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Soft rings based on soft binary operations

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Abstract

This paper aims to study soft rings using the concept of soft binary operations as a quadruple ($\mathfrak{R}$, $*$, $\triangle$, P) involving of a non-empty set R and soft binary operations ($*$, P) and ($\triangle$, P) satisfying all the ring axioms in softness settings. We study soft rings, along with related concepts like soft subrings and soft ideals. Their properties and relationships between soft rings and classical rings are examined, and elucidated by examples. Furthermore, soft Boolean rings and prime soft ideals are defined and associated theorems are proved.

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1. Introduction

Molodtsov [1] in 1999 proposed the theory of soft sets for modelling uncertainty and vagueness. He also revealed applications in areas like stability and regularization, game theory, operations research and soft analysis. These days, studies on soft set theory are developing and gained significant attention in various fields, including Mathematics, Computer Science, Economics, Engineering, Environment and Decision Making [2], [3], [4], [5], [6], [7], [8] defined soft groups as a soft set (F, P) over ordinary group G where for each $\theta \in P$, $\mathrm{F}(\theta)$ is a subgroup of G. [9] introduced soft groups using soft elements as defined in [10]. [11] extended and defined soft groups on soft elements and binary operations.

In the above discussions, groups are defined based on classical binary operation. However, Weldetekle et al. [12] defined soft groups $(\mathrm{G}, *, P)$ on a non-empty set G and soft binary operation $(*, P)$ satisfying group axioms using softness settings introduced by [13]. Based on Molodtsov's definition of soft sets, [14], [15] introduced soft subrings and soft ideals of a ring.

This paper introduces soft rings based on soft binary operations as a quadruple $(\mathfrak{R}, *, \triangle, P)$ consisting of a non-empty set $\mathfrak{R}$ and soft binary operations $(*, P)$ and $(\triangle, P)$ satisfying all the ring axioms in a softness settings. We state and prove theorems on soft rings along with related concepts like soft subrings and soft ideals, and their properties are illustrated by examples.

In Section 2, some important definitions and related results on soft set theory are presented. In Section 3, we define soft rings and state illustrative examples. Moreover, we prove the characterization theorem: $(\mathfrak{R}, +, \cdot, P)$ is a soft ring iff $(SE_P(\mathfrak{R}), \tilde{+}, \tilde{\cdot})$ is a ring. In Section 4, we define soft subrings, soft prime and soft maximal ideals.

2. Preliminaries

We briefly recall the following fundamental definitions and related results published on soft set theory which are required in our work. In this paper, unless explicitly stated, U denotes non-empty initial universe set, P set of parameters suitable for U.

Definition 2.1: [1] Given a non-empty set X. A pair (F, P) is called soft set over X if F is a mapping from P to the power set of X.

Definition 2.2: [16] Let (F, P) and (M, P) be soft sets over X.

(1) (F, P) is said to be contained in (M, P), written as $(\mathrm{F}, P) \mathrel{\breve{\subseteq}} (\mathrm{M}, P)$ iff for each $\alpha \in P, q \in \mathrm{M}(\alpha)$ whenever $q \in \mathrm{F}(\alpha)$.

(2) (F, P) is said to be equal to (M, P) written as $(\mathrm{F}, P) \mathrel{\breve{=}} (\mathrm{M}, P)$ iff $(\mathrm{F}, P) \mathrel{\breve{\subseteq}} (\mathrm{M}, P)$ and $(\mathrm{M}, P) \mathrel{\breve{\subseteq}} (\mathrm{F}, P)$.

Definition 2.3: [16] The soft set (F, P) over X with $\mathrm{F}(\alpha) = \{\emptyset\}$ for all $\alpha \in P$ is called the null soft set over X, and denoted by $(0_X, P)$. Moreover, (F, P) with $\mathrm{F}(\alpha) = X$ for all $\alpha \in P$ is called the absolute soft set over X, and denoted by $(1_X, P)$.

Definition 2.4: [17] Let V and W be sets. A soft relation (F, P) from V to W is defined as a soft set over $V \times W$. For $\alpha \in P$ and $(v, w) \in V \times W$, we write $(\alpha, v, w) \in \mathrm{F}$ to mean $(v, w) \in \mathrm{F}(\alpha)$.

Definition 2.5: [13] A soft relation (F, P) from V to W is said to be a soft mapping from V to W if :

(1) for all $\alpha \in P$ and $s \in V$, there exists $t \in W$ such that $(\alpha, s, t) \in \mathrm{F}$;

(2) for all $\alpha \in P, s \in V$ and $t_1, t_2 \in W$, $(\alpha, s, t_1) \in \mathrm{F}$ and $(\alpha, s, t_2) \in \mathrm{F}$ imply $t_1 = t_2$.

Definition 2.6: [12] A soft binary operation on a non-empty set X is a soft mapping from $X \times X$ to X.

Lemma 2.7: [12] If $(\mathrm{G}, *, P)$ is a soft group then for each $\alpha \in P$, $s, t, z \in \mathrm{G}$:

(1) $s *_\alpha t = e_\theta$ iff $t = s^{-\alpha}$ and $s = t^{-\alpha}$;

(2) $s *_\alpha t = z$ iff $s^{-\alpha} * z = t$;

(3) $s *_\alpha t = z$ iff $z * t^{-\alpha} = s$; and

(4) $s *_\alpha t = z$ iff $t^{-\alpha} * s^{-\alpha} = z^{-\alpha}$. Moreover, if G is an abelian soft group then $s *_\alpha t = z \Leftrightarrow s^{-\alpha} * t^{-\alpha} = z^{-\alpha}$.

3. Soft Rings

In modern mathematics, particularly within the framework of soft set theory, a soft ring is a concept introduced to deal with uncertain and imprecise data. The study of soft rings and related structures like soft ideals helps in applying the theory of soft sets to algebraic concepts, providing a new way to analyze and model algebraic systems in the presence of uncertainty. In this section, we define soft rings based on soft binary operations and illustrate by examples.

Definition 3.1: Let $(*, P)$ and $(\triangle, P)$ be soft binary operations on U. Then $(\triangle, P)$ is said to be:

1. *left distributive* over $(*, P)$ if for every $\alpha \in P$ and $x, y, z, u, v, w, p, q \in U$, $x *_\alpha y = u, z \triangle_\alpha u = v$ and $z \triangle_\alpha x = w, z \triangle_\alpha y = p, w *_\alpha p = q$ imply $v = q$.
2. *right distributive* over $(*, P)$ if for all $\alpha \in P$ and $x, y, z, u, v, w, p, q \in U$, $x *_\alpha y = u, u \triangle_\alpha z = v$ and $x \triangle_\alpha z = w, y \triangle_\alpha z = p, w *_\alpha p = q$ imply $v = q$.
3. *distributive* over $(*, P)$ if it is both left and right distributive over $(*, P)$.

