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Rings and near-rings with right homoderivations

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Abstract

This paper studies several algebraic identities in rings and near-rings by introducing a new class of mappings known as right homoderivations. The obtained results provide meaningful insights and contribute to the ongoing development of research in this field.

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1. Introduction

The research of near-rings started with the study of algebraic structures analogous to rings but lacking some of the axioms that define a ring. The concept of a near-ring emerged in the 20th century as a means of studying structures in which only one of the two ring operations (addition or multiplication) does not fully satisfy standard ring properties. Over the decades, near-ring theory has evolved and established connections with various areas of mathematics, becoming an important branch of algebra in its own right.

The theory of identities in near-rings is an important area within the broader framework of near-ring theory. Identities in this context refer to algebraic

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expressions involving the operations of a near-ring that hold for all elements of the near-ring. These identities help classify and understand the structure of near-rings by providing constraints on how the operations interact with each other.

The theory of identities in near-rings provides a framework for understanding the algebraic properties of these structures. By studying the identities that hold in different classes of near-rings, mathematicians gain insight into the classification and behavior of near-rings, extending the applicability of algebraic structures beyond the classical concept of rings. The theory continues to evolve and has connections to areas such as module theory, functional analysis, and cryptography.

$(\mathcal{N},+,\cdot)$ denotes a near-ring in this study. If the equality $w_1 \cdot 0 = 0$ for $w_1 \in \mathcal{N}$ holds, which is a property that comes naturally from the left distributive law, then $\mathcal{N}$ is *zero-symmetric,*. $\mathcal{N}$ is termed *3-prime*, if, for $w_1, w_2 \in \mathcal{N}$, $w_1 \mathcal{N} w_2 = \{0\} \Rightarrow w_1 = 0$ or $w_2 = 0$. Furthermore, if for $w \in \mathcal{N}$, $2w = 0 \Rightarrow w = 0$, then $\mathcal{N}$ is termed *2-torsion free*. For $a, b \in \mathcal{N}$, the well-known *Jordan* and *Lie products* will be denoted as $w_1 \circ w_2 = w_1 w_2 + w_2 w_1$ and $[w_1, w_2] = w_1 w_2 - w_2 w_1$ respectively; further information about near-rings can be found in [10].

In [1, 9], the commutativity of multiplication in rings and near-rings was studied, along with specific algebraic conditions involving derivations, generalized derivations, right derivations, generalized right derivations, homoderivations, and generalized homoderivations. In [9], El Sofy introduced the notion "homoderivation" in the context of ring theory, which blends the fundamental notions of derivations and ring homomorphisms. His work established several key results concerning ring commutativity. Subsequent studies, such as [3], [4], [5], [6], [7], and [8], have extended this line of inquiry to near-rings, particularly focusing on their commutativity properties and the structure of Jordan right ideals by employing identities involving both homoderivations and generalized homoderivations.

Furthermore, Majeed and Farhan introduced a novel class of additive mappings known as right derivations, denoted by $\psi: \mathcal{N} \to \mathcal{N}$, which satisfy the identity $\psi(w_1 w_2) = \psi(w_1) w_2 + \psi(w_2) w_1$ for every $w_1, w_2 \in \mathcal{N}$. They derived significant results concerning the commutativity of $\mathcal{N}$ under right derivations that adhere to specific algebraic conditions [6]. Inspired by the approach of El Sofi and Boua on homoderivations (see, [9] and [3]). This, study aims to merge the notions of "homomorphism" and "right derivation" into a unified framework termed a "right homomorphism" for near-rings and rings.

Definition 1: Consider $\mathcal{A}$ to be a ring or (near-ring), $w_1, w_2 \in \mathcal{A}$, and ψ to be an additive map of $\mathcal{A}$.

