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Quantum and $\mathcal{LCD}$ codes from the direct product of r-copies of a non-local ring

Pooja Soni *
Department of Mathematics
Indira Gandhi University
Meerpur
Rewari 123401
Haryana
India

and

Department of Mathematics
Lovely Professional University
Phagwara 144411
Punjab
India

Manju Pruthi †
Department of Mathematics
Indira Gandhi University
Meerpur
Rewari 123401
Haryana
India

Abstract

This work focuses on constructing enhanced quantum error-correcting codes ($\mathcal{Q}$E$\mathcal{CC}$'s) based on the algebraic structure $\mathbb{R}_r$, which is defined as the direct product of r-copies of a non-local ring R, through the application of cyclic codes ($\mathcal{CC}$'s) over R through the CSS construction method. We discuss the structure of $\mathbb{R}_r$, highlights the fundamental features of R, and investigate the distance-preserving properties of the gray map over $\mathbb{R}_r$. We compare

* *E-mail:* **sonipooja2796@gmail.com (Corresponding Author)**
† *E-mail:* **manju.pruthi@yahoo.com**

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our codes with existing ones. Additionally, we generate several new and some optimal Linear complementary dual ($\mathcal{LCD}$) codes. We extend the study of Gowdhaman and Mohan [1], who have constructed $Q\mathcal{ECC}$'s over the ring $\mathbb{R}$. This work advances the design of efficient codes for quantum computing. We use magma for some computation.

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Keywords: *$Q\mathcal{ECC}$, $\mathcal{CC}$, $\mathcal{LCD}$, Direct product (DP).*

1. Introduction

Communication is a crucial part of our lives, involving the exchange of information between individuals through various noisy channels that can introduce errors. Safe data transmission is possible by proper error correction and detection techniques [2].

Quantum computation [3] provides a significant speed advantage over classical methods, particularly in solving tricky problems. Quantum bits (qubits) entanglement nature ensures secure transmission. Their superposition property makes them capable of storing more information than classical bits. Quantum data can be secured by $Q\mathcal{ECC}$'s. The first $Q\mathcal{ECC}$ was discovered by [3] in 1995. In 1998, [4] introduced a crucial technique for constructing $Q\mathcal{ECC}$'s by classical codes. Over the years, research has advanced toward the development of more efficient and optimized $Q\mathcal{ECC}$s, as well as their generalization over finite fields [5] and rings [6–10], utilizing cyclic codes. $\mathcal{LCD}$ codes also play a vital role in error correction and detection due to their complementary relationship with their dual. They provide error free transmission and may be derived from $\mathcal{CC}$'s by imposing specific conditions on their generator polynomials. Significant contributions in this area were carried out by [11-15], who constructed both $\mathcal{LCD}$ codes and $Q\mathcal{ECC}$'s by $\mathcal{CC}$'s over non-chain rings. Furthermore, [16] defined the $\mathcal{CC}$'s over a DP of rings, while [17] derived $Q\mathcal{ECC}$'s from $\mathcal{CC}$'s over the ring $F_p \times (F_p + vF_p)$.

By the above motivation, we extend the work of Gowdhaman and Mohan [1], who have constructed $Q\mathcal{ECC}$'s over the ring $\mathbb{R}$. Here, we construct a new structure $\mathbb{R}_r$, the DP of r-copies of the ring, $\mathbb{R} = \mathcal{F}_p\,[e_1, e_2]/\langle {e_1}^3 - e_1, {e_2}^3 - e_2, e_1e_2 - e_2e_1\rangle$, from which we derive novel and enhanced $Q\mathcal{ECC}$'s, along with new and optimized $\mathcal{LCD}$ codes, utilizing $\mathcal{CC}$'s over $\mathbb{R}$. This work represents a step forward in coding theory, offering new possibilities for error correction.

Structure of this study: Section 1 provides an overview of quantum coding, highlighting its background, significance, and contributions to quantum information theory. Section 2 defines fundamental terms, the structure of $\mathbb{R}_r$, and the essential properties of $\mathbb{R}$. It also introduces the Gray map, $\varphi : {\mathbb{R}_r}^{\sigma} \to \mathcal{F}_p^{9\sigma}$ and discusses its distance-preserving properties. Section 3 covers the important results of $\mathcal{CC}$'s over $\mathbb{R}$. Section 4 uses the results from the previous section to generate $Q\mathcal{ECC}$'s from $\mathbb{R}_r$ via the CSS construction. Section 5 examines the structure of $\mathcal{LCD}$ codes derived from $\mathbb{R}_r$. Section 6 demonstrates

the application of $\mathbb{R}_r$ to develop more effective $\mathcal{QECC}$'s than those found in existing literature, along with several novel and optimized $\mathcal{LCD}$ codes, using Magma. Finally, section 7 concludes the study.

2. Structure of $\mathbb{R}_r$, and Gray Map

Let $\mathbb{R}_r = \underbrace{\mathbb{R} \times \mathbb{R} \times \ldots \times \mathbb{R}}_{r\ times} = \{(X_1|X_2|\ldots|X_r)\}$ represents the DP of r-copies of the ring $\mathbb{R} = \mathcal{F}_p[e_1, e_2]/\langle {e_1}^3 - e_1, {e_2}^3 - e_2, e_1e_2 - e_2e_1\rangle = \mathcal{F}_p + e_1\mathcal{F}_p + e_1^2\mathcal{F}_p + e_2\mathcal{F}_p + e_2^2\mathcal{F}_p + e_1e_2\mathcal{F}_p + e_1^2e_2\mathcal{F}_p + e_1e_2^2\mathcal{F}_p + e_1^2e_2^2\mathcal{F}_p$, $p = q^e$ for $e \in \mathbb{N}$, $q \geq 3$ is a prime, and $\mathcal{F}_p$ denotes finite field. Each $X_i \in \mathbb{R}$ can be expressed, $X_i = a_{i1} + a_{i2}e_1 + a_{i3}e_1^2 + a_{i4}e_2 + a_{i5}e_2^2 + a_{i6}e_1e_2 + a_{i7}e_1^2e_2 + a_{i8}e_1e_2^2 + a_{i9}\, e_1^2e_2^2$ such that $a_{i\ell} \in \mathcal{F}_p\ \forall\ i \in \{1\ to\ r\}, \ell \in \{1\ to\ 9\}$. The ring $\mathbb{R}$ is non-local, commutative, non-chain, and finite such that $|\mathbb{R}| = p^9$. The idempotent elements of the ring $\mathbb{R}$ are: $I_1 = \frac{(e_1^2+e_1)(e_2^2+e_2)}{4}$, $I_2 = \frac{(e_1^2-e_1)(e_2^2-e_2)}{4}$, $I_3 = \frac{(e_1^2-e_1)(e_2^2+e_2)}{4}$, $I_4 = \frac{(e_1^2+e_1)(e_2^2-e_2)}{4}$, $I_5 = \frac{(e_1^2-e_1)(1-e_2^2)}{2}$, $I_6 = \frac{(e_1^2+e_1)(1-e_2^2)}{2}$, $I_7 = \frac{(1-e_1^2)(e_2^2-e_2)}{2}$, $I_8 = \frac{(1-e_1^2)(e_2^2+e_2)}{2}$, $I_9 = (1-e_1^2)(1-e_2^2)$ such that $I_\ell^2 = I_\ell$, $I_\ell I_{e'} = I_{e'}I_\ell = 0$ for $e' \neq \ell, \ell, e' \in \{1\ to\ 9\}$. Using Chinese remainder theorem, the decomposition of $\mathbb{R}$ is, $\mathbb{R} = \bigoplus_{\ell=1}^{9} I_\ell\mathbb{R} \cong \bigoplus_{\ell=1}^{9} I_\ell\mathcal{F}_p$. Any $\eta \in \mathbb{R}$ may be uniquely written as $\eta = \sum_{\ell=1}^{9} \eta_\ell I_\ell$, $\eta_\ell \in \mathcal{F}_p$, $\ell \in \{1, 2, \ldots, 9\}$.