Definition 3.2: Let $(*, P)$ and $(\triangle, P)$ be soft binary operations on a non-empty set $\mathfrak{R}$. Then $(\mathfrak{R}, *, \triangle, P)$ is called a *soft ring* if the following properties are satisfied:

(i) $(\mathfrak{R}, *, P)$ is an abelian soft group,
(ii) $(\triangle, P)$ is associative, and
(iii) $(\triangle, P)$ is distributive over $(*, P)$

Example 3.3: Let $\mathfrak{R} = \mathbb{R}$ and $P = \mathbb{Z}$. For each $\theta \in P$, $p, q, z \in \mathfrak{R}$, define soft binary operations $(*, P)$ and $(\triangle, P)$ as:

$$p *_\theta q = z \iff z = p + q + \theta, \text{ and}$$

$$p \triangle_\theta q = z \iff z = -\theta.$$

Therefore, $(\mathfrak{R}, *, \triangle, P)$ is a soft ring.

Example 3.4: Suppose $\mathfrak{R} = M_{2x2}(\mathbb{Z})$ and $P = \mathbb{Z}$. Let $\theta \in P$ and $L, M, N \in \mathfrak{R}$ where

$$L = \begin{bmatrix} l_1 & l_2 \\ l_3 & l_4 \end{bmatrix},\ M = \begin{bmatrix} m_1 & m_2 \\ m_3 & m_4 \end{bmatrix}, \text{ and } N = \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix}$$

Define the soft binary operations $(*, P)$ and $(\triangle, P)$ as

$$L *_\theta M = N \text{ iff } \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix} = \begin{bmatrix} l_1 + m_1 & l_2 + m_2 - \theta \\ l_3 + m_3 - \theta & l_4 + m_4 \end{bmatrix}, \text{ and}$$

$$L \triangle_\theta M = N \text{ iff } \begin{bmatrix} n_1 & n_2 \\ n_3 & n_4 \end{bmatrix} = \begin{bmatrix} l_1 m_1 & \theta \\ \theta & l_4 m_4 \end{bmatrix}.$$

Therefore $(\mathfrak{R}, *, \triangle, P)$ is a soft ring.

Remark 3.5: Suppose $(\mathfrak{R}, *, \triangle, P)$ is soft ring.

- The *zero elements* of $\mathfrak{R}$ are denoted by 0_θ for $\theta \in P$.
- For $r \in \mathfrak{R}$, the inverses of r are denoted by $-r^\theta$ for all $\theta \in P$.
- The soft binary operation $(*, P)$ is known as soft addition, and is denoted by $(+, P)$.
- The soft binary operation $(\triangle, P)$ is known as soft multiplication, and is denoted by $(\cdot, P)$.

Lemma 3.6: (Elementary Properties of Soft Rings) *Consider a soft ring* $(\mathfrak{R}, +, \cdot, P)$, $\theta \in P$ *and* $x, z, a, b, u, v, w \in \mathfrak{R}$. *Then*

(a) $0_\theta \cdot_\theta x = 0_\theta$ and $x \cdot_\theta 0_\theta = 0_\theta$.

(b) *If* $a \cdot_\theta b = z, a \cdot_\theta - b^\theta = u$ *and* $-a^\theta \cdot_\theta b = v$ *then* $u = v = -z^\theta$.

(c) *If* $-a^\theta \cdot_\theta - b^\theta = z$ *and* $a \cdot_\theta b = w$ *then* $z = w$.

Definition 3.7: Suppose $(\mathfrak{R}, +, \cdot, P)$ is soft ring.

1. $(\mathfrak{R}, +, \cdot, P)$ is said to be a commutative soft ring if $(\cdot, P)$ is commutative on $\mathfrak{R}$. That is for all $\theta \in P, a, b, x, y \in \mathfrak{R}, a \cdot_\theta b = x$, $b \cdot_\theta a = y$ imply $x = y$.
2. If for every $\theta \in P$, there exits e in $\mathfrak{R}$ such that $y \cdot_\theta e = y$ and $e \cdot_\theta y = y$ for every $y \in \mathfrak{R}$, we call e θ-identity (or θ-unity) element of $\mathfrak{R}$ and denote it by 1_θ. Then, $\mathfrak{R}$ is said to be a soft ring with *unity* if for each $\theta \in P$, $\exists 1_\theta \in \mathfrak{R}$ so that $y \cdot_\theta 1_\theta = y, 1_\theta \cdot_\theta y = y$ for each $y \in \mathfrak{R}$.
3. If $\mathfrak{R}$ is soft ring with unity, $y \in \mathfrak{R}$ is said to be θ*-unit element* of $\mathfrak{R}$ if there is $y^{-\theta}$ in $\mathfrak{R}$ so that $y \cdot_\theta y^{-\theta} = 1_\theta$ and $y^{-\theta} \cdot_\theta y = 1_\theta$.

Notation:

- We call the identity elements of $\mathfrak{R}$ corresponding to $(\cdot, P)$ *unity elements* of $\mathfrak{R}$ and denote them by 1_θ for $\theta \in P$.
- For $r \in \mathfrak{R}$ and $\theta \in P$, the *inverses* of r corresponding to $(\cdot, P)$ are denoted by $r^{-\theta}$.

Lemma 3.8: *Suppose* $(\mathfrak{R}, +, \cdot, P)$ *is a soft ring with unity. Then for each* $\theta \in P$ *and* $a \in \mathfrak{R}$*:*

(i) $a \cdot_\theta - 1_\theta^\theta = -a^\theta$

(ii) $-1_\theta^\theta \cdot_\theta - 1_\theta^\theta = 1_\theta$, where -1_θ^θ stands for $-(1_\theta)^\theta$.

Definition 3.9: For $\kappa \in P$, we define $U(\mathfrak{R})(\kappa) = \{s \in \mathfrak{R}: s \cdot_\kappa s^{-\kappa} = 1_\kappa$ and $s^{-\kappa} \cdot_\kappa s = 1_\kappa$ for some $s^{-\kappa} \in \mathfrak{R}\}$, which is the collection of all κ-unit elements of $\mathfrak{R}$.

Let $(\mathfrak{R}, +, \cdot, P)$ be soft ring with unity. Then

1. $\mathfrak{R}$ is said to be a soft division ring if $\mathfrak{R} \backslash U(\mathfrak{R})(\kappa) = \{0_\kappa\}$ for each $\kappa \in P$.
2. $\mathfrak{R}$ is said to be a *soft field* if $\mathfrak{R}$ is commutative soft division ring.

The following theorem establishes the algebraic structure relationships between soft rings and classical rings. We begin by introducing the concept of soft elements and defining two binary operations on the collection of soft elements. Then, we state and prove a theorem that provides the sufficient and necessary conditions relating soft rings to their classical counterparts, marking an important result in this paper.

Now, we consider the collection of soft elements of $\mathfrak{R}$ in order to state and prove neccesary and sufficient conditions of a quadruple $(\mathfrak{R}, +, \cdot, P)$ to be a soft ring, commutative soft ring, commutative soft ring with unity and a soft Boolean ring.

Remark 3.11: [6] A *soft element* in $\mathfrak{R}$ is a soft set (F, P) over $\mathfrak{R}$ where the cardinality of $\mathrm{F}(\theta)$ is 1 $for\ all\ \theta \in P$, which means $\mathrm{F}(\theta)$ is singleton set for every $\theta \in P$.

Notation: The soft elements are denoted by $\breve{a}, \breve{b}, \breve{c}, \ldots$, and collection of all such elements of $\mathfrak{R}$ by $SE_P(\mathfrak{R})$. Every element of $\mathfrak{R}$ can be considered as a soft element over $\mathfrak{R}$.