1. ψ is called a right homoderivation if

$$\psi(w_1 w_2) = \psi(w_1)\psi(w_2) + \psi(w_1) w_2 + \psi(w_2) w_1.$$

2. The map ψ is termed a Jordan right homoderivation if

$$\psi(w_1^2) = \psi(w_1)\psi(w_1) + 2\psi(w_1)w_1.$$

Example 1: Consider the commutative ring $\mathcal{A}$ with $Char(\mathcal{A}) \neq 2$. Define $\mathcal{B}$ and $\varphi: \mathcal{B} \to \mathcal{B}$ by:

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & w_1 & w_2 \\ 0 & 0 & 0 \\ 0 & 0 & w_3 \end{pmatrix} \mid w_1, w_2, w_3 \in \mathcal{A} \right\}, \varphi \begin{pmatrix} 0 & w_1 & w_2 \\ 0 & 0 & 0 \\ 0 & 0 & w_3 \end{pmatrix} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & -w_3 \end{pmatrix}$$

Clearly, $\mathcal{B}$ is a ring and φ serves as a right homoderivation on $\mathcal{B}$. However, φ is neither a homomorphism nor a right derivation.

It is obvious that any right homoderivation inherently satisfies the conditions of a Jordan right homoderivation. Unfortunately, the reverse is not true.

Example 2: Define $\mathcal{B}$ and h by:

$$\mathcal{B} = \left\{ \begin{pmatrix} 0 & m & n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} \mid m, n, p \in \mathcal{A} \right\}, h \begin{pmatrix} 0 & m & n \\ 0 & 0 & p \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & m & n \\ 0 & 0 & -p \\ 0 & 0 & 0 \end{pmatrix}.$$

Then h is a Jordan right homoderivation of the ring $\mathcal{B}$, which is not a right homoderivation.

This paper begins by exploring a new type of mapping on rings, particularly prime rings, and derives significant results, which are presented in Lemmas 1, 2, 3, 4, and 5. Furthermore, the commutativity of the prime ring $\mathcal{A}$ is examined under the assumption that, if ψ is a right homoderivation of $\mathcal{A}$ that satisfies $(i)\ \psi(w_1w_2 - w_2w_1) = w_1w_2 - w_2w_1$, $(ii)\ \psi(w_1w_2 + w_2w_1) = w_1w_2 + w_2w_1$, and $(iii)\ \psi(w_1w_2 + w_2w_1) = w_1w_2 - w_2w_1$ for all $w_1, w_2 \in \mathcal{A}$.

The second part of this study is devoted to examining these mappings within the framework of near-rings, utilizing the identities $(i)\ \psi(w_1)w_2 - w_2\psi(w_1) \subseteq \mathcal{Z}(\mathcal{N})$, $(ii)\ \psi(w_1w_2 - w_2w_1) = 0$, $(iii)\ \psi(w_1w_2 + w_2w_1) = 0$, and $(iv)\ \psi(w_1w_2 + w_2w_1) = w_1w_2 - w_2w_1$ for all $w_1, w_2 \in \mathcal{N}$, where $\mathcal{Z}(\mathcal{N})$ is the multiplicative center of $\mathcal{N}$.

2. Some preliminaries

For this article, a prime ring is denoted by $\mathcal{R}$ and a 3-prime near-ring by $\mathcal{N}$.

Lemma 1: *If φ is a right homoderivation of $\mathcal{R}$, then $\varphi(\mathcal{Z}(\mathcal{R}))$ is included in $\mathcal{Z}(\mathcal{R})$.*

Proof: Let z be an element of $\mathcal{Z}(\mathcal{R})$. Let's calculate $\varphi(tz)$ and $\varphi(zt)$ by two different methods; also, by their subtraction, we obtain

$$[\varphi(z), \varphi(t)] = 0\ forall\ t \in \mathcal{R} \tag{1}$$

For $t = xz$, in (1) and using it with simple simplification, we find that $\varphi(z)\varphi(z)x = \varphi(z)x\varphi(z)$. Now, replacing x by tx for an arbitrary $t \in \mathcal{R}$ and

applying the same relation again yields $\varphi(z)\mathcal{R}[\varphi(z),x]=\{0\}$. So, $\varphi(\mathcal{Z}(\mathcal{R}))$ is contained in $\mathcal{Z}(\mathcal{R})$ by primeness of $\mathcal{R}$.