Definition 2.1: A code, $\mathbb{C} \subseteq \mathbb{R}^{\sigma_1} \times \mathbb{R}^{\sigma_2} \times \ldots \times \mathbb{R}^{\sigma_r} = \mathbb{R}_r^{\sigma}$ is said to be an $\mathbb{R}$-submodule of $\mathbb{R}_r^{\sigma}$, if it is a σ-length linear code (LC) of over $\mathbb{R}$, where $\sigma = (\sigma_1 + \sigma_2 + \ldots + \sigma_r) \in \mathbb{N}$.

2.1. *The Gray Map on* $\mathbb{R}_r^{\sigma}$

It provides a systematic approach for mapping the elements of ring to vectors over $\mathcal{F}_p$, ensuring the preservation of the properties of the algebraic system. This method is crucial for the efficient processing and analysis of $\mathcal{CC}$'s.

Define a Gray map, $\varphi_1 : \mathbb{R} \to \mathcal{F}_p^9$ by $\varphi_1(\eta) = (\eta_1, \eta_2, \eta_3, \ldots, \eta_9)$, $\eta = \sum_{\ell=1}^{9} \eta_\ell I_\ell$, $\eta_\ell \in \mathcal{F}_p$, $\ell \in \{1\ to\ 9\}$. Its extension map, $\varphi_2 : \mathbb{R}_r \to \mathcal{F}_p^{9r}$ by, $\varphi_2(\mathbb{S}_1|\mathbb{S}_2|\ldots|\mathbb{S}_r) = (\eta_{11}, \eta_{12}, \ldots, \eta_{19}, \eta_{21}, \eta_{22}, \ldots, \eta_{29}, \ldots, \eta_{r1}, \eta_{r2}, \ldots, \eta_{r9})$, where $\mathbb{S}_i = \sum_{\ell=1}^{9} \eta_{i\ell} I_\ell$, $i \in \{1\ to\ r\}$, which can be simply defined to σ-length as:

Definition 2.2: Gray map, $\varphi : \mathbb{R}_r^{\sigma} \to \mathcal{F}_p^{9\sigma}$ by, $\varphi(\mathbb{T}_1|\,\mathbb{T}_2|\ldots|\mathbb{T}_r) = (\mathfrak{u}_{11}, \mathfrak{u}_{12}, \ldots, \mathfrak{u}_{19}, \mathfrak{u}_{21}, \mathfrak{u}_{22}, \ldots, \mathfrak{u}_{29}, \ldots, \mathfrak{u}_{r1}, \mathfrak{u}_{r2}, \ldots, \mathfrak{u}_{r9})$, where $\mathbb{T}_i = \sum_{\ell=1}^{9} \mathfrak{u}_{i\ell} I_\ell$, $\mathfrak{u}_{i\ell} = (\mathfrak{u}_{i\ell 1}, \mathfrak{u}_{i\ell 2}, \ldots, \mathfrak{u}_{i\ell\sigma_i})$, $i \in \{1\ to\ r\}$, $\ell \in \{1\ to\ 9\}$.

Note: Let $\mathbb{S}$ be a ring. The Hamming weight $\omega_H(\varepsilon)$ of $\varepsilon = (\varepsilon_1, \varepsilon_2, \ldots, \varepsilon_n) \in \mathbb{S}^n$ is defined as $\omega_H(\varepsilon) = \sum_{\varepsilon_b \neq 0} 1$, where $b \in \{1\ to\ n\}$. Also, the Hamming distance between ε and $\varepsilon' \in \mathbb{S}^n$ is $d_H(\varepsilon, \varepsilon') = \omega_H(\varepsilon - \varepsilon')$.

Definition 2.3: Lee weight of $\eta \in \mathbb{R}$ is defined as, $\omega_L(\eta) = \omega_H(\varphi_1(\eta))$. Also, the Lee weight of $\eta' \in \mathbb{R}_r^\sigma$ is the sum of the Lee weights of its coordinates. The Lee distance between $\eta'' \& \eta' \in \mathbb{R}_r^\sigma$ is defined as, $d_L(\eta'', \eta') = \omega_L(\eta'' - \eta')$.

Theorem 2.1: *Mapping $\varphi : \mathbb{R}_r^\sigma \to \mathcal{F}_p^{9\sigma}$ is $\mathcal{F}_p$-linear, bijective & preserves distance.*

Proof: Clearly, φ is bijective and linear over $\mathcal{F}_p$. Also, for all $\eta'', \eta' \in \mathcal{R}_r{}^\sigma$, we have $d_L(\eta'', \eta') = \omega_L(\eta'' - \eta') = \omega_H(\varphi(\eta'' - \eta')) = \omega_H(\varphi(\eta'') - \varphi(\eta')) = d_H(\varphi(\eta''), \varphi(\eta'))$, which shows that φ preserves distance.

Lemma 2.1: *[15] Let $\mathbb{C} \subseteq \mathbb{R}_r^\sigma$ be a LC of length σ. Then, $\varphi(\mathbb{C})$ be an $\mathcal{F}_p$- LC, $[9\sigma, k, d]$, where d represents the minimum Lee distance (MLD) of $\mathbb{C}$, & $k = log_p|\mathbb{C}|$.*

3. $\mathcal{CC}$'s over the ring $\mathbb{R}$

Let $\mathbb{C} \subseteq \mathbb{R}^n$ be a LC over $\mathbb{R}$, and define $\mathbb{C}_\ell = \{a_\ell \in \mathcal{F}_p^n \mid \sum_{e'=1}^{9} a_{e'} I_{e'} \in \mathbb{C}, a_{e'} \in \mathcal{F}_p^n\}$. Then, $\mathbb{C}_\ell \subseteq \mathcal{F}_p^n$ is a LC over $\mathcal{F}_p \; \forall \; \ell \in \{1 \; to \; 9\}$. Additionally, $\mathbb{C}$ and $\varphi(\mathbb{C})$ can be written as $\mathbb{C} = \bigoplus_{\ell=1}^{9} \mathbb{C}_\ell I_\ell$ and $(\mathbb{C}) = \bigotimes_{\ell=1}^{9} \mathbb{C}_\ell$, respectively. The generator matrix (GM) of $\mathbb{C}$ is written as $\begin{bmatrix} I_1\mathbb{G}_1 \\ I_2\mathbb{G}_2 \\ \vdots \\ I_9\mathbb{G}_9 \end{bmatrix}$, where $\mathbb{G}_\ell$ denotes the GM of $\mathbb{C}_\ell$.

Remark: For a LC, $\mathrm{M}_{j'}$, define $\bigoplus_{\mathrm{j}}^{k'} M_{\mathrm{j}} = \{ \sum_{\mathrm{j}=1}^{k'} b_{\mathrm{j}}' \mid b_{\mathrm{j}}' \in \mathrm{M}_{\mathrm{j}} \}$, and $\bigotimes_{\mathrm{j}=1}^{k'} \mathrm{M}_{\mathrm{j}} = \{(b_1', b_2', \ldots, b_{k'}') \mid b_{\mathrm{j}}' \in \mathrm{M}_{\mathrm{j}} \}$, for $\mathrm{j} \in \{1 \; to \; k'\}$.