Given soft binary operations $(+, P)$ and $(\cdot, P)$ on $\mathfrak{R}$, define the operations $\tilde{+}$ and $\tilde{\cdot}$ on $SE_P(\mathfrak{R})$ by :

$$\breve{a} \tilde{+} \breve{b} = \breve{c} \Leftrightarrow a_\theta +_\theta b_\theta = c_\theta \text{ for all } \theta \in P \text{ and}$$

$$\breve{a} \tilde{\cdot} \breve{b} = \breve{d} \Leftrightarrow a_\theta \cdot_\theta b_\theta = d_\theta \text{ for all } \theta \in \mathrm{P}.$$

Theorem 3.12: *Suppose $(+, P)$ and $(\cdot, P)$ are soft binary operations on $\mathfrak{R}$. Then $(\mathfrak{R}, +, \cdot, P)$ is soft ring iff $(SE_P(\mathfrak{R}), \tilde{+}, \tilde{\cdot})$ is a ring.*

***Proof**:* In [11], it is proved that $(\mathfrak{R}, +, P)$ is abelian soft group iff $(SE_P(\mathfrak{R}), \tilde{+})$ is abelian group. Thus, it remains to prove $\tilde{\cdot}$ is well-defined, and $(\cdot, P)$ is soft distributive over $(+, P)$ iff $\tilde{\cdot}$ is distributive over $\tilde{+}$.

(i) To show $\tilde{\cdot}$ is well-defined: Suppose $\breve{x}, \breve{y}, \breve{z}, \breve{a}, \breve{b}, \breve{c}$ are in $SE_P(\mathfrak{R})$ where $\breve{x} \tilde{\cdot} \breve{y} = \breve{z}$, $\breve{a} \tilde{\cdot} \breve{b} = \breve{c}$ and $\breve{x} = \breve{a}, \breve{y} = \breve{b}$.

Then $\breve{x}(\theta) = \breve{a}(\theta)$ and $\breve{y}(\theta) = \breve{b}(\theta)$ for all $\theta \in P$, and thus $x_\theta = a_\theta$ and $y_\theta = b_\theta$ for all $\theta \in P$. Hence $a_\theta \cdot_\theta b_\theta = c_\theta$, $x_\theta \cdot_\theta y_\theta = z_\theta\ for\ all\ \theta \in P$. Since $(\cdot, P)$ is well-defined, $c_\theta = z_\theta$, for all $\theta \in P$. Hence $\breve{c} = \breve{z}$.Therefore $\tilde{\cdot}$ is well-defined.

(ii) To show $(\cdot, P)$ is soft distributive over $(+, P)$ $\Leftrightarrow \tilde{\cdot}$ is distributive over $\tilde{+}$: Given $\breve{a}, \breve{b}, \breve{c}, \breve{x}, \breve{y}, \breve{p}, \breve{q}, \breve{w}$ in $SE_P(\mathfrak{R})$ where $\breve{b} \tilde{+} \breve{c} = \breve{x}, \breve{a} \tilde{\cdot} \breve{x} = \breve{y}, \breve{a} \tilde{\cdot} \breve{b} = \breve{p}, \breve{a} \tilde{\cdot} \breve{c} = \breve{q}$ and $\breve{p} \tilde{+} \breve{q} = \breve{w}$. We prove $\breve{y} = \breve{w}$, that is $\breve{y}(\theta) = \breve{w}(\theta)\ for\ all$ $\theta \in P$.

For each $\theta \in P$, we get $b_\theta +_\theta c_\theta = x_\theta, a_\theta \cdot_\theta x_\theta = y_\theta, a_\theta \cdot_\theta b_\theta = p_\theta, a_\theta \cdot_\theta c_\theta = q_\theta$ and $p_\theta +_\theta q_\theta = w_\theta$. By left distributivity of $(\cdot, P)$ over $(+, P)$ we have $y_\theta = w_\theta$. It follows that $\breve{y} = \breve{w}$ and hence $\tilde{\cdot}$ is left distributive over $\tilde{+}$. Similraly, we can show that $\tilde{\cdot}$ is right distributive over $\tilde{+}$.

Conversely, let $x, y, z, u, v, w, p, q \in \mathfrak{R}$ and for every $\theta \in P, x +_\theta y = u, z \cdot_\theta u = v, z \cdot_\theta x = p$, $z \cdot_\theta y = q$ and $p +_\theta q = w$. We show that $v = w$. Consider soft elements $\breve{x}, \breve{y}, \breve{z}, \breve{u}, \breve{v}, \breve{w}, \breve{p}$, and $\breve{q}$ that are defined as $\breve{x}(\theta) = \{x\}, \breve{y}(\theta) = \{y\}, \breve{z}(\theta) = \{z\}, \breve{u}(\theta) = \{u\}, \breve{v}(\theta) = \{v\}, \breve{w}(\theta) = \{w\}, \breve{p}(\theta) =$

$\{p\}, \breve{q}(\theta) = \{q\}$ for each $\theta \in P$. Then, $\breve{x} \tilde{+} \breve{y} = \breve{u}, \breve{z} \tilde{\cdot} \breve{u} = \breve{v}, \breve{z} \tilde{\cdot} \breve{x} = \breve{p}, \breve{z} \tilde{\cdot} \breve{y} = \breve{q}, \breve{p} \tilde{+} \breve{q} = \breve{w}$. Since $\tilde{\cdot}$ is left distributive over $\tilde{+}$, $\breve{v} = \breve{w}$ and hence $v = w$. This shows that $(\cdot, P)$ is left ditributive over $(+, P)$. Similarly one can show that $(\cdot, P)$ is right ditributive over $(+, P)$.

Lemma 3.13: *$(\mathfrak{R}, +, \cdot, P)$ is commutative soft ring iff $(SE_P(\mathfrak{R}), \tilde{+}, \tilde{\cdot})$ is commutative ring.*

Lemma 3.14: *$(\mathfrak{R}, +, \cdot, P)$ is soft ring with unity iff $(SE_P(\mathfrak{R}), \tilde{+}, \tilde{\cdot})$ is ring with unity.*

Remark 3.15: The ring $(SE_P(\mathfrak{R}), \tilde{+}, \tilde{\cdot})$ will never be a division ring because $\breve{a} \in SE_P(\mathfrak{R})$ defined by

$$\breve{a}(\theta) = \begin{cases} a, if\ \theta = \gamma \\ 0_\theta, if\ \theta \neq \gamma, \end{cases}, \text{ where } a \in \mathfrak{R}, a \neq 0_\gamma$$

is a non-zero element of $SE_P(\mathfrak{R})$ but not a θ-unit element.

Definition 3.16: Let $(\mathfrak{R}, +, \cdot, P)$ be a soft ring and $\theta \in P$. $a \in \mathfrak{R}$ is called θ-idempotent element, if $a \cdot_\theta a = a$.

Example 3.17: For any soft ring $(\mathfrak{R}, +, \cdot, P)$, 0_θ is θ-idempotent element.

Let $Id(\mathfrak{R})(\theta)$ be the set of all θ-idempotent elements in $\mathfrak{R}$.

Definition 3.18: A soft ring $(\mathfrak{R}, +, \cdot, P$ is said to be a soft Boolean ring, if $\mathfrak{R} = Id(\mathfrak{R})(\theta)$ for all $\theta \in P$.