Lemma 2: *If ϕ is a right homoderivation of $\mathcal{R}$ with a multiplicative center $\mathcal{Z}(\mathcal{R})\neq\{0\}$, then either $\mathcal{R}$ is commutative or $\phi_{|\mathcal{Z}(\mathcal{R})}=0$.*

Proof: Letting $x,y\in\mathcal{R}$ and $z\in\mathcal{Z}(\mathcal{R})-\{0\}$. By calculating $\phi(x(yz))$ and $\phi((xy)z)$ and comparing them $\phi(z)(xy-yx)=0$. So, $\phi(z)\mathcal{R}(xy-yx)=\{0\}$. Then either $\mathcal{R}$ becomes commutative or $\phi_{|\mathcal{Z}(\mathcal{R})}=0$ by primeness of $\mathcal{R}$.

Lemma 3: *Consider $g\neq 0$ to be a right homoderivation of $\mathcal{R}$. If $\exists u_0\in\mathcal{R}$ such that $g(u_0)=0$, then $u_0\in\mathcal{Z}(\mathcal{R})$.*

Proof: Let us calculate $g((mn)u_0)$ and $g(m(nu_0)$ in two different ways. By using the definition of right homoderivation g, we obtain $g(n)mu_0=g(n)u_0m$. Substituting nm for m, where $n\in\mathcal{R}$, we get $g(n)\mathcal{R}(u_0m-mu_0)=\{0\}\,\forall\, n,m\in\mathcal{R}$, since $g\neq 0$ and $\mathcal{R}$ is prime, we conclude that $u_0\in\mathcal{Z}(\mathcal{R})$.

Lemma 4:

(a) ([2], Lemma 1.2(iii)) *Assume that $z\in\mathcal{Z}(\mathcal{R})$, $x\in\mathcal{N}$ with the property that $xz\in\mathcal{Z}(\mathcal{R})$. Then $\mathcal{Z}(\mathcal{R})$ contains x.*

(b) ([2], Lemma 1.5) *$\mathcal{N}$ is a commutative ring if and only if $\mathcal{Z}(\mathcal{N})$ includes a nontrivial semigroup (left or right) ideal.*

Lemma 5: *$\mathcal{N}$ must be zero-symmetric if it admits a right homoderivation $h_1\neq 0$.*

Theorem 1: *If $\mathcal{N}$ admits a right homoderivation $g\neq 0$ satisfying $g(\mathcal{N})$ is included in $\mathcal{Z}(\mathcal{N})$. Then, $\mathcal{N}$ is a commutative ring.*

Proof: If $\mathcal{Z}(\mathcal{N})=\{0\}$, it follows that $g=0$, which contradicts the given condition that $g\neq 0$. Therefore, there is $w\in\mathcal{Z}(\mathcal{N})-\{0\}$ such that $g(w)+g(w)\in\mathcal{Z}(\mathcal{N})$. So, $(\mathcal{N},+)$ is abelian.

We now calculate $g(zxy)$ in two different ways, and under the assumption that $(\mathcal{N},+)$ is abelian, we obtain $g(z)(xy-yx)=0$, so $g(z)\mathcal{N}(xy-yx)=\{0\}$, then $g_{|\mathcal{Z}(\mathcal{R})}=0$ or $\mathcal{N}$ is a commutative ring by 3-primeness of $\mathcal{N}$.

If $g_{|\mathcal{Z}(\mathcal{R})}=0$, then $0=g^2(xy)=2g(y)g(x)$. This implies that $g=0$, which contradicts our assumption.

