cup Z(G))]$ is isomorphism to K_2 and similarly if $|C_G(a_i) - (S \cup Z(G))| > 2$ then $\Gamma(G, S)[C_G(a_i) - (S \cup Z(G))]$ is isomorphism to $K_{|C_G(a_i) - (S \cup Z(G))|}$, as required. Conversely, if $\Gamma(G, S)[C_G(a_i) - (S \cup Z(G))]$ is isomorphism to $K_{|C_G(a_i) - (S \cup Z(G))|}$ then $|C_G(a_i) - (S \cup Z(G))| \geq 1$. Since $|C_G(a_i) - (S \cup Z(G))| = |C_G(a_i) - [(S \cup Z(G)) \cap C_G(a_i)]| = |C_G(a_i)| - |C_G(a_i) \cap S| - |Z(G)|$ we have $|C_G(a_i)| \geq |C_G(a_i) \cap S| + |Z(G)| + 1$. Hence, $|C_G(a_i)| > |C_G(a_i) \cap S| + |Z(G)|$ and the proof is completed.

In the following theorem, we investigate whenever the graph is a star.

Theorem 2.13: *Let G be a non-abelian group and $x, a_1, \ldots, a_m, y_1, \ldots, y_m \in G - Z(G)$. Then the graph $\Gamma(G,S)$ is isomorphism to the star graph with at most $m+1$ vertices if both of the following conditions hold:*

(i) $S = (G - Z(G)) - \{x, y_1, \ldots, y_m\}$

(ii) $[C_G(a_1) \cap \ldots \cap C_G(a_m)] - S \cup \{a_1, \ldots, a_m\} = \{x\}$ and for every $1 \leq i \leq m$ we have $y_i \in C_G(a_i) - Z(G) \cup \{x\}$.

Proof: Since $x, y_1, \ldots, y_m \in G - Z(G)$ and according to the condition (i), the graph $\Gamma(G,S)$ has at most $m+1$ vertices. And according to the condition (ii), for every $1 \leq i \leq m$ there exists at most m centralizers $C_G(a_i)$ so that two members $x, y_i \in C_G(a_i)$ are vertices of $\Gamma(G,S)$ and every y_i is adjacent to x. Hence, the graph $\Gamma(G,S)$ is isomorphism to the star graph with at most $m+1$ vertices, as required.

In the following theorem, we determine the clique number of graph $\Gamma(G,S)$.

Theorem 2.14: *Let G be a non-abelian group and $S = \{a_1, a_2, \ldots, a_k\}$ be a proper subset of $G - Z(G)$. Then the clique number of the graph $\Gamma(G,S)$ is*

$$\omega(\Gamma(G,S)) = max\{\left|C_G(a_i) - \left(S \cup Z(G)\right)\right| \text{ such that } 1 \leq i \leq k\}$$

Proof: We can see that $\omega(\Gamma(G,S)) \geq max\{|C_G(a_i) - (S \cup Z(G))|; 1 \leq i \leq k\}$, by theorem 2.12, and if $\omega(\Gamma(G,S)) > max\{|C_G(a_i) - (S \cup Z(G))|; 1 \leq i \leq k\}$, then there exist some $x \in V(\Gamma(G,S))$ such that $x \notin C_G(a_i)$ for every $1 \leq i \leq k$ and x adjacent to all of vertices in clique. So, for every $y \in C_G(a_i)$ implies that $[x, a_i] = [y, a_i] = 1$ and which is a contradiction.

Hence $\omega(\Gamma(G,S)) = max\{\left|C_G(a_i) - \left(S \cup Z(G)\right)\right| \text{ such that } 1 \leq i \leq k\}$, as required.

In the following lemma, we find the independence number of graph $\Gamma(G,S)$ when S is a singleton set.

Lemma 2.15: *Let G be a non-abelian group and $S = \{a\}$ be a proper subset of $G - Z(G)$. Then the independence number of graph $\Gamma(G,S)$ is*

(i) $\alpha(\Gamma(G,S)) = |V|$ if $|C_G(a) - (S \cup Z(G))| \leq 1$

(ii) $\alpha(\Gamma(G,S)) = |V - [C_G(a) - (S \cup Z(G))]| + 1$ if $|C_G(a) - (S \cup Z(G))| > 1$.

Proof: If $|C_G(a) - (S \cup Z(G))| \geq 2$, then the graph $\Gamma(G,S)$ has a clique with at least two vertices. Therefore, the sum of isolated vertices and one vertex as a representative of the clique should be counted to find the independence number and the proof (ii) follows. If $|C_G(a) - (S \cup Z(G))| \leq 1$, then the graph has no clique. Hence, the size of the independence number is equal to the number of vertices.

In the following lemma, we investigate the independence number of graph $\Gamma(G,S)$ when S is has two members.

Lemma 2.16: *Let G be a non-abelian group and $S=\{a,b\}$ be a proper subset of $G-Z(G)$. Then the independence number of graph $\Gamma(G,S)$ is one of the following values:*

(i) $\alpha(\Gamma(G,S))=|V-[C_G(a)\cup C_G(b)-(S\cup Z(G))]|+2$, if $|C_G(a)-(S\cup Z(G))|\geq 2$ and $|C_G(b)-(S\cup Z(G))|\geq 2$.

(ii) $\alpha(\Gamma(G,S))=|V-[C_G(a)-(S\cup Z(G))]|+1$, if $|[C_G(a)-(S\cup Z(G))]|\geq 2$ and $|C_G(b)-(S\cup Z(G))|\leq 1$.

(iii) $\alpha(\Gamma(G,S))=|V-[C_G(b)-(S\cup Z(G))]|+1$, if $|[C_G(b)-(S\cup Z(G))]|\geq 2$ and $|C_G(a)-(S\cup Z(G))|\leq 1$.