Proposition 3.19: Given soft Boolean ring $(\mathfrak{R}, +, \cdot, P)$. Then

(i) for every $\theta \in P$, any $s \in \mathfrak{R}, s +_\theta s = 0_\theta$, and

(ii) $(\mathfrak{R}, +, \cdot, P)$ is commutative soft ring.

Remark 3.20: From (i) above, we see that every element of $\mathfrak{R}$ is its own inverse. That is $-a^\theta = a$ for all $\theta \in P$ and for all $a \in \mathfrak{R}$.

Theorem 3.21: *Suppose $(\mathfrak{R}, +, \cdot, P)$ is a commutative soft ring with unity. $(\mathfrak{R}, +, \cdot, P)$ is soft Boolean ring iff $(SE_P(\mathfrak{R}), \tilde{+}, \tilde{\cdot})$ is a Boolean ring.*

4. Soft Subrings and Soft Ideals

Soft subrings and soft ideals are a subject of study in abstract algebra, particularly for characterizing algebraic structures. The concept of soft ideals is more general than that of classical ideals, providing a tool to study parameterized structures. Operations like soft union and soft intersection affect soft ideals. For instance, the intersection of two soft ideals is generally a soft ideal, similar to the classical case. Concepts like prime soft ideals and maximal soft ideals have been defined and explored to extend their classical ring theory counterparts.

Definition 4.1: Suppose $(\mathfrak{R},+,\cdot,P)$ is soft ring. A soft set (S,P) over $\mathfrak{R}$ is called a *soft subring* of $\mathfrak{R}$ if, for each $\theta\in P;\ a,b\in S(\theta);\ x,y\in\mathfrak{R}$,

i) $0_\theta\in S(\theta)$,

ii) if $a+_\theta-b^\theta=x$, then $x\in S(\theta)$, and

iii) if $a\cdot_\theta b=y$, then $y\in S(\theta)$.

Example 4.2: The soft set $(Z_{\mathfrak{R}},P)$ where $Z_{\mathfrak{R}}:P\to P(\mathfrak{R}),Z_{\mathfrak{R}}(\theta)=\{0_\theta\}\ for\ all\ \theta\in P$ is a soft subring of $\mathfrak{R}$.

Example 4.3: The absolute soft set $(1_{\mathfrak{R}},P)$ where $1_{\mathfrak{R}}:P\to P(\mathfrak{R}),1_{\mathfrak{R}}(\theta)=\mathfrak{R}$ for each $\theta\in P$ is a soft subring of $\mathfrak{R}$.

Example 4.4: Consider the soft ring $(\mathfrak{R},*,\triangle,\mathbb{Z})$ in Example 3.3. Suppose (S,P) is soft set over $\mathfrak{R}$ so that for $\alpha\in\mathbb{Z},S(\alpha)=\{p+q\sqrt{|\theta|}:p,q\in\mathbb{Z}\}$. Hence (S,P) is soft subring of $\mathfrak{R}$.

Example 4.5: Consider the soft ring in Example 3.4, $\mathfrak{R}=M_{2x2}(\mathbb{Z}),P=\mathbb{Z}$. Now, define the soft set (S,P) over $\mathfrak{R}$ by: for each $\alpha\in P,S(\alpha)=\{\begin{bmatrix} s & \alpha\\ \alpha & t\end{bmatrix}:s,t\in\mathbb{Z}\}$. Then (S,P) is soft subring of $\mathfrak{R}$.

Lemma 4.6: *Suppose* $(\mathfrak{R},+,\cdot,P)$ *is soft ring and* $a\in\mathfrak{R}$. *Define the soft set* (S,P) *over* $\mathfrak{R}$ *by* $S(\theta)=\{r\in\mathfrak{R}:a\cdot_\theta r=0_\theta\}$ *for each* $\theta\in P$. *Then* (S,P) *is soft subring of* $\mathfrak{R}$.

Definition 4.7: Suppose $(\mathfrak{R},+,\cdot,P)$ is soft ring. The center of $\mathfrak{R}$ denoted as $(Z_P(\mathfrak{R}),P)$ is soft set over $\mathfrak{R}$ which is defined by, for each $\kappa\in P$

$$Z_P(\mathfrak{R})(\kappa)=\{s\in\mathfrak{R}\mid s\cdot_\kappa p=q\ \text{iff}\ p\cdot_\kappa s=q\ for\ each\ p,q\in\mathfrak{R}\}.$$

Theorem 4.8: *Given soft ring* $(\mathfrak{R},+,\cdot,P)$. *Then, the center* $(Z_P(\mathfrak{R}),P)$ *is a soft subring of* $\mathfrak{R}$.

Definition 4.9: Suppose $(\mathfrak{R},+,\cdot,P)$ is a soft ring. The soft set (I,P) is said to be:

a) left soft ideal of $\mathfrak{R}$ if $\forall\theta\in P,a,b\in I(\theta)$ and $r,x,y\in\mathfrak{R}$
   - i) $0_\theta\in I(\theta)$,
   - ii) if $a+_\theta-b^\theta=x$, then $x\in I(\theta)$ and
   - iii) if $r\cdot_\theta a=y$, then $y\in I(\theta)$.

b) right soft ideal of $\mathfrak{R}$ if for each $\theta\in P,a,b,\in I(\theta)$ and $r,x,y\in\mathfrak{R}$
   - i) $0_\theta\in I(\theta)$,
   - ii) if $a+_\theta-b^\theta=x$, then $x\in I(\theta)$ and
   - iii) if $a\cdot_\theta r=y$, then $y\in I(\theta)$.

c) *soft ideal* of $\mathfrak{R}$, if (I,P) is both left and right ideal of $\mathfrak{R}$.

Example 4.10: Given soft ring $(\mathfrak{R}, +, \cdot, P)$ and $d \in \mathfrak{R}$. Consider the soft set (I, P) where $\forall \theta \in P, I(\theta) = \{s \in \mathfrak{R} | t \cdot_\theta d = s$ for some $t \in \mathfrak{R}\}$. Hence, (I, P) is left soft ideal of $\mathfrak{R}$.

Example 4.11: Consider the soft subring in Example 4.4, $\mathfrak{R} = M_{2x2}(\mathbb{Z}), P = \mathbb{Z}$. The soft set (S, P) over $\mathfrak{R}$ is also soft ideal of $\mathfrak{R}$.

Lemma 4.12: (Intersections of soft ideals)

1. *Intersection of two soft ideals is a soft ideal.*
2. *Arbitrary intersections of soft ideals is also a soft ideal.*

Remark 4.13: Suppose $(\mathfrak{R}, +, \cdot, P)$ is soft ring. Let (I, P) and (J, P) are soft ideals of $\mathfrak{R}$.

1. We define the soft set $(I + J, P)$ over $\mathfrak{R}$ by

$(I + J)(\theta) = \{s \in \mathfrak{R}: p +_\theta q = s$ for some $p \in I(\theta), q \in J(\theta)\}$

Claim: $I + J$ is the smallest soft ideal of $\mathfrak{R}$ containing I and J.

Proof: Since $0_\theta \in I(\theta)\ and\ 0_\theta \in J(\theta)\ \forall\ \theta \in P$, $0_\theta +_\theta 0_\theta = 0_\theta$.

This implies $0_\theta \in I + J\ \forall \theta \in P$.