3. Right homoderivations on rings

Proposition 1: If $\mathcal{R}$ is commutative of characteristic not 2, then every Jordan right homoderivation is also a right homoderivation.

Proof: Consider ψ is a Jordan right homoderivation of $\mathcal{R}$, then for all $w_1, w_2 \in \mathcal{R}$

$$\psi((w_1+w_2)^2) = \psi(w_1+w_2)\psi(w_1+w_2) + \psi(w_1+w_2)(w_1+w_2) + \psi(w_1+w_2)(w_1+w_2).$$

Which gives

$$\psi(w_1w_2) = \psi(w_1)\psi(w_2) + \psi(w_1)w_2 + \psi(w_2)w_1.$$

Therefore, ψ is a right homoderivation.

Theorem 2: *If $\psi \neq 0$ is a homoderivation of $\mathcal{R}$ with $char(\mathcal{R}) \neq 3$ satisfying $\psi([w_1, w_2]) = [w_1, w_2]$ for all $w_1, w_2 \in \mathcal{R}$, then $\mathcal{R}$ is commutative.*

Proof: Assuming

$$\psi([w_1, w_2]) = [w_1, w_2] \; \forall \; w_1, w_2 \in \mathcal{R}. \tag{2}$$

Placing $w_2 = w_2w_1$ in (2) and using the definition of ψ, we find

$$-\psi(w_1)[w_1, w_2] = [w_1, w_2]\psi(w_1). \tag{3}$$

Putting $w_2 = w_2w_1$ in (3) and applying (3) once again yields

$$[w_1, w_2][\psi(w_1), w_1] = 0.$$

Putting $w_2 = w_2w$, where $w \in \mathcal{R}$, we deduce that

$$[w_1, w_2]\mathcal{R}[\psi(w_1), w_1] = \{0\}.$$

Thus w_1 belongs to $\mathcal{Z}(\mathcal{R})$ or $\psi(w_1)w_1 = w_1\psi(w_1)$. In both scenarios, the conclusion agrees with the preceding cases

$$\psi(w_1)w_1 = w_1\psi(w_1).$$

By calculating $\psi(w_1(w_2w_2))$ and $\psi((w_1w_2)w_2)$ and comparing them, we conclude that

$$\psi(w_2)\psi(w_2)w_1 + \psi(w_2)w_2w_1 + \psi(w_2)w_2w_1 = \psi(w_2)w_1\psi(w_2) + \psi(w_2)w_1w_2 + \psi(w_2)w_1w_2.$$

Letting $w_2 = [v_1, v_2]$ and $Char(\mathcal{R}) \neq 3$, we find

$$[v_1, v_2]w_1[v_1, v_2] = [v_1, v_2][v_1, v_2]w_1.$$

Putting $w_1 = w_1r$, so

$$[v_1, v_2]\mathcal{R}([v_1, v_2]r - r[v_1, v_2]) = \{0\},$$

which reduces to $[v_1, v_2] \in \mathcal{Z}(\mathcal{R})$. So

$$v_1[v_1, v_2] = [v_1, v_1v_2] \in \mathcal{Z}(\mathcal{R}),$$

then

$$[v_1, v_2]\mathcal{R}[v_1, v_2] = \{0\}.$$

Therefore, $\mathcal{R}$ becomes commutative.

Theorem 3: *If $\mathcal{R}$ admits a right homoderivation Φ with $Char(\mathcal{R}) \neq 2$ or 3 and satisfies*

(a) $\Phi(w_1 \circ w_2) = w_1 \circ w_2$ *for all* $w_1, w_2 \in \mathcal{R}$, or

(b) $\Phi(w_1 \circ w_2) = [w_1, w_2]$, *for all* $w_1, w_2 \in \mathcal{R}$
Then $\Phi = 0$.