Definition 3.1: A code, $\mathbb{C} \subseteq \mathbb{R}_r^\sigma = \mathbb{R}^{\sigma_1} \times \mathbb{R}^{\sigma_2} \times \ldots \times \mathbb{R}^{\sigma_r}$ can be represented as $(\mathbb{C}_1|\mathbb{C}_2|\ldots|\mathbb{C}_r)$, where $\mathbb{C}_i \subseteq \mathbb{R}^{\sigma_i}$, and $\mathbb{C}_i = \bigoplus_{\ell=1}^{9} \mathbb{C}_{i,\sigma_i}^{(\ell)} I_\ell$ for $i \in \{1 \; to \; r\}$.

Definition 3.2: The inner product of $y, x \in \mathbb{R}_r^\sigma$, is defined as, $\langle y, x \rangle = [\sum_{j=1}^{\sigma_1}(a_{11j}e_{11j}) + \sum_{j=1}^{\sigma_2}(a_{21j}e_{21j}) + \cdots + \sum_{j=1}^{\sigma_r}(a_{r1j}e_{r1j})]I_1 + [\sum_{j=1}^{\sigma_1}(a_{12j}e_{12j}) + \sum_{j=1}^{\sigma_2}(a_{22j}e_{22j}) + \cdots + \sum_{j=1}^{\sigma_r}(a_{r2j}e_{r2j})]I_2 + \ldots\ldots + [\sum_{j=1}^{\sigma_1}(a_{19j}e_{19j}) + \sum_{j=1}^{\sigma_2}(a_{29j}e_{29j}) + \ldots + \sum_{j=1}^{\sigma_r}(a_{r9j}e_{r9j})]I_9$, where $y = (\mathbb{T}_1|\,\mathbb{T}_2|\ldots|\mathbb{T}_r)$, $\mathbb{T}_i = \sum_{\ell=1}^{9} a_{i\ell} I_\ell$, $a_{i\ell} = (a_{i\ell 1}, a_{i\ell 2}, \ldots, a_{i\ell\sigma_i})$, and $x = (\mathbb{H}_1|\,\mathbb{H}_2|\ldots|\mathbb{H}_r)$, $\mathbb{H}_i = \sum_{\ell=1}^{9} e_{i\ell} I_\ell$, $e_{i\ell} = (e_{i\ell 1}, e_{i\ell 2}, \ldots, e_{1\ell\sigma_i})$ for $i \in \{1 \; to \; r\}$, $\ell \in \{1 \; to \; 9\}$.

Definition 3.3: A LC $\mathbb{C} \subseteq \mathbb{R}_r^\sigma$ is called a σ-length $\mathcal{CC}$ over $\mathbb{R}$ if, for every codeword $(\mathbb{a}_1, \mathbb{a}_2, \ldots, \mathbb{a}_\sigma) \in \mathbb{C}$, we have $(\mathbb{a}_\sigma, \mathbb{a}_1, \ldots, \mathbb{a}_{(\sigma-1)}) \in \mathbb{C}$. Furthermore, the dual code of $\mathbb{C} \subseteq \mathbb{R}_r^\sigma$ is $\mathbb{C}^\perp = \{ x \mid \forall \; y \in \mathbb{C}, \langle y, x \rangle = 0 \}$.

To grasp the definition of a $\mathcal{CC}$, $\mathbb{C} \subseteq \mathbb{R}_r^\sigma$, let us consider any vector $u \in \mathbb{R}_r^\sigma$, *i.e.,*

$$u = \Big(\big(b_{111}, b_{112}, \dots, b_{11\,\sigma_1}\big)I_1 + (b_{121}, b_{122}, \dots, b_{12\,\sigma_1})I_2 + \cdots + (b_{191}, b_{192}, \dots, b_{19\sigma_1})I_9\Big| (b_{211}, b_{212}, \dots, b_{21\sigma_2})I_1 + (b_{221}, b_{222}, .., b_{22\sigma_2})I_2 + \cdots + (b_{291}, b_{292}, \dots, b_{29\sigma_2})I_9|\dots\dots\dots\dots\dots\dots|(b_{r11}, b_{r12}, \dots, b_{r1\sigma_r})I_1 + (b_{r21}, b_{r22}, \dots, b_{r2\,\sigma_r})I_2 + \dots + \big(b_{r91}, b_{r92}, \dots, b_{r9\sigma_r}\big)I_9\Big),$$ in the form, $u = (b_{111}I_1 + b_{121}I_2 + \cdots + b_{191}I_9, b_{112}I_1 + b_{122}I_2 + \cdots + b_{192}I_9, \dots, b_{11\,\sigma_1}I_1 + b_{12\,\sigma_1}I_2 + \cdots + b_{19\,\sigma_1}I_9, b_{211}I_1 + b_{221}I_2 + \cdots + b_{291}I_9, b_{212}I_1 + b_{222}I_2 + \cdots + b_{292}I_9, \dots, b_{21\,\sigma_2}I_1 + b_{22\,\sigma_2}I_2 + \cdots + b_{29\,\sigma_2}I_9, \dots\dots\dots\dots\dots, b_{r11}I_1 + b_{r21}I_2 + \cdots + b_{r91}I_9, b_{r12}I_1 + b_{r22}I_2 + \cdots + b_{r92}I_9, \dots, b_{r1\,\sigma_r}I_1 + b_{r2\,\sigma_r}I_2 + \cdots + b_{r9\,\sigma_r}I_9)$.

Theorem 3.1: *Let* $\mathbb{C} \subseteq \mathbb{R}^n$ *be a LC. Then,* $\mathbb{C} = \bigoplus_{\ell=1}^{9} \mathbb{C}_\ell I_\ell$ *is a* $\mathcal{CC}$ *iff each* $\mathbb{C}_\ell \subseteq \mathcal{F}_p^n$ *is a* $\mathcal{CC}$.

Proof: Let $\xi \in \mathbb{C}$ be any element, $= \sum_{\ell=1}^{9} \xi_\ell I_\ell = (\sum_{\ell=1}^{9} \xi_{\ell,0}\, I_\ell, \sum_{\ell=1}^{9} \xi_{\ell,1}\, I_\ell, \dots, \sum_{\ell=1}^{9} \xi_{\ell,(n-1)}\, I_\ell)$, where $\xi_\ell = \big(\xi_{\ell,0}, \xi_{\ell,1}, \dots, \xi_{\ell,(n-1)}\big) \in \mathbb{C}_\ell \; \forall\, \ell \in \{1, 2, \dots, 9\}$. Suppose $\mathbb{C}_\ell$ is a $\mathcal{CC}$ over $\mathcal{F}_p \; \forall\, \ell \in \{1, 2, \dots, 9\}$, then $\alpha(\xi_\ell) = \alpha(\xi_{\ell,0}, \xi_{\ell,1}, \dots, \xi_{\ell,(n-1)}) = \big(\xi_{\ell,(n-1)}, \xi_{\ell,0}, \dots, \xi_{\ell,(n-2)}\big) \in \mathbb{C}_\ell$, where α is cyclic shift map. Now, $(\xi) = \alpha\big(\sum_{\ell=1}^{9} \xi_{\ell,0}\, I_\ell, \sum_{\ell=1}^{9} \xi_{\ell,1}\, I_\ell, \dots, \sum_{\ell=1}^{9} \xi_{\ell,(n-1)}\, I_\ell\big) = (\sum_{\ell=1}^{9} \xi_{\ell,(n-1)}\, I_\ell, \sum_{\ell=1}^{9} \xi_{\ell,0}\, I_\ell\, , \dots, \sum_{\ell=1}^{9} \xi_{\ell,(n-2)}\, I_\ell) = I_1\alpha(\xi_1) + I_2\alpha(\xi_2) + \cdots + I_9\, \alpha(\xi_9) \in \mathbb{C}$. Therefore, $\mathbb{C}$ be a $\mathcal{CC}$ over $\mathbb{R}$. Suppose that $\mathbb{C}$ is a $\mathcal{CC}$ over $\mathbb{R}$. Then, for $\xi \in \mathbb{C}$ $\alpha(\xi) = I_1\, \alpha(\xi_1) + I_2\, \alpha(\xi_2) + \cdots + I_9\, \alpha(\xi_9) \in \mathbb{C} \Rightarrow \alpha(\xi_\ell) \in \mathbb{C}_\ell$. Hence, each $\mathbb{C}_\ell$ be a $\mathcal{CC}$ over $\mathcal{F}_p$.