For $\theta \in P, let\ x, y \in (I + J)(\theta)$. Then there exist $a_1, a_2 \in I(\theta)\ and\ b_1$, $b_2 \in J(\theta)$ where $a_1 +_\theta b_1 = x\ , a_2 +_\theta b_2 = y$.

Let $c_1 \in I(\theta)$ and $c_2 \in J(\theta)$ such that $a_1 +_\theta - a_2^\theta = c_1$ and $b_1 +_\theta - b_2^\theta = c_2$.

Let $s \in \mathfrak{R}$ such that $x +_\theta - y^\theta = s$. Then, $c_1 +_\theta c_2 = s$. It follows that $s \in (I + J)(\theta)$.

Assume $s \in \mathfrak{R}$, $x \in (I + J)(\theta)$. Thus there exist $a \in I(\theta)$, $b \in J(\theta)$ so that $a +_\theta b = x$. Suppose $s \cdot_\theta a = u\ and\ s \cdot_\theta b = v$ for some u and v. Then there exist $w, z \in \mathfrak{R}$ where $u +_\theta v = w$, $s \cdot_\theta x = z$. Since $(\cdot, P)$ is distibutive over $(+, P)$, $w = z$. Since $w \in (I + J)(\theta)$, $z \in (I + J)(\theta)$. Hence $(I + J, P)$ is a soft ideal of $\mathfrak{R}$.

Let $\theta \in P$ and $a \in I(\theta)$. Since $0_\theta \in J(\theta)$ and $a +_\theta 0_\theta = a$, we have $a \in (I + J)(\theta)$. Thus, $I(\theta) \subseteq (I + J)(\theta)$. Similarly, we can show that $J(\theta) \subseteq (I + J)(\theta)$ for each $\theta \in P$. Hence $(I, P) \,\tilde{\subseteq}\, (I + J, P)$ and $(J, P) \,\tilde{\subseteq}\, (I + J, P)$. Therefore, both I and J are contained in $I + J$.

Suppose (K, P) is a soft ideal of $\mathfrak{R}$ such that $(I, P) \,\tilde{\subseteq}\, (K, P)$ and $(J, P) \,\tilde{\subseteq}\, (K, P)$. We show $(I + J, P) \,\tilde{\subseteq}\, (K, P)$. Let $\theta \in P$ and let $x \in (I + J)(\theta)$. Thus, $a \in I(\theta)$ and $b \in J(\theta)$ where $a +_\theta b = x$. Since $I(\theta) \subseteq K(\theta)$ and $J(\theta) \subseteq K(\theta)$, we get $a, b \in K(\theta)$ and hence $x \in K(\theta)$. This proves our claim.

2. Define the soft set (IJ, P) over $\mathfrak{R}$ by :

$IJ(\theta) = \{x \in \mathfrak{R}$: there exist sequences $a_1, a_2, \ldots, a_n$ in $I(\theta)$, $b_1, b_2, \ldots, b_n$ in $J(\theta)$, $c_1, c_2, \ldots, c_n$; $c_1 = x_1, x_2, \ldots, x_n = x$ in $\mathfrak{R}$ so that $a_i \cdot_\theta b_i =$

$c_i, x_j +_\theta c_{j+1} = x_{j+1}$ for $i = 1,2,\ldots,n$ and $j = 1,2,\ldots,n-1; n \in \mathbb{N}\}$

Claim: (IJ, P) is a soft ideal of $\mathfrak{R}$.

Proof: First, we prove $0_\theta \in IJ(\theta)$ for each $\theta \in P$. Consider the sequences $a_1, a_2, \ldots, a_n$ in $I(\theta)$ where $a_i = 0_\theta\ \forall i, b_1, b_2, \ldots, b_n\ in\ J(\theta), c_1, c_2, \ldots, c_n$ and $c_1 = x_1, x_2, \ldots, x_n = 0_\theta$ in $\mathfrak{R}$ such that $a_i \cdot_\theta b_i = c_i, x_j +_\theta c_{j+1} = x_{j+1}$. Then, $0_\theta \in IJ(\theta)$.

Now, let $x, y \in IJ(\theta)$. We want to show that $-x^\theta \in IJ(\theta)$ and if $x +_\theta y = z$ for some $z \in \mathfrak{R}$, then $z \in IJ(\theta)$. Since $x \in IJ(\theta)$, there exist sequences $a_1, a_2, \ldots, a_n$ in $I(\theta)$ and $b_1, b_2, \ldots, b_n$ in $J(\theta)$, $c_1, c_2, \ldots, c_n$ and $c_1 = x_1, x_2, \ldots, x_n = x$ in $\mathfrak{R}$ where $a_i \cdot_\theta b_i = c_i$ and $x_j +_\theta c_{j+1} = x_{j+1}$. Consider the sequences $-a_1^\theta, -a_2^\theta, \ldots, -a_n^\theta \in I(\theta)$, $b_1, b_2, \ldots, b_n$ in $J(\theta)$, $-c_1^\theta, -c_2^\theta, \ldots,$ $-c_n^\theta$ and $-c_1^\theta = -x_1^\theta, \ldots, -x_n^\theta = -x^\theta$ in $\mathfrak{R}$, we will have $-a_i^\theta \cdot_\theta b_i = -c_i^\theta$ and $-x_j^\theta +_\theta - c_{j+1}^\theta = -x_{j+1}^\theta$. Thus, $-x^\theta \in IJ(\theta)$.

Since $y \in IJ(\theta)$, there exist sequences $a_{n+1}, \ldots, a_k$ in $I(\theta)$ and $b_{n+1}, \ldots, b_k$ in $J(\theta)$, $c_{n+1}, \ldots, c_k$, and $c_{n+1} = x_{n+1}, x_{n+2}, \ldots, x_k = y$ such that $a_i \cdot_\theta b_i = c_i$ and $x_j +_\theta c_{j+1} = x_{j+1}$ for $i = n+1, n+2, \ldots, k$ and $j = n+1, n+2, \ldots, k-1$.

Suppose $x +_\theta y = z$ for some $z \in \mathfrak{R}$. We need to show that $z \in IJ(\theta)$. Now consider the sequences $a_1, a_2, \ldots, a_n, a_{n+1}, \ldots, a_k$ in $I(\theta)$, $b_1, b_2, \ldots, b_n, b_{n+1}, \ldots, b_k$ in $J(\theta)$, $c_1, c_2, \ldots, c_n, c_{n+1}, .., c_k$ in $\mathfrak{R}$ and the sequences $c_1 = x_1 = y_1, x_2 = y_2, \ldots, x_n = y_n, x + c_{n+1} = y_{n+1}, .., y_k = z$ such that $a_i \cdot_\theta b_i = c_i$ and $y_j +_\theta c_{j+1} = y_{j+1}$ for $i = 1,2,\ldots,k$, $j = 1,2,\ldots k-1$. Thus, $z \in IJ(\theta$.