Proof: (a) Suppose that Φ satisfies

$$\Phi(\alpha \circ \beta) = \alpha \circ \beta \ \forall\ \alpha, \beta \in \mathcal{R}. \tag{4}$$

Placing $\beta = \beta\alpha$ in (4), we find

$$(\alpha \circ \beta)\Phi(\alpha) = -\Phi(\alpha)(\alpha \circ \beta) \tag{5}$$

Changing β to $\beta\alpha$ in (5) and reapplying it yields $(\alpha \circ \beta)([\Phi(\alpha), \alpha]) = 0$, then

$$\alpha\beta([\Phi(\alpha), \alpha]) = -\beta\alpha([\Phi(\alpha), \alpha]) \ \forall\ \alpha, \beta \in \mathcal{R}. \tag{6}$$

Now take $\beta = t\beta$ in (6), we find

$$[\alpha, t]\mathcal{R}([\Phi(\alpha), \alpha]) = \{0\}.$$

From primeness of $\mathcal{R}$,

$$\Phi(\alpha)\alpha = \alpha\Phi(\alpha) \ \forall\ \alpha \in \mathcal{R}.$$

Using the same method that was used previously, we get

$$\Phi(\beta)\alpha\Phi(\beta) + \Phi(\beta)\alpha\beta + \Phi(\beta)\alpha\beta = \Phi(\beta)\Phi(\beta)\alpha + \Phi(\beta)\beta\alpha + \Phi(\beta)\beta\alpha \ \forall\ \alpha, \beta \in \mathcal{R}.$$

Let $\beta = m \circ n$, where $m, n \in \mathcal{R}$ in the above equation. With our hypothesis, we arrive at $(m \circ n)\alpha(m \circ n) = (m \circ n)(m \circ n)\alpha$. Substituting αs for α in the previous relationship, we find $(m \circ n)\mathcal{R}[(m \circ n), s] = \{0\}$, by using primeness of $\mathcal{R}$, we get $m \circ n \in \mathcal{Z}(\mathcal{R})$. So, $m(m \circ n) \in \mathcal{Z}(\mathcal{R})$ for all $m, n \in \mathcal{R}$, which implies $[m, \alpha]\mathcal{R}(m \circ n) = \{0\}$. So, $\mathcal{R}$ becomes commutative by using primeness of $\mathcal{R}$. In this case, from our hypothesis $\alpha\beta = \Phi(\alpha\beta)$. Replacing β by βz, we find $\Phi(z)\alpha\beta = 0$, so $\Phi = 0$ due to the primeness of $\mathcal{R}$.

(b) Suppose that g satisfies

$$g(xy + yx) - (xy - yx) = 0. \tag{7}$$

Set $x = y$ in (7), we get $g(x^2) = 0$, and it follows that $x^2 \in \mathcal{Z}(\mathcal{R})$ by using Lemma 3.

Now replace x by x^2 in (7), so

$$g(y)x^2 = 0 \Rightarrow g(y)\mathcal{R}x^2 = \{0\}.$$

This implies either $g = 0$ or $x^2 = 0$. Assume that $x^2 = 0$; then

$$0 = (x + xy)(x + xy) = xyx \ \forall\ x, y \in \mathcal{R},$$

which leads to a contradiction due to the primeness of $\mathcal{R}$.

4. Right homoderivations on near-rings

Theorem 4: *Consider* $h \neq 0$ *a right homoderivation of* $\mathcal{N}$ *that satisfies* $[h(x_1), x_2] \subseteq \mathcal{Z}(\mathcal{N})$ *for all* $x_1, x_2 \in \mathcal{N}$, *then* $\mathcal{N}$ *must be a commutative ring.*

Proof: Suppose $\mathcal{Z}(\mathcal{N}) = \{0\}$, so $h(\mathcal{N})$ is included in $\mathcal{Z}(\mathcal{N})$. So $h = 0$, a contradiction.