Corollary 3.1: *A LC,* $\mathbb{C} = (\mathbb{C}_1|\, \mathbb{C}_2|\dots|\, \mathbb{C}_r) \subseteq \mathbb{R}_r^{\sigma}$ *is a* $\mathcal{CC}$ *iff each* $\mathbb{C}_{i,\sigma_i}^{(\ell)} \subseteq \mathcal{F}_p{}^{\sigma_i}$ *is a* $\mathcal{CC}$ *over* $\mathcal{F}_p \; \forall\, i \in \{1\ to\ r\}$, *and* $\ell \in \{1\ to\ 9\}$, *where* $\mathbb{C}_i = \bigoplus_{\ell=1}^{9} \mathbb{C}_{i,\sigma_i}^{(\ell)} I_\ell \subseteq \mathbb{R}^{\sigma_i}$.

Corollary 3.2: *The dual code* $\mathbb{C}^{\perp} = \bigoplus_{l=1}^{9} \mathbb{C}_\ell^{\perp} I_\ell$ *of a* $\mathcal{CC}$, $\mathbb{C} = \bigoplus_{\ell=1}^{9} I_\ell\, \mathbb{C}_\ell$ *is also a cyclic code, where* $\mathbb{C}_\ell^{\perp}$ *denotes the dual code of* $\mathbb{C}_\ell$ *for* $\ell \in \{1\ to\ 9\}$.

Theorem 3.2: *Let* $\mathbb{C} = \bigoplus_{\ell=1}^{9} I_\ell\, \mathbb{C}_\ell$ *be a* $\mathcal{CC}$ *over* $\mathbb{R}$. *Then,* $\mathbb{C} = \langle I_1\mathbb{g}_1(t), I_2\, \mathbb{g}_2(t), \dots, I_9\mathbb{g}_9(t)\rangle$, *where* $\mathbb{C}_\ell = \langle \mathbb{g}_\ell(t)\, \rangle$, $\ell \in \{1\ to\ 9\}$.

Proof: Given that $\mathbb{C} = \bigoplus_{\ell=1}^{9} I_\ell\, \mathbb{C}_\ell$ is a $\mathcal{CC}$, it follows from Theorem 3.1 that each $\mathbb{C}_\ell$ is also $\mathcal{CC}$. Further, $\mathbb{C} = \sum_{l=1}^{9} I_l\langle \mathbb{g}_l(t)\rangle \Rightarrow \mathbb{C} \subseteq \langle I_1\mathbb{g}_1(t), I_2\, \mathbb{g}_2(t), \dots,\, I_9\mathbb{g}_9(t)\rangle$ (1). Let $(t) \in \langle I_1\mathbb{g}_1(t), I_2\, \mathbb{g}_2(t), \dots, I_9\mathbb{g}_9(t)\rangle$, then $\exists\ \mathbb{h}_\ell{}'(t) \in \mathcal{F}_p[t]$ such that $\mathbb{g}(t) = \sum_{\ell=1}^{9} I_\ell\, \mathbb{g}_\ell(t)\mathbb{h}_\ell{}'(t)$, which gives $\mathbb{g}(t) \in \bigoplus_{\ell=1}^{9} I_\ell\mathbb{C}_\ell = \mathbb{C} \Rightarrow \langle I_1\mathbb{g}_1(t), I_2\, \mathbb{g}_2(t), \dots,\, I_9\mathbb{g}_9(t)\rangle \subseteq \mathbb{C}$ (2). Hence, from (1) and (2), $\mathbb{C} = \langle I_1\mathbb{g}_1(t), I_2\, \mathbb{g}_2(t), \dots, I_9\mathbb{g}_9(t)\rangle$.

Theorem 3.3: *Let* $\mathbb{C} = \bigoplus_{\ell=1}^{9} I_\ell\mathbb{C}_\ell$ *be an n-length* $\mathcal{CC}$ *over* $\mathbb{R}$. *Then,* $\mathbb{C} = \langle \mathbb{f}(t)\rangle$ *such that* $\mathbb{f}(t)|(t^n - 1)$, *where* $\mathbb{f}(t) = \sum_{\ell=1}^{9} I_\ell\, \mathbb{g}_\ell(t)$.

Proof: Let $\mathbb{C}' = \langle \sum_{\ell=1}^{9} I_\ell \, \mathbb{g}_\ell(t) \rangle = \langle \mathbb{f}(t) \rangle$. Then $\mathbb{C} \supseteq \mathbb{C}'$. Now, $I_{e'} \, \mathbb{f}(t) = I_{e'} \{\sum_{\ell=1}^{9} I_\ell \, \mathbb{g}_\ell(t)\} = I_{e'} \mathbb{g}_{e'}(t)$ for all $\ell, e' \in \{1 \; to \; 9\}$, $\left(using \; {I_\ell}^2 = I_\ell, I_\ell I_{e'} = 0 \text{ for } \ell \neq e'\right)$. Hence, $I_{e'} \, \mathbb{g}_{e'}(t) \in \langle \mathbb{f}(t) \rangle$ for all $\ell, e' \in \{1, 2, \ldots, 9\}$, and thus $\mathbb{C}' \supseteq \mathbb{C}$. Therefore, $\mathbb{C} = \langle \mathbb{f}(t) \rangle$. Also, since $\mathbb{C}_\ell = \langle \, \mathbb{g}_\ell(t) \, \rangle$, so $\mathbb{g}_\ell(t) | (t^n - 1)$. Thus, $\exists \; \mathbb{h}_\ell(t) \in \mathcal{F}_p[t] / \langle t^n - 1 \rangle$ such that $\mathbb{g}_\ell(t) \mathbb{h}_\ell(t) = (t^n - 1) \; \forall \; \ell \in \{1, 2, .., 9\}$, Hence, $I_\ell \, \mathbb{g}_\ell(t) \mathbb{h}_\ell(t) = I_\ell \, (t^n - 1) \Rightarrow I_\ell \, \mathbb{f}(t) \mathbb{h}_\ell(t) = I_\ell \, (t^n - 1)$ $\forall \; \ell \in \{1, 2, \ldots, 9\}$. Thus, $[\sum_{\ell=1}^{9} \mathbb{h}_\ell(t) I_\ell] \, \mathbb{f}(t) = (t^n - 1)$, $(using \; \sum_{\ell=1}^{9} I_\ell = 1)$. Hence, $\mathbb{f}(t) | \, (t^n - 1)$.

4. $\mathcal{Q}$E$\mathcal{CC}$'s from $\mathbb{R}_r$, Using $\mathcal{CC}$'s over $\mathbb{R}$

Here, we discuss the development of $\mathcal{Q}$E$\mathcal{CC}$'s from $\mathbb{R}_r$, using $\mathcal{CC}$'s over $\mathbb{R}$ via CSS construction. To do so, we give some important results.