Suppose $r \in \mathfrak{R}$ and $r \cdot_\theta x = w$. We want to show that $w \in IJ(\theta)$. There exist sequences $a_1^*, a_2^*, \ldots, a_n^*$ in $I(\theta)$ where $r \cdot_\theta a_i = a_i^*$; $i = 1,2,\ldots,n$; $b_1, b_2, \ldots, b_n$ in $J(\theta)$ and $c_1^* = x_1^*, \ldots, c_2^*, \ldots, x_n^* = w$ in $\mathfrak{R}$ where $a_i^* \cdot_\theta b_i = c_i^*$ and $x_j^* +_\theta c_{j+1}^* = x_{j+1}^*$ for $i = 1,2,\ldots,n$; $j = 1,2,\ldots n-1$. Hence (IJ, P) is soft ideal of $\mathfrak{R}$.

3. We define the soft set (I^2, P) by: for $\theta \in P$,

$I^2(\theta) = \{x \in \mathfrak{R}:$ there exist sequences $a_1, a_2, \ldots, a_n$, $b_1, b_2, \ldots, b_n$ in $I(\theta)$, $c_1, c_2, \ldots, c_n$ and $c_1 = x_1, x_2, \ldots, x_n = x$ in $\mathfrak{R}$ where $a_i \cdot_\theta b_i = c_i$ and $x_j +_\theta c_{j+1} = x_{j+1}$; $i = 1,2,\ldots,n$; $j = 1,2,\ldots n-1\}$, and the soft set (I^n, P) by $(I^n, P) \cong (I^{n-1}I, P)$, $\forall n \in \mathbb{N}, n \geq 2$. Consequently (I^n, P) is soft ideal, $\forall n \in \mathbb{N}, n \geq 2$.

Definition 4.14: Suppose $(*, P)$ is soft binary operation on a non-empty set M. Then the triple $(M, *, P)$ is said to be a *soft monoid* if

i) $(*, P)$ is associative, and

ii) for each $\theta \in P$, there exists identity element $e_\theta \in M$ such that $a *_\theta e_\theta = a$ and $e_\theta * a = a$ for all $a \in M$.

Lemma 4.15: *Suppose* $(\mathfrak{R},+,\cdot,P)$ *is soft ring with unity. For each* $\theta \in P$, *consider* $U(\mathfrak{R})(\theta)=\{p \in \mathfrak{R}: p \cdot_\theta q = 1_\theta,\ \ q \cdot_\theta p = 1_\theta \text{ for some } q \in \mathfrak{R}\}$. *If* $\mathfrak{R}^* = \bigcup_{\theta \in P} U(\mathfrak{R})(\theta)$, *then* $(\mathfrak{R}^*,\cdot,P)$ *is a soft monoid.*

Proof: $(\cdot,P)$ is associative on $\mathfrak{R}^*$ because $(\cdot,P)$ is associative on $\mathfrak{R}$ for each $\theta \in P$. Since $1_\theta \in U(\mathfrak{R})(\theta)$ for each $\theta \in P$, we have $1_\theta \in \mathfrak{R}^*$. Hence $(U(\mathfrak{R}),\cdot,P)$ is a soft monoid.

Consider the soft set $(U(\mathfrak{R}),P)$ over $\mathfrak{R}^*$.

Lemma 4.16: $(U(\mathfrak{R}),P)$ *is a soft submonoid of* $\mathfrak{R}^*$.

Lemma 4.17: *Suppose* (J,P) *is soft ideal of soft ring* $\mathfrak{R}$ *with unity and* $\theta \in P$. *If* $1_\theta \in J(\theta)$ *then* $J(\theta)=\mathfrak{R}$.

Proof: Suppose $1_\theta \in J(\theta)$. Clearly, $J(\theta)$ contanied in $\mathfrak{R}$. Now, let $p \in \mathfrak{R}$. Then $p \cdot_\theta 1_\theta = p$. Since (J,P) is soft ideal of $\mathfrak{R}$, we have $p \in J(\theta)$. Therefore $J(\theta)=\mathfrak{R}$.

Note that if $1_\theta \in J(\theta)$ for each $\theta \in P$, then $(J,P) \cong (1_\mathfrak{R},P)$.

Theorem 4.18: *Suppose* $(\mathfrak{R},+,\cdot,P)$ *is commutative soft ring with unity and* (I,P) *is soft ideal of* $\mathfrak{R}$. *Then,* $\mathfrak{R}$ *is soft field iff for each* $\theta \in P$, *either* $I(\theta)=\{0_\theta\}$ *or* $I(\theta)=\mathfrak{R}$.

Lemma 4.19: *If* (I,P) *and* (J,P) *are soft ideals of* $\mathfrak{R}$ *and* $(I,P) \cap (J,P) \cong (0_\mathfrak{R},P)$ *then for each* $\theta \in P, a \in I(\theta)$ *and* $b \in J(\theta), a \cdot_\theta b = 0_\theta$.

Definition 4.20: For soft set (S,P) over $\mathfrak{R}$, define subset $\hat{S}$ of $SE_P(\mathfrak{R})$ by:

$$\hat{S}=\{\hat{x} \in SE_P(\mathfrak{R}): \hat{x}(\theta) \subseteq S(\theta)\ \forall \theta \in P\}.$$

Theorem 4.21: *A soft set* (S,P) *over* $\mathfrak{R}$ *is a soft subring of* $\mathfrak{R}$ *iff* $\hat{S}$ *is a subring of* $SE_P(\mathfrak{R})$.

Proof: Assume (S,P) is a soft subring of $\mathfrak{R}$. We have $0_\theta \in S(\theta)$ for all $\theta \in P$. Thus $\breve{0} \in \hat{S}$. Let $\breve{p},\breve{q}$ be elements of $\hat{S}$, and $\breve{r}$ and $\breve{s}$ be soft elements of $\mathfrak{R}$ where $\breve{p}\tilde{+}(-\breve{q})=\breve{r}$ and $\breve{p} \tilde{\cdot} \breve{q}=\breve{s}$. Then for each $\theta \in P, \breve{p}(\theta)+_\theta-\breve{q}(\theta)=\breve{r}(\theta)$ and $\breve{p}(\theta)\cdot_\theta \breve{q}(\theta)=\breve{s}(\theta)$. Since $\breve{p}(\theta)$ and $\breve{q}(\theta)$ are in $S(\theta)$ and (S,P) is soft subring, we get $\breve{r}(\theta) \in S(\theta)$ and $\breve{s}(\theta) \in S(\theta)$. Thus $\breve{r}$, $\breve{s} \in \hat{S}$ and hence $\hat{S}$ is subring of $SE_P(\mathfrak{R})$.