Hence, $\mathcal{Z}(\mathcal{N}) \neq \{0\}$, and with the given hypotheses, we find $[[h(x_1), x_2], t] = 0$. Placing $h(x_1)x_2$ for x_2 in the previous expression, then

$$[h(x_1), x_2]\mathcal{N}[h(x_1), x_2] = \{0\}.$$

Since $\mathcal{N}$ is 3-prime, $\mathcal{Z}(\mathcal{N})$ contains $h(\mathcal{N})$. So, $\mathcal{N}$ becomes a commutative ring.

Theorem 5: *Consider $h \neq 0$ is a right homoderivation of $\mathcal{N}$ that satisfies $h(x_1) \circ x_2 \in \mathcal{Z}(\mathcal{N})$ for all $x_1, x_2 \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Proof: Consider

$$h(x_1) \circ x_2 \in \mathcal{Z}(\mathcal{N}) \text{ for all } x_1, x_2 \in \mathcal{N}. \tag{8}$$

$y\mathcal{N}(-th(-x) + h(-x)t) = \{0\}$. So, $h = 0$, which contradicts $h \neq 0$, so $\exists z \in \mathcal{Z}(\mathcal{N})\backslash\{0\}$. Substituting x_2 with z in (8), we find

$$z(h(x_1) + h(x_1)) \in \mathcal{Z}(\mathcal{N}) \tag{9}$$

An application of Lemma 4 (a), leads to

$$h(2x_1) \in \mathcal{Z}(\mathcal{N}). \tag{10}$$

Substituting $2x_1$ for x_1 in (8), then (10) forces that

$$h(x_1 + x_1)(x_2 + x_2) \in \mathcal{Z}(\mathcal{N}).$$

According to Lemma 4(a), either $x_2 + x_2 \in \mathcal{Z}(\mathcal{N})$ or $h(x_1 + x_1) = 0$. Due to the 2-torsion freeness, $h(x_1) = 0$, which contradicts the fact that $h \neq 0$. Therefore, $x_2 + x_2 \in \mathcal{Z}(\mathcal{N})$ which implies that $x_2^2 + x_2^2 \in \mathcal{Z}(\mathcal{N}) \; \forall \; x_2 \in \mathcal{N}$, so, $(x_2 + x_2)\mathcal{N}(x_2t - tx_2) = \{0\}$. We derive $\mathcal{N} \subseteq \mathcal{Z}(\mathcal{N})$ from the 3-primeness of $\mathcal{N}$. According to Lemma 4 (b), we conclude that $\mathcal{N}$ is a commutative ring.

Theorem 6: *Consider $h \neq 0$ is a right homoderivation of $\mathcal{N}$ that satisfies $h([x, y]) = 0$ for all $x, y \in \mathcal{N}$, then $\mathcal{N}$ is a commutative ring.*

Proof: Consider

$$h([x, y]) = 0 \; \forall \; x, y \in \mathcal{N}. \tag{11}$$

Using $y = xy$ in (11) and applying it to the hypothesis, then

$$h(x)yx = h(x)xy \tag{12}$$

By setting $y = yw$ in (12) and applying (12), we find $h(x)\mathcal{N}(xw - wx) = \{0\}$. Therefore, $x \in \mathcal{Z}$ *or* $h(x) = 0$. Suppose $\exists \, x_0 \in \mathcal{Z}$, and taking $y = x_0y$ in (11), we get

$$h(x_0)xy = h(x_0)yx \tag{13}$$

For $y = ty$ (13), we find $h(x_0)\mathcal{N}[x, y] = \{0\}$. So, $\mathcal{N}$ is contained in $\mathcal{Z}(\mathcal{N})$ or $h(x_0) = 0$ by using 3-primeness of $\mathcal{N}$. From Lemma 4 (b), we find the desired result.

For a prime ring $\mathcal{R}$, the previous theorem directly leads to the next result.