Definition 4.1: [12] "A $\mathcal{Q}$E$\mathcal{CC}$, $[[n, \mathbb{K}', d'']]_p$ is a $p^{\mathbb{K}'}$ −dimensional subspace of a p^n −dimensional complex Hilbert space, represented as $\underbrace{\mathbb{C}^p \otimes \mathbb{C}^p \otimes \ldots \otimes \mathbb{C}^p}_{n \; times}$, where p is a prime power and d'' is the minimum Hamming distance of a $\mathcal{Q}$E$\mathcal{CC}$. A $\mathcal{Q}$E$\mathcal{CC}$ has the ability to correct up to $\lfloor (d'' - 1)/2 \rfloor$ errors."

Theorem 4.1: *Let $\mathbb{C} \subseteq \mathbb{R}_r^\sigma$ be a LC. Then, the code $\varphi(\mathbb{C})$ is $\mathcal{F}_p$- linear. Furthermore, $|\mathbb{C}| = |\varphi(\mathbb{C})|$ and $\varphi(\mathbb{C}^\perp) = (\varphi(\mathbb{C}))^\perp$.*

Proof. The linearity of $\mathbb{C}$ directly implies the linearity of $\varphi(\mathbb{C})$. Additionally, given that φ is bijective, we conclude that $|\mathbb{C}| = |\varphi(\mathbb{C})|$. To establish that $\varphi(\mathbb{C}^\perp) = (\varphi(\mathbb{C}))^\perp$, we need to prove, $\varphi(\mathbb{C}^\perp) \subseteq [\varphi(\mathbb{C})]^\perp$, $|\varphi(\mathbb{C}^\perp)| = |[\varphi(\mathbb{C})]^\perp|$, which we will prove as follows:

Let $y = (\mathbb{T}_1 | \, \mathbb{T}_2 | \ldots | \mathbb{T}_r) \in \mathbb{C}^\perp$, where $\mathbb{T}_i = \sum_{\ell=1}^{9} u_{i\ell} I_\ell$ such that $u_{i\ell} = \left(u_{i\ell 1}, u_{i\ell 2}, \ldots, u_{i\ell \, \sigma_i}\right)$, and $x = (\mathbb{H}_1 | \, \mathbb{H}_2 | \ldots | \mathbb{H}_r) \in \mathbb{C}$, where $\mathbb{H}_i = \sum_{\ell=1}^{9} e_{i\ell} I_\ell$ such that $e_{i\ell} = \left(e_{i\ell 1}, e_{i\ell 2}, \ldots, e_{i\ell \, \sigma_i}\right)$ for all $i \in \{1 \; to \; r\}$, $\ell \in \{1 \; to \; 9\}$.

Now,

$\langle y, x \rangle = \left[\sum_{j=1}^{\sigma_1} \left(u_{11j} e_{11j}\right) + \sum_{j=1}^{\sigma_2} \left(u_{21j} e_{21j}\right) + \cdots + \sum_{j=1}^{\sigma_r} \left(u_{r1j} e_{r1j}\right)\right] I_1 +$
$\left[\sum_{j=1}^{\sigma_1} (u_{12j} e_{12j}) + \sum_{j=1}^{\sigma_2} (u_{22j} e_{22j}) + \cdots + \sum_{j=1}^{\sigma_r} (u_{r2j} e_{r2j})\right] I_2 + \ldots\ldots\ldots +$
$\left[\sum_{j=1}^{\sigma_1} \left(u_{19j} e_{19j}\right) + \sum_{j=1}^{\sigma_2} \left(u_{29j} e_{29j}\right) + \ldots + \sum_{j=1}^{\sigma_r} \left(u_{r9j} e_{r9j}\right)\right] I_9 = 0$ gives
$\sum_{j=1}^{\sigma_1} \left(u_{11j} e_{11j}\right) + \sum_{j=1}^{\sigma_2} \left(u_{21j} e_{21j}\right) + \cdots + \sum_{j=1}^{\sigma_r} \left(u_{r1j} e_{r1j}\right) = 0,$
$\sum_{j=1}^{\sigma_1} (u_{12j} e_{12j}) + \sum_{j=1}^{\sigma_2} (u_{22j} e_{22j}) + \cdots + \sum_{j=1}^{\sigma_r} (u_{r2j} e_{r2j}) = 0, \ldots,$
$\sum_{j=1}^{\sigma_1} \left(u_{19j} e_{19j}\right) + \sum_{j=1}^{\sigma_2} \left(u_{29j} e_{29j}\right) + \ldots + \sum_{j=1}^{\sigma_r} \left(u_{r9j} e_{r9j}\right) = 0 \Rightarrow \; \varphi(y) \varphi(x) =$
0. Thus, $\varphi(\mathbb{C}^\perp) \subseteq [\varphi(\mathbb{C})]^\perp$. Now, $|\mathbb{C}^\perp| = \prod_{i=1}^{r} \left|{\mathbb{C}_i}^\perp\right| = \prod_{i=1}^{r} p^{9\sigma_i - \sum_{\ell=1}^{9} {k_i}^{(\ell)}}$,
where ${k_i}^{(\ell)}$ is the dimension of $\mathcal{F}_p$ −LC, $\mathbb{C}_{i,\sigma_i}^{(\ell)}$, $i \in \{1 \; to \; r\}$. Bijection of $\varphi \Rightarrow$
$|\varphi(\mathbb{C}^\perp)| = |\mathbb{C}^\perp| = \prod_{i=1}^{r} p^{9\sigma_i - \sum_{\ell=1}^{9} {k_i}^{(\ell)}}$. Also, $|\varphi(\mathbb{C})| = |\mathbb{C}| =$
$\prod_{i=1}^{r} p^{\sum_{\ell=1}^{9} {k_i}^{(\ell)}}$ gives $|[\varphi(\mathbb{C})]^\perp| = p^{9\sigma - \mathbb{A}}$, where
$\mathbb{A} = \sum_{i=1}^{r} {k_i}^{(1)} + \sum_{i=1}^{r} {k_i}^{(2)} + \ldots + \sum_{i=1}^{r} {k_i}^{9}$. Hence, $|\varphi(\mathbb{C}^\perp)| = |[\varphi(\mathbb{C})]^\perp|$.

Corollary 4.1: *Let* $\mathbb{C} \subseteq \mathcal{R}_r{}^{\sigma}$ *be a LC. Then,* $\mathbb{C}^{\perp} \subseteq \mathbb{C} \Rightarrow \{\varphi(\mathbb{C})\}^{\perp} \subseteq \varphi(\mathbb{C})$.

Lemma 4.1: *Let* $\mathbb{C} = \langle \mathbb{f}(t) \rangle$ *be an n-length* $\mathcal{CC}$ *over* $\mathcal{F}_p$. *Then,* $\mathbb{C}^{\perp} \subseteq \mathbb{C} \Leftrightarrow t^n - 1 \equiv 0\big(mod\ \mathbb{f}(t)\mathbb{f}^*(t)\big)$, $\mathbb{f}^*(t)$ *is the reciprocal polynomial (RP) of* $\mathbb{f}(t)$.