Conversly, assume $\hat{S}$ is subring of $SE_P(\mathfrak{R})$. Thus, $\breve{0} \in \hat{S}$ which shows that $0_\theta \in S(\theta)$ for all $\theta \in P$. Now for $\theta \in P$, take $p,q \in S(\theta)$ and $u,v \in \mathfrak{R}$ so that $p+_\theta-q^\theta=u$ and $p \cdot_\theta q = v$. Here we show that u and v belong to $S(\theta)$. Take soft elements $\breve{p},\breve{q},\breve{u}$ and $\breve{v}$ that are defined as:

$\breve{p}(\delta)=\{p\}, \breve{q}(\delta)=\{q\}\ \forall \lambda \in P$ and

$$\breve{u}(\delta) = \begin{cases} \{u\}, if\ \delta = \theta \\ \{u_\delta\}, if\ \delta \neq \theta, for\ u_\delta \in \mathfrak{R}, p +_\delta - q^\delta = u_\delta, and \end{cases}$$

$$\breve{v}(\delta) = \begin{cases} \{v\}, if\ \delta = \theta \\ \{v_\delta\}, if\ \delta \neq \theta, for\ v_\delta \in \mathfrak{R}, p \cdot_\delta q = v_\delta. \end{cases}$$

Then $\breve{p}$ and $\breve{q}$ are in $\hat{S}$. Since $\hat{S}$ is subring of $SE_P(\mathfrak{R})$, $\breve{p}\tilde{+}(-\breve{q})$ and $\breve{p}\,\tilde{\cdot}\,\breve{q}$ are in $\hat{S}$. Suppose $\breve{p}\tilde{+}(-\breve{q}) = \breve{u}$ and $\breve{p}\,\tilde{\cdot}\,\breve{q} = \breve{v}$. Thus, $\breve{u}, \breve{v} \in \hat{S}$ and hence $\breve{u}(\theta), \breve{v}(\theta) \in S(\theta)$ for each $\theta \in P$. Therefore, $u, v \in S(\theta)$.

Theorem 4.22: *A soft set (I, P) over $\mathfrak{R}$ is soft ideal of $\mathfrak{R}$ iff $\hat{I}$ is ideal of $SE_P(\mathfrak{R})$.*

Definition 4.23: Suppose $(\mathfrak{R}, +, \cdot, P)$ is soft ring.

1. Soft ideal (I, P) of $\mathfrak{R}$ is called *proper soft ideal* if for each $\theta \in P, I(\theta) \subsetneqq \mathfrak{R}$.
2. Soft ideal $(\wp, P)$ of $\mathfrak{R}$ is called *prime soft ideal* of $\mathfrak{R}$ if
   i) $(\wp, P)$ is proper, and
   ii) for any soft ideals (I, P) and (J, P) of $\mathfrak{R}$, $(IJ, P) \,\breve{\subseteq}\, (\wp, P)$ implies either $I(\theta) \subseteq \wp(\theta)$ or $J(\theta) \subseteq \wp(\theta)$ for each $\theta \in P$.
3. Soft ideal (I, P) of $\mathfrak{R}$ is called *maximal* if
   i) (I, P) is proper, and
   ii) for evrey soft ideal (J, P) of $\mathfrak{R}$, if $(I, P) \,\breve{\subseteq}\, (J, P)$ then for each $\theta \in P, I(\theta) = J(\theta)$ or $J(\theta) = \mathfrak{R}$.

Theorem 4.24: *Suppose $(\mathfrak{R}, +, \cdot, P)$ is soft ring and $(\wp, P)$ be proper soft ideal of $\mathfrak{R}$. Let, for $\theta \in P$ and $a, b, c \in \mathfrak{R}, a \cdot_\theta b = c, c \in \wp(\theta)$ implies $a \in \wp(\theta)$ or $b \in \wp(\theta)$. Hence, $(\wp, P)$ is prime soft ideal of $\mathfrak{R}$.*

Notation: Given soft ring $(\mathfrak{R}, +, \cdot, P), a, x \in \mathfrak{R}$. For non-negative integer n, if there is a sequence $0_\theta = x_0, x_1, \ldots, x_n = x$ in $\mathfrak{R}$ so that $x_i +_\theta a = x_{i+1})$, for $i = 0,1,2, \ldots, n-1$, we write $n \cdot_\theta a = x$. For a negative integer n, $n \cdot_\theta a = x$ means $-n \cdot_\theta - a^\theta = x$.

Theorem 4.25: *Suppose $(\mathfrak{R}, +, \cdot, P)$ is commutative soft ring. If $(\wp, P)$ is prime soft ideal of $\mathfrak{R}$, then for $\theta \in P, s, t \in \mathfrak{R}, s \cdot_\theta t = v$ and $v \in \wp(\theta)$ imply that $s \in \wp(\theta)$ or $t \in \wp(\theta)$.*

Proof: Consider the soft sets (I^a, P) and (I^b, P) that are defined by: for $\theta \in P$,

$I^a(\theta) = \{y \in \mathfrak{R} \mid r \cdot_\theta a = u, n +_\theta a = x$ and $u +_\theta x = y$ for some $r, u, x \in \mathfrak{R},\ n \in \mathbb{Z}\}$

$I^b(\theta) = \{z \in \mathfrak{R} \mid r' \cdot_\theta b = v, m +_\theta b = x'$ and $u +_\theta x' = z$ for some $r', v, x' \in \mathfrak{R},\ m \in \mathbb{Z}\}$

Then (I^a, P) and (I^b, P) are soft ideals of $\mathfrak{R}$. By Remark 4.13, $(I^a I^b, P)$ is soft ideal in $\mathfrak{R}$. Moreover, $(I^a I^b, P) \,\breve{\subseteq}\, (\wp, P)$. Since $(\wp, P)$ is prime soft ideal in $\mathfrak{R}$, we have for each $\theta \in P$, either $I^a(\theta) \subseteq \wp(\theta)$ or $I^b(\theta) \subseteq \wp(\theta)$ and hence either $a \in \wp(\theta)$ and $b \in \wp(\theta)$.

Theorem 4.26: *Suppose $(\mathfrak{R}, +, \cdot, P)$ is a commutative soft ring with unity and $(\wp, P)$ is a soft ideal in $\mathfrak{R}$. Consider $\widehat{\wp} = \{\breve{a} \in SE_P(\mathfrak{R}) : \breve{a}(\theta) \subseteq \wp(\theta)$ for all $\theta \in P\}$. If $\widehat{\wp}$ is prime ideal of $(SE_P(\mathfrak{R}), \tilde{+}, \tilde{\cdot})$ then $(\wp, P)$ is prime soft ideal in $\mathfrak{R}$.*

***Proof*:** Assume (I, P) and (J, P) are soft ideals in $\mathfrak{R}$ where $(IJ, P) \tilde{\subseteq} (\wp, P)$. We will show that for each $\theta \in P$, $I(\theta) \subseteq \wp(\theta)$ or $J(\theta) \subseteq \wp(\theta)$. Let $\kappa \in P$ be arbitrary. If $I(\kappa) \subseteq \wp(\kappa)$, we are done. Suppose $I(\kappa) \nsubseteq \wp(\kappa)$. Then, there exists $p \in I(\kappa) \backslash \wp(\kappa)$. Let $q \in J(\kappa)$ and consider the soft elements :

$$\breve{p}(\theta) = \begin{cases} \{p\}, if\ \theta = \kappa \\ \{0_\theta\}, if\ \theta \neq \kappa \end{cases}$$

$$\breve{q}(\theta) = \begin{cases} \{q\}, if\ \theta = \kappa \\ \{0_\theta\}, if\ \theta \neq \kappa \end{cases}$$

Suppose $p \cdot_\kappa q = t$ for $t \in \mathfrak{R}$. Since $p \in I(\kappa)$ and $q \in J(\kappa)$, $t \in \wp(\kappa)$. Thus, $\breve{p} \,\tilde{\cdot}\, \breve{q} \in \widehat{\wp}$. Since $\widehat{\wp}$ is prime ideal of $SE_P(\mathfrak{R})$, either $\breve{p} \in \widehat{\wp}$ or $\breve{q} \in \widehat{\wp}$. Since $\breve{p}(\kappa) = p \notin \wp(\kappa)$, we have $\breve{p} \notin \widehat{\wp}$ and hence $\breve{q} \in \widehat{\wp}(\kappa)$. Thus, $q \in \wp(\kappa)$ as required.