Corollary 1: *$\mathcal{R}$ is commutative if a right homoderivation $\delta \neq 0$ of $\mathcal{R}$ that satisfies $\delta([m,n]) = 0$ for every $m,n \in \mathcal{R}$.*

Corollary 2: *If $\delta \neq 0$ is a right homoderivation of $\mathcal{R}$ satisfying $[\delta(m), \delta(n)] = 0$ for every $m,n \in \mathcal{R}$, then $\mathcal{R}$ must be commutative.*

Theorem 7: *Consider $\mathcal{N}$ is 2-torsion free. There is no right homoderivation $h \neq 0$ of $\mathcal{N}$ satisfying*

(a) $h(xy + yx) = 0 \; \forall \; x,y \in \mathcal{N}$ or,

(b) $h(xy - yx) - (xy + yx) = 0 \; \forall \; x,y \in \mathcal{N}$.

Proof: (a) Assuming that

$$h(xy + yx) = 0 \; \forall \; x,y \in \mathcal{N}. \tag{14}$$

Taking $y = xy$ in (14), then

$$-h(x)yx = h(x)xy \; \forall \; x,y \in \mathcal{N}. \tag{15}$$

Placing $y = yt$ in (15) and applying it, we infer $h(x)yt(-x) = h(x)y(-x)t$. Taking $-w$ in place of x, we get $h(-w)\mathcal{N}[w,t] = \{0\}$, so

$$w \in \mathcal{Z}(\mathcal{N}) \; or \; h(w) = 0 \; \forall \; x \in \mathcal{N}. \tag{16}$$

If $\exists x_0 \in \mathcal{Z}(\mathcal{N})$, by substituting x_0 for x in (14), we find $h(2x_0y) = 0$, so $h(x_0y) = 0$, therefore

$$h(x_0)h(y) + h(x_0)y + h(y)x_0 = 0$$

Taking $y = uv + vu$, it follows that

$$h(x_0)(uv + vu) = 0,$$

so

$$h(x_0)uv = -h(x_0)vu \; \forall \; u,v \in \mathcal{N}.$$

By substituting vt for v, then

$$h(x_0)uvt = -h(x_0)vtu,$$

so

$$h(x_0)\,\mathcal{N}[-u,t] = \{0\}.$$

So either $-u \in \mathcal{Z}(\mathcal{N})$ or $h(x_0) = 0$. From (16) with Lemma 4 (b), we deduce $\mathcal{N}$ is a commutative ring. So, (14) gives $h(xy) = 0$. Putting $y = uy$, we find

$$h(x)\,\mathcal{N}y = \{0\}$$

So, $h = 0$ by 3-primeness of $\mathcal{N}$, which is a contradiction.

(b) For $h(xy - yx) = xy + yx$. Putting x in place of y, we find $x^2 = 0$, so $0 = x(x + y)^2 = xyx$. Using $\mathcal{N}$ is 3-prime, we obtain a contradiction $\mathcal{N} = \{0\}$.

The next example shows that "$\mathcal{N}$ is 3-prime" cannot be removed in all theorems in this section.

Example 3: Considering a left near-ring $\mathcal{S}$ that is zero-symmetric, define $\mathcal{N}$, $\Phi: \mathcal{N} \to \mathcal{N}$ by:

$$\mathcal{N} = \left\{ \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \mid \alpha_1, \alpha_2, 0 \in \mathcal{S} \right\},$$

$$\Phi \begin{pmatrix} 0 & \alpha_1 & \alpha_2 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & \alpha_1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

Clearly, Φ is a right homoderivation of $\mathcal{N}$ that satisfies $(i)\ \Phi(M) \in \mathcal{Z}(\mathcal{N})$, $(ii)\ \Phi(M)N - N\Phi(M) \in \mathcal{Z}(\mathcal{N})$, $(iii)\ \Phi(M) \circ N \in \mathcal{Z}(\mathcal{N})$, $(iv)\ \Phi(MN - NM) = 0$, $(v)\ \Phi(MN + NM) = 0$, and $(vi)\ \Phi(MN - NM) = MN + NM$. However, $\mathcal{N}$ is not a commutative ring.

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