Proof: Let $\mathbb{C}^{\perp} = \langle\ s'^*(t)\rangle$,where $s'(t) = (t^n - 1)/\mathbb{f}(t)$, $s'^*(t)$ is a RP of $s'(t)$. Now, $\mathbb{C}^{\perp} \subseteq \mathbb{C} \Leftrightarrow \exists\ l'(t) \in \mathcal{F}_p[t]$, such that $s'^*(t) = l'(t)\mathbb{f}(t) \Leftrightarrow \mathbb{f}^*(t)\mathbb{f}(t) = \frac{(t^n-1)}{s'^*(t)}\mathbb{f}(t) = \frac{(t^n-1)\mathbb{f}(t)}{l'(t)\mathbb{f}(t)} = \frac{(t^n-1)}{l'(t)} \Leftrightarrow \mathbb{f}^*(t)\mathbb{f}(t)\ l'(t) = (t^n - 1) \Leftrightarrow (t^n - 1)|\mathbb{f}^*(t)\mathbb{f}(t) \Leftrightarrow t^n - 1 \equiv 0(mod\ \mathbb{f}(t)\mathbb{f}^*(t))$. □

Theorem 4.2: *Let* $\mathbb{C} = \bigoplus_{\ell=1}^{9} \mathbb{C}_\ell I_\ell$ *is an n-length* $\mathcal{CC}$ *over* $\mathbb{R}$. *Then,* $\mathbb{C} \supseteq \mathbb{C}^{\perp} \Leftrightarrow t^n - 1 \equiv 0\big(mod\ \mathbb{g}_\ell(t)\mathbb{g}_\ell^*(t)\big)\ \forall\ \ell \in \{1\ to\ 9\}$, *where* $\mathbb{C}_\ell = \langle \mathbb{g}_\ell(t) \rangle$, $\mathbb{g}_\ell^*(t)$ *is the RP of* $\mathbb{g}_\ell(t)$.

Proof: Suppose $\mathbb{g}_\ell(t)\mathbb{g}_\ell^*(\mathrm{t})|(t^n - 1)$. Then, by Lemma 4.1, $\mathbb{C}_\ell^{\perp} \subseteq \mathbb{C}_\ell$, $\ell \in \{1\ to\ 9\}$. Hence, $\mathbb{C}_\ell^{\perp} I_\ell \subseteq \mathbb{C}_\ell I_\ell$. Therefore, $\mathbb{C}^{\perp} = \bigoplus_{\ell=1}^{9} \mathbb{C}_\ell^{\perp} I_\ell \bigoplus_{\ell=1}^{9} \mathbb{C}_\ell I_\ell = \mathbb{C}$. Conversely, suppose $\mathbb{C}^{\perp} \subseteq \mathbb{C} \Rightarrow \bigoplus_{\ell=1}^{9} \mathbb{C}_\ell^{\perp} I_\ell \subseteq \bigoplus_{\ell=1}^{9} \mathbb{C}_\ell I_\ell$. Then, $\mathbb{C}_\ell^{\perp} \subseteq \mathbb{C}_\ell$. Using Lemma 4.1, we get $\mathbb{g}_\ell(t)\mathbb{g}_\ell^*(\mathrm{t})|(t^n - 1)$ for $\ell \in \{1\ to\ 9\}$. □

Theorem 4.3: *Let* $\mathbb{C} \subseteq \mathbb{R}_r^{\sigma}$ *be a* $\mathcal{CC}$ *over* $\mathbb{R}$. *Then,* $\mathbb{C}^{\perp} \subseteq \mathbb{C}$ *iff* ${\mathbb{C}_i}^{\perp} \subseteq \mathbb{C}_i\ \forall\ i \in \{1\ to\ r\}$, *where* $\mathbb{C}_i \subseteq \mathbb{R}^{\sigma_i}$.

Proof: Let us assume that $\mathbb{C}^{\perp} \subseteq \mathbb{C}$. Then, we have $(\mathbb{C}_1^{\perp}|\mathbb{C}_2^{\perp}|\ldots|\mathbb{C}_r^{\perp}) \subseteq (\mathbb{C}_1|\mathbb{C}_2|\ldots|\mathbb{C}_r)$. Therefore, $\mathbb{C}_i^{\perp} \subseteq \mathbb{C}_i\ \forall\ i \in \{1\ to\ r\}$. Suppose $\mathbb{C}_i^{\perp} \subseteq \mathbb{C}_i\ \forall\ i \in \{1\ to\ r\}$. Then, $(\mathbb{C}_1^{\perp}|0|\ldots|0) \subseteq (\mathbb{C}_1|0|\ldots|0)$,…,$(0|0|\ldots|\mathbb{C}_r^{\perp}) \subseteq (0|0|\ldots|\mathbb{C}_r)$. As a result, we conclude that $(\mathbb{C}_1^{\perp}|\mathbb{C}_2^{\perp}|\ldots|\mathbb{C}_r^{\perp}) \subseteq (\mathbb{C}_1|\mathbb{C}_2|\ldots|\mathbb{C}_r)$, which leads to $\mathbb{C}^{\perp} \subseteq \mathbb{C}$.

Corollary 4.2: *Suppose* $\mathbb{C} \subseteq \mathbb{R}_r^{\sigma}$ *is a LC. Then,* $\mathbb{C} \supseteq \mathbb{C}^{\perp}$ *iff* $\mathbb{C}_{i,\sigma_i}^{(\ell)}$ *contains its dual over* $\mathcal{F}_p^{\sigma_i}\ \forall\ i \in \{1\ to\ r\}$, $\ell \in \{1\ to\ 9\}$.

Corollary 4.3: *If* $\mathbb{C}^{\perp} = \bigoplus_{\ell=1}^{9} \mathbb{C}_\ell^{\perp} I_\ell$ *is a* $\mathcal{CC}$, *where* $\mathbb{C}_\ell^{\perp} = \langle \mathbb{v}_\ell(t) \rangle$, $\ell \in \{1, 2, \ldots, 9\}$, *then* $\mathbb{C}^{\perp} = \langle \mathbb{v}(t) \rangle = \langle \sum_{\ell=1}^{9} I_\ell \mathbb{v}_\ell(t) \rangle$, *and* $\mathbb{v}(t)|(t^n - 1)$.

Theorem 4.4: *(CSS Construction, [15]) "Let* $\mathbb{C}_1 = [n, \mathbb{K}_1, d_1]_p$ *and* $\mathbb{C}_2 = [n, \mathbb{K}_2, d_2]_p$ *be two LC's over* $\mathcal{F}_p$ *such that* $\mathbb{C}_2^{\perp} \subseteq \mathbb{C}_1$, *then* $\exists$ *a* $\mathcal{QECC}$, $[[n, \mathbb{K}_1 + \mathbb{K}_2 - n, \geq d']]_p$, *where* $d' = min\{d_1, d_2\}$. *If* $\mathbb{C}_1^{\perp} \subseteq \mathbb{C}_1$, *then* $\exists$ *a* $\mathcal{QECC}$, $[[n, 2\mathbb{K}_1 - n, \geq d_1]]_p$. *"*

Theorem 4.5: *Consider* $\mathbb{C} \subseteq \mathbb{R}_r^{\sigma}$ *be a* $\mathcal{CC}$ *over* $\mathbb{R}$, *where* $\sigma = \sigma_1 + \sigma_2 + \cdots + \sigma_r$. *If* $\mathbb{C}_i \supseteq \mathbb{C}_i^{\perp}\ \forall\ i \in \{1\ to\ r\}$, *where* $\mathbb{C}_i \subseteq \mathbb{R}^{\sigma_i}$, *then* $\exists$ *a* $\mathcal{QECC}$, $\big[[9\sigma, 2\mathbb{K} - 9\sigma, d_L]\big]_p$, $\mathbb{K}$ *and* d_L *represent the dimension and MLD of* $\varphi(\mathbb{C})$, *respectively*

Proof: Suppose $\mathbb{C}_i^{\perp} \subseteq \mathbb{C}_i \; \forall \; i \in \{1 \; to \; r\}$, where $\mathbb{C}_i \subseteq \mathbb{R}^{\sigma_i}$. By above results, we have $\mathbb{C}^{\perp} \subseteq \mathbb{C}$, and hence $\varphi(\mathbb{C})$ is dual containing LC, $[9\sigma, \mathbb{K}, d_L]$ over $\mathcal{F}_p$. Consequently, by applying the CSS construction to $\varphi(\mathbb{C})$, we get a $\mathcal{Q}\mathrm{E}\mathcal{CC}$, $[[9\sigma, 2\mathbb{K} - 9\sigma, d_L]]_p$.