(iv) $\alpha\big(\Gamma(G,S)\big)=|V|$, if $|C_G(a)-(S\cup Z(G))|\leq 1$ and $|C_G(b)-(S\cup Z(G))|\leq 1$.

Proof: (i) If $[(C_G(a)\cap C_G(b))-(S\cup Z(G))]=\emptyset$ therefore the biggest independent set contains isolated vertices and one representative from each clique. So $\alpha(\Gamma(G,S))=|V-[C_G(a)\cup C_G(b)-(S\cup Z(G))]|+2$.

If $[(C_G(a)\cap C_G(b))-(S\cup Z(G))]\neq\emptyset$ so we claim that there exist two vertices $x\in C_G(a)-(S\cup Z(G))$ and $y\in C_G(b)-(S\cup Z(G))$ such that $x,y\notin (C_G(a)\cap C_G(b))-(S\cup Z(G))$, otherwise $a=b$ and is a contrary. Then the biggest independent set contains isolated vertices and x and y which are representative from each clique.

Hence, $\alpha(\Gamma(G,S))=|V-[C_G(a)\cup C_G(b)-(S\cup Z(G))]|+2$ as required. (ii) In this case, the graph has a clique with at least two vertices and therefore the biggest independent set contains isolated vertices and one representative from members of $C_G(a)-(S\cup Z(G))$. So, $\alpha(\Gamma(G,S))=|V-[C_G(a)-(S\cup Z(G))]|+1$ as required.

So $\alpha(\Gamma(G,S))=|V-[C_G(a)\cup C_G(b)-(S\cup Z(G))]|+1$

(iii) The proof is similar to part (ii).

(iv) It follows from Lemma 2. 15.

In the following theorem, when S is a set with more than two members, we can give a lower bound for the independence number of graph $\Gamma(G,S)$.

Theorem 2.17: *Suppose that G is a non-abelian group and $S=\{a_1,a_2,\dots,a_k\}$ is a proper subset of $G-Z(G)$. Then we have:*

$$\alpha(\Gamma(G,S))\geq |V-[C_G(a_1)\cup C_G(a_2)\cup\ldots\cup C_G(a_k)]|+k$$

Proof: It is similar to the proof of Lemma 2.16.

The following proposition, is a consequence of Lemma 2.16. which gives a result for domination number of $\Gamma(G,S)$.

Proposition 2.18: *Let G be a non-abelian group and $S = \{a\}$ be a proper subset of $G - Z(G)$. Then $\gamma(\Gamma(G,S)) = \alpha(\Gamma(G,S))$, where $\gamma(\Gamma(G,S))$ is the domination number of $\Gamma(G,S)$.*

In the following theorem, we check states which the graph is connected.

Theorem 2.19: Let G be a non-abelian group and $S = \{a_1, a_2, \dots, a_k\}$ be a proper subset of $G - Z(G)$. Then in the following cases the graph $\Gamma(G,S)$ is disconnected.

(i) $|C_G(a_i) - (G - Z(G))| = 1$ for some $a_i \in S$.

(ii) $V - \bigcup_1^k C_G(a_i) \neq \emptyset$

(iii) If $|C_G(a_i) - (S \cup Z(G))| \geq 2$ and $|C_G(a_j) - (S \cup Z(G))| \geq 2$ then for some $a_i, a_j \in S$ such that $1 \leq i,j \leq k$, $|[C_G(a_i) \cap C_G(a_j)] - (S \cup Z(G))| = 0$.

Proof: If condition (i) is true, the graph has no isolated vertex, and if condition (ii) is true, both cliques have a commonality and are connected.

The following theorem is a generalization of Theorem 3.1. in [9].

Theorem 2.20: *Let G be a non-abelian group and $S = \{a_1, a_2, \dots., a_k\}$ be a subset of $G - Z(G)$. Then the graph $\Gamma(G,S)$ is planner if the following conditions exist:*

(i) *For every $1 \leq i \leq k$ to have $|C_G(a_i) - (S \cup Z(G))| \leq 4$.*

(ii) $|(C_G(a_i) \cap C_G(a_j)) - (S \cup Z(G))| \leq 1$ *such that $1 \leq i \neq j \leq k$.*

Proof: According to the definition of planar graph, $\Gamma(G,S)$ must not have subgraphs isomorphism with K_5 and $K_{3,3}$. For this purpose, if condition (i) is satisfied, the graph does not have a subdivision isomorphism with K_5. By establishing condition (ii), the graph does not have a subdivision isomorphism with $K_{3,3}$, so the graph will be planner.

In the following theorem, we check conditions which we have an outer planar graph.

Theorem 2.21: *Let G be a non-abelian group and $S = \{a_1, a_2, \dots., a_k\}$ be a proper subset of $G - Z(G)$. Then the graph $\Gamma(G,S)$ is outer planar graph if the following conditions exist:*

(i) *For every $1 \leq i \leq k$ to have $|C_G(a_i) - (S \cup Z(G))| \leq 3$.*

(ii) $|(C_G(a_i) \cap C_G(a_j) \cap C_G(a_l)) - (S \cup Z(G))| \leq 1$ *for every $1 \leq i,j,l \leq k$ such that all three are distinct.*

Proof: According to the definition, for $\Gamma(G,S)$ to be outer planar, it must not have subgraphs isomorphism with K_4 and $K_{2,3}$. Establishing condition (i) makes

$\Gamma(G,S)$ not have a subgraph isomorphism with K_4 and establishing condition (ii) makes $\Gamma(G,S)$ not have a subgraph isomorphism with $K_{2,3}$.

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