Theorem 4.27: *Suppose $(\mathfrak{R}, +, \cdot, P)$ is commutative soft ring with unity and (μ, P) is soft ideal of P. Consider $\hat{\mu} = \{\breve{a} \in SE_P(\mathfrak{R}) : \breve{a}(\theta) \subseteq \mu(\theta)$ for every $\theta \in P\}$. If $\hat{\mu}$ is maximal ideal of $(SE_P(\mathfrak{R}), \tilde{+}, \tilde{\cdot})$, then (μ, P) is maximal soft ideal in $\mathfrak{R}$.*

***Proof*:** Suppose (J, P) is soft ideal of $\mathfrak{R}$ such that $(\mu, P) \tilde{\subseteq} (J, P)$. We need to show that for each $\theta \in P$ either $\mu(\theta) = J(\theta)$ or $J(\theta) = \mathfrak{R}$. Let $\gamma \in P$ be arbitrary. If $\mu(\gamma) = J(\gamma)$, we are done. Suppose $\mu(\gamma) \nsubseteq J(\gamma)$. Then there exists $b \in J(\gamma) \backslash \mu(\gamma)$. Now we show that $J(\gamma) = \mathfrak{R}$. Clearly $J(\gamma) \subseteq \mathfrak{R}$. Now, let $r \in \mathfrak{R}$. Consider the soft elements defined as:

$$\breve{r}(\theta) = \begin{cases} \{r\}, if\ \theta = \gamma \\ \{0_\theta\}, if\ \theta \neq \gamma \end{cases}$$

$$\breve{b}(\theta) = \begin{cases} \{b\}, if\ \theta = \gamma \\ \{0_\theta\}, if\ \theta \neq \gamma \end{cases}$$

Since $b \notin \mu(\gamma)$, we have $\breve{b} \notin \hat{\mu}$. Thus, by maximality of $\hat{\mu}$, we obtain $\hat{\mu} \tilde{+} (\breve{b}) = SE_P(\mathfrak{R})$. Since $\breve{1} \in SE_P(\mathfrak{R}), \breve{1}$ is expressed as $\breve{1} = \breve{t} \tilde{+} \breve{a} \,\tilde{\cdot}\, \breve{b}$ for some $\breve{t} \in \hat{\mu}$ and $\breve{a} \in SE_P(\mathfrak{R})$. Then $\breve{r} \,\tilde{\cdot}\, \breve{1} = \breve{r} \,\tilde{\cdot}\, \breve{t} \tilde{+} \breve{r} \,\tilde{\cdot}\, (\breve{a} \,\tilde{\cdot}\, \breve{b})$ and $\breve{r} = \breve{r} \,\tilde{\cdot}\, \breve{t} \tilde{+} (\breve{r} \,\tilde{\cdot}\, \breve{a}) \,\tilde{\cdot}\, \breve{b}$. Suppose $\breve{r} \,\tilde{\cdot}\, \breve{t} = \breve{x}, \breve{r} \,\tilde{\cdot}\, \breve{a} = \breve{y}$ *and* $\breve{y} \,\tilde{\cdot}\, \breve{b} = \breve{z}$. Thus $\breve{x} \tilde{+} \breve{z} = \breve{r}$. Moreover, $r_\theta \cdot_\theta t_\theta = x_\theta, r_\theta \cdot_\theta a_\theta = y_\theta, y_\theta \cdot_\theta b_\theta = z_\theta$, for all $\theta \in P$. In particular, if $\theta = \gamma$ then $r_\gamma \cdot_\gamma t_\gamma = x_\gamma, r_\gamma \cdot_\gamma a_\gamma = y_\gamma, y_\gamma \cdot_\gamma b_\gamma = z_\theta$. From $\hat{t} \in \hat{\mu}$, we get $\hat{t}(\theta) \subseteq \mu(\theta) \subseteq J(\theta)$ for all $\theta \in P$. This implies $t_\gamma \in J(\gamma)$ and hence $x_\gamma \in J(\gamma)$. Since $b \in J(\gamma)$, $z_\gamma \in J(\gamma)$ and also from $\breve{x} \tilde{+} \breve{z} = \breve{r}$, we have

$x_\theta +_\theta z_\theta = r_\theta$ for all $\theta \in P$ and inparticular for $\theta = \gamma$, $x_\gamma +_\gamma z_\gamma = r_\gamma$, which implies that $r_\gamma \in J(\gamma)$ and hence $r \in J(\gamma)$. Therefore, $J(\gamma) = \mathfrak{R}$.

Theorem 4.28: *Every commutative soft ring with unity has a maximal soft ideal.*

Proof: Assume $\mathcal{A}$ is the set of proper soft ideals of $\mathfrak{R}$, ordered by soft inclusion $\tilde{\sqsubseteq}$. Then $(\mathcal{A}, \tilde{\sqsubseteq})$ is a poset. Let $\mathcal{C}$ be a chain in $\mathcal{A}$. Let $(I, P) = \bigcup_{(J,P)\in\mathcal{C}} (J, P)$.

Claim: (I, P) is a proper soft ideal.

i) For $\theta \in P$, $0_\theta \in J(\theta) \subseteq I(\theta)$.

ii) Suppose x_1, x_2 are in $I(\theta)$. Then, there are (J_1, P) and (J_2, P) in $\mathcal{C}$ such that $x_1 \in J_1(\theta)$ and $x_2 \in J_2(\theta)$. Since $\mathcal{C}$ is a chain, either $(J_1, P) \tilde{\sqsubseteq} (J_2, P)$ or $(J_2, P) \tilde{\sqsubseteq} (J_1, P)$. With out loss of generality, assume $(J_1, P) \tilde{\sqsubseteq} (J_2, P)$. Then $x_1 \in J_2(\theta)$, so $x_1 +_\theta x_2 = y$ implies that $y \in J_2(\theta) \subseteq I(\theta)$.

iii) Let $a \in \mathfrak{R}, s \in I(\theta)$ and $a \cdot_\theta s = t$. Then there exists (J, P) in $\mathcal{C}$ so that $s \in J(\theta)$, and hence $t \in J(\theta) \subseteq I(\theta)$. Therefore, (I, P) is a soft ideal of $\mathfrak{R}$.

iv) It remains to show that (I, P) is proper. For each (J, P) in $\mathcal{C}$, and for each $\theta \in P, 1_\theta \notin J(\theta)$, so $1_\theta \notin I(\theta)$. Thus, $I(\theta) \neq \mathfrak{R}$ for each $\theta \in P$, and hence (I, P) is proper soft ideal of $\mathfrak{R}$. The soft ideal (I, P) is upper bound of $\mathcal{C}$. Therefore, by Zorn's Lemma, $\mathcal{P}$ has maximal element, say $(\mathcal{M}, P)$.

5. Conclusion

In this paper, we defined soft rings based on soft binary operations and have investigated their properties such as soft subrings, soft ideals, and gave related examples. Also, we investigated and proved results on the algebraic structure of soft Boolean rings and prime soft ideals. Moreover, properties of idempotent elements and the ideal structure of $SE_P(\mathfrak{R})$ is studied.

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