5. $\mathcal{LCD}$ codes from $\mathbb{R}_r$

A LC, $\mathbb{C}$ is referred to as an $\mathcal{LCD}$ code if $\mathbb{C}^{\perp} \cap \mathbb{C} = \{0\}$. This code can be constructed by applying specific constraints to the generator polynomials (GP) of $\mathcal{CC}$.

Lemma 5.1: *[18] Let $\mathbb{C} = \langle \mathbb{f}(t) \rangle$ is a $\mathcal{CC}$ such that $\mathbb{C} \subseteq \mathcal{F}_p^n$. Then, if $hcf(n, p) \neq 1$, then $\mathbb{C}$ is an $\mathcal{LCD}$ code $\Leftrightarrow$ $\mathbb{f}(t)$ is self-reciprocal as well as each monic irreducible factor of $\mathbb{f}(t)$ appears with the same multiplicity in both $t^n - 1$ and $\mathbb{f}(t)$. Also, if $hcf(n, p) = 1$, then $\mathbb{C}$ is an $\mathcal{LCD}$ code $\Leftrightarrow$ it is a reversible code.*

Theorem 5.1: *A $\mathcal{CC}$, $\mathbb{C} \subseteq \mathbb{R}^n$, expressed as $\mathbb{C} = \bigoplus_{\ell=1}^{9} \mathbb{C}_\ell I_\ell$ is an $\mathcal{LCD}$ code if each $\mathbb{C}_\ell \subseteq \mathcal{F}_p^n$ is an $\mathcal{LCD}$ code $\forall \; \ell \in \{1 \; to \; 9\}$.*

Proof: It is clear that, $\mathbb{C}^{\perp} \cap \mathbb{C} = \{0\}$ iff $\mathbb{C}_\ell \cap \mathbb{C}_\ell{}^{\perp} = \{0\}$.

Corollary 5.1: *Consider a cyclic code, $\mathbb{C} \subseteq \mathbb{R}_r^{\sigma}$ expressed as $(\mathbb{C}_1 | \, \mathbb{C}_2 | \ldots | \, \mathbb{C}_r)$, where $\mathbb{C}_i \subseteq \mathcal{R}^{\sigma_i}$ and $\mathbb{C}_i = \bigoplus_{\ell=1}^{9} \mathbb{C}_{i,\sigma_i}^{(\ell)} I_\ell$. Then, $\mathbb{C}$ is an $\mathcal{LCD}$ code iff $\mathbb{C}_{i,\sigma_i}^{(\ell)} \subseteq \mathcal{F}_p{}^{\sigma_i}$ is an $\mathcal{LCD}$ code $\forall \; i \in \{1 \; to \; r\}$, $\ell \in \{1 \; to \; 9\}$.*

Corollary 5.2: *A $\mathcal{CC}$, $\mathbb{C} \subseteq \mathbb{R}^n$ is an $\mathcal{LCD}$ code $\Leftrightarrow$ the polynomial $\mathbb{g}_\ell(t)$ is self reciprocal & every monic irreducible factor of $\mathbb{g}_\ell(t)$ appears with the same multiplicity in both $(t^n - 1)$ and $\mathbb{g}_\ell(t)$, where $\mathbb{C} = \langle \sum_{\ell=1}^{9} I_\ell \, \mathbb{g}_\ell(t) \rangle$, $\mathbb{g}_\ell(t) \in \mathcal{F}_p[t]$, $\mathbb{g}_\ell(t) | (t^n - 1)$, $j \in \{1 \; to \; 9\}$, $hcf(n, p) \neq 1$.*

Theorem 5.2: *A $\mathcal{CC}$, $\mathbb{C} \subseteq \mathbb{R}^n$ is an $\mathcal{LCD}$ code $\Leftrightarrow$ each $\mathbb{C}_\ell \subseteq \mathcal{F}_p^n$ is a reversible code, where $\mathbb{C} = \bigoplus_{\ell=1}^{9} \mathbb{C}_\ell I_\ell$, $hcf(n, p) = 1$.*

Proof: This follows directly from the above results.

Lemma 5.2: *For a LC, $\mathbb{C} \subseteq \mathbb{R}^n$, we have $\varphi(\mathbb{C} \cap \mathbb{C}^{\perp}) = \varphi(\mathbb{C}) \cap \varphi(\mathbb{C}^{\perp})$.*

Theorem 5.3: *A LC, $\mathbb{C} \subseteq \mathbb{R}^n$ is an $\mathcal{LCD}$ code $\Leftrightarrow$ $\varphi(\mathbb{C}) \subseteq \mathcal{F}_p^{9n}$ is an $\mathcal{LCD}$ code.*

6. $\mathcal{Q}\mathrm{E}\mathcal{CC}$'s and $\mathcal{LCD}$ codes derived from $\mathbb{R}_r$

Using the above results, we design several novel and improved $\mathcal{Q}\mathrm{E}\mathcal{CC}$'s from $\mathbb{R}_r$, utilizing $\mathcal{CC}$'s over $\mathbb{R}$. Furthermore, we find some optimum and several novel $\mathcal{LCD}$ codes. We use Magma for some computation. As discussed above, a $\mathcal{CC}$, $\mathbb{C} \subseteq \mathbb{R}_r^{\sigma}$ is given by $\mathbb{C} = (\mathbb{C}_1 | \, \mathbb{C}_2 | \ldots | \, \mathbb{C}_r)$, where $\mathbb{C}_i \subseteq \mathbb{R}^{\sigma_i}$ and $\mathbb{C}_i =$

$\bigoplus_{\ell=1}^{9} \mathbb{C}_{i,\sigma_i}^{(\ell)} I_\ell$, $i \in \{1\ to\ r\}$. The GP of $\mathbb{C}_i$ be $\mathbb{g}_{i,\sigma_i}^{(1)}(t)I_1 + \cdots + \mathbb{g}_{i,\sigma_i}^{(9)}(t)I_9$, where $\mathbb{g}_{i,\sigma_i}^{(\ell)}(t)$ is a GP of the code $\mathbb{C}_{i,\sigma_i}^{(\ell)}$ for $\ell \in \{1\ to\ 9\}$, $i \in \{1\ to\ r\}$. Utilizing above results, we obtained the parameters of $\varphi(\mathbb{C})$ and $Q\mathrm{E}\mathcal{CC}$'s.

Example 6.1: Let $p = 29, \sigma_1 = 8, \sigma_2 = 14,\ \sigma_3 = 12,\ r = 3$. Then, we have $\mathbb{R}_r^{\sigma} = \frac{\mathbb{R}[t]}{\langle t^8-1\rangle} \times \frac{\mathbb{R}[t]}{\langle t^{14}-1\rangle} \times \frac{\mathbb{R}[t]}{\langle t^{12}-1\rangle}$, $t^8 - 1 = (t+1)(t+12)\ldots(t^2+17)$, $t^{14} - 1 = (t+1)(t+4)\ldots(t+28)$, $t^{12} - 1 = (t+1)(t+12)\ldots(t^2+28t+1)$. Here, $\mathbb{g}^{(\ell)}{}_{1,\sigma_1}(t) = t^2 + 17$, $\mathbb{g}^{(\ell)}{}_{2,\sigma_2}(t) = (t+4)(t+9), \mathbb{g}^{(\ell)}{}_{3,\sigma_3}(t) = t^2 + 17t + 28, \ell \in \{1\ to\ 9\}$. Then, LC, $\varphi(\mathbb{C})$ be $[306, 252, 4]$, and a $Q\mathrm{E}\mathcal{CC}$ be $[[306, 198, 4]]_5$ (New).

6.2. *Computation Table*

First and second columns of Table 1 as show below, represents the length n, and σ_i of a code $\mathbb{C} \subseteq \mathcal{R}_r{}^{\sigma}$ and $\mathbb{C}_{i,\sigma_i}^{(\ell)}$, respectively. Next two columns of the table represent the GP, $\mathbb{g}_{i,\sigma_i}^{(\ell)}$(t), of the code $\mathbb{C}_{i,\sigma_i}^{(\ell)}$ for $\ell \in \{1\ to\ 9\}$. Last three columns represent the parameters of a linear code $\varphi(\mathbb{C})$, novel and improved $Q\mathrm{E}\mathcal{CC}$'s, and existing $Q\mathrm{E}\mathcal{CC}$'s. The polynomial $(t^2 + 3t + 12)(t + 1)$ is written as (13)(12))*(11). Table 2 lists several new and some optimal (opt.) $\mathcal{LCD}$ codes, with parameters given by φ(ℂ).

7. Conclusion

In this paper, we derived novel and improved $Q\mathrm{E}\mathcal{CC}$'s from $\mathbb{R}_r$, the DP of r-copies of a finite, non-local ring $\mathbb{R} = \mathcal{F}_p[e_1, e_2]/\langle {e_1}^3 - e_1, {e_2}^3 - e_2, e_1e_2 - e_2e_1\rangle$, utilizing $\mathcal{CC}$'s over $\mathbb{R}$ via CSS construction. We explored the structure of $\mathbb{R}_r$, the properties of gray map, leading to the development of new and optimized Linear complementary dual ($\mathcal{LCD}$) codes. We extend the study of Gowdhaman and Mohan [1], who have constructed $Q\mathrm{E}\mathcal{CC}$'s over the ring $\mathbb{R}$. This work advances the design of efficient codes for quantum computing. We use magma for some computation.

Table 1

Parameters of $Q\mathrm{E}\mathcal{CC}$'s from the DP of $\mathbb{R}$

n	σ_i	$\mathbb{g}^{(\ell)}{}_{1,\sigma_1} = \mathbb{g}^{(e')}{}_{1,\sigma_1}$ $\ell, e' \in \{1\ to\ 9\}$	$\mathbb{g}^{(\ell)}{}_{1,\sigma_2} = \mathbb{g}^{(e')}{}_{1,\sigma_2}$ $\ell, e' \in \{1\ to\ 9\}$	$\varphi(\mathbb{C})$	New $Q\mathrm{E}\mathcal{CC}$'s	Existing $Q\mathrm{E}\mathcal{CC}$'s
351	$\sigma_1 = 39$	(14)*(14)	…..	[351, 333, 3]	$[[351, 315, 3]]_{13}$	$[[351, 279, 3]]_{13}$ [1]
216	$\sigma_1 = 24$	107	……	[216, 198, 4]	$[[216, 180, 4]]_{19}$	New

Contd...

180	$\sigma_1 = 8$, $\sigma_2 = 12$	10(17)	1(17)(28)	[180, 144, 5]	$[[180, 108, 5]]_{23}$	New
189	$\sigma_1 = 5$, $\sigma_2 = 16$	14	102	[189, 162, 4]	$[[189, 135, 4]]_5$	$[[189, 117, 3]]_5$[1]
135	$\sigma_1 = 15$	111	…..	[135, 117, 4]	$[[135, 99, 4]]_5$	$[[135, 63, 3]]_5$[1]
180	$\sigma_1 = 8$, $\sigma_2 = 12$	1(12)	1(12)(28)	[180, 153, 5]	$[[180, 126, 5]]_{29}$	New
279	$\sigma_1 = 16$, $\sigma_2 = 15$	(12)*(103)	111	[279, 234, 5]	$[[279, 189, 5]]_5$	[[279, 171 [1]
270	$\sigma_1 = 14$, $\sigma_2 = 16$	(14)*(19)	10(12)	[270, 234, 4]	$[[270, 198, 4]]_{29}$	New
108	$\sigma_1 = 12$,	1(17)(28)	…..	[108, 90, 4]	$[[108, 72, 4]]_{29}$	New
180	$\sigma_1 = 20$	1(14)(28)	…..	[180,162,5]	$[[180, 144, 5]]_{29}$	New

Table 2

Parameters of $\mathcal{LCD}$ codes from the DP of $\mathbb{R}$

n	σ_i	$\mathbb{g}^{(\ell)}_{1,\sigma_1} = \mathbb{g}^{(e')}_{1,\sigma_1}$ $\ell, e' \in \{1\ to\ 9\}$	$\mathbb{g}^{(\ell)}_{1,\sigma_2} = \mathbb{g}^{(e')}_{1,\sigma_2}$ $\ell, e' \in \{1\ to\ 9\}$	$\varphi(\mathbb{C})$	Observation
81	$\sigma_1 = 4, \sigma_2 = 5$	101	16	$[81, 54, 5]_7$	New
63	$\sigma_1 = 3, \sigma_2 = 4$	111	101	$[63, 27, 5]_{11}$	New
99	$\sigma_1 = 8$	11	……	$[99, 90, 3]_{11}$	Opt.
81	$\sigma_1 = 9$	111	…..	$[81, 63, 5]_{11}$	New
63	$\sigma_1 = 3, \sigma_2 = 4$	111	11	$[63, 36, 4]_{17}$	New
63	$\sigma_1 = 3, \sigma_2 = 4$	1(16)	11	$[63, 45, 3]_{17}$	New
90	$\sigma_1 = 6, \sigma_2 = 4$	111	11	$[90, 63, 5]_{17}$	New
81	$\sigma_1 = 4, \sigma_2 = 5$	101	151	$[81, 45, 4]_{19}$	New
63	$\sigma_1 = 3, \sigma_2 = 4$	111	101	$[63, 27, 5]_{23}$	New
27	$\sigma_1 = 3$	111	…….	$[27, 9, 5]_{23}$	New
99	$\sigma_1 = 5, \sigma_2 = 6$	1(22)	111	$[99, 72, 4]_{23}$	New
27	$\sigma_1 = 3$	111	…….	$[27, 9, 5]_{29}$	New
108	$\sigma_1 = 4, \sigma_2 = 8$	11	11	$[108, 90, 3]_{31}$	New
72	$\sigma_1 = 8$	101	……	$[72, 54, 5]_{31}$	New
72	$\sigma_1 = 8$	11	……	$[72, 63, 3]_{37}$	Opt.
54	$\sigma_1 = 6$	(11)*(1(36))	…….	$[54, 36, 5]_{37}$	New
27	$\sigma_1 = 3$	111	…….	$[27, 9, 5]_{41}$	New
36	$\sigma_1 = 4$	101	…….	$[36, 18, 5]_{43}$	New

Contd...

27	$\sigma_1 = 3$	111	…….	$[27, 9, 5]_{47}$	New
54	$\sigma_1 = 6$	111	…….	$[54, 36, 5]_{53}$	New
126	$\sigma_1 = 5, \sigma_2 = 9$	11111	111	$[126, 72, 6]_{53}$	New
45	$\sigma_1 = 5$	11111	…..	$[45, 9, 6]_{53}$	New
90	$\sigma_1 = 6, \sigma_2 = 4$	111	101	$[90, 54, 6]_{59}$	New
27	$\sigma_1 = 3$	111	……	$[27, 18, 5]_{59}$	Opt.

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