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On star double graphs

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Abstract

The Mycielskian operation $\mu(\mathcal{G})$, introduced in [1] and later referred to as the Star Shadow Graph $S^*(\mathcal{G})$ in [2], preserves triangle-freeness while increasing the chromatic number by one. Motivated by this classical construction, we introduce a new graph operation called the Star Double Graph $D^*(\mathcal{G})$. We investigate its structural properties, including its chromatic number, triangle count, and its relationship to the Mycielskian. In addition, we compute several topological indices for both Star Shadow and Star Double Graphs, specifically $H(\mathcal{G})$, $W(\mathcal{G})$, $M_1(\mathcal{G})$, and $M_2(\mathcal{G})$. These results provide new insights into variations of the Mycielskian and their potential applications in graph theory and related fields.

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1. Introduction

Let $\mathcal{G} = (V, E)$ be a simple connected graph. [3] calls $\mathcal{G}$ triangle-free if it contains no 3-cycle. It also defines graph coloring as the allocation of vertex labels such that neighboring vertices share different labels. The smallest number of colors or labels required to achieve such is denoted by $\chi(\mathcal{G})$.

The Mycielskian construction in [1] produces from $\mathcal{G}$ a new graph $\mu(\mathcal{G})$ with vertex set $\mathcal{V} = V \cup V' \cup \{s\}$, where $V' = \{\mathfrak{g}' | \mathfrak{g} \in V\}$, and edge set $\mathcal{E} = E \cup \{\mathfrak{g}\mathfrak{h}' | \mathfrak{g}\mathfrak{h} \in E\} \cup \{s\mathfrak{g}' | \mathfrak{g} \in V\}$. [1] showed that $\mu(\mathcal{G})$ preserves the triangle-freeness of $\mathcal{G}$, and if $\chi(\mathcal{G}) = \varepsilon$, then $\chi(\mu(\mathcal{G})) = \varepsilon + 1$. [2] called $\mu(\mathcal{G})$ the Star Shadow Graph of $\mathcal{G}$, denoted $S^*(\mathcal{G})$, where the vertices in V' are called shadow vertices, and s is called the star vertex. Related investigations and extensions can be found in [4], [5], and [6].

The significance of the Mycielskian construction lies in its ability to enlarge a graph while preserving triangle-freeness, and at the same time increasing the chromatic number in the smallest possible way.

We introduce a new variation called the Star Double Graph $D^*(\mathcal{G})$, inspired by $S^*(\mathcal{G})$. We study its structural properties and derive expressions for several topological indices, including $H(\mathcal{G})$ (as defined in [7]), $M_1(\mathcal{G})$, $M_2(\mathcal{G})$ (as defined in [8]), and $W(\mathcal{G})$ (as defined in [6]), for both $S^*(\mathcal{G})$ and $D^*(\mathcal{G})$.

Throughout this paper, $\mathcal{G} = (V, E)$ is simple and connected, with κ vertices, λ edges, t triangles, and $\text{diam}(G) = d$

2. Preliminary Concepts

Topological indices are numerical invariants widely used in graph theory and its applications, particularly in chemistry and network analysis. They translate graph structure into scalar values that capture connectivity patterns (see [6, 8]). Depending on their formulation, these indices may be degree-based, distance-based, or spectral. Below, we recall several classical examples.

Definition 2.1: (Harary Index). In [7], $H(\mathcal{G})$ is a distance-based invariant defined as,

$$H(\mathcal{G}) = \frac{1}{2}\sum_{a=1}^{\kappa} \sum_{b=1}^{\kappa} (d_{\mathcal{G}}(\mathfrak{g}_a, \mathfrak{g}_b))^{-1}, a \neq b,$$

where $d_{\mathcal{G}}(\mathfrak{g}_a, \mathfrak{g}_b)$ is the distance between vertices $\mathfrak{g}_a$ and $\mathfrak{g}_b$ in $\mathcal{G}$.

Definition 2.2: (Wiener Index). In [6], $W(\mathcal{G})$ is a distance-based invariant defined as,

$$W(\mathcal{G}) = \frac{1}{2}\sum_{a=1}^{\kappa} \sum_{b=1}^{\kappa} d_{\mathcal{G}}(\mathfrak{g}_a, \mathfrak{g}_b), a \neq b$$

Definition 2.3: (First Zagreb Index). $M_1(\mathcal{G})$ is a degree-based invariant defined in [8] as follows,

$$M_1(\mathcal{G}) = \sum_{a=1}^{\kappa} (\deg(\mathfrak{g}_a))^2.$$

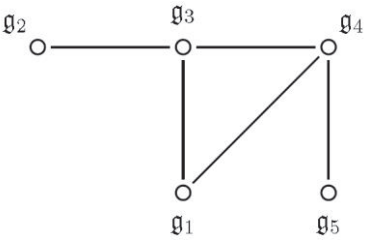
An equivalent form, also noted in [8], is

$$M_1(\mathcal{G}) = \sum_{\mathfrak{g}_a\mathfrak{g}_b \in E(\mathcal{G})} (\deg(\mathfrak{g}_a) + \deg(\mathfrak{g}_b)).$$

Definition 2.4: (Second Zagreb Index). $M_2(\mathcal{G})$ is another degree-based invariant, and it is defined in [8] as follows,

$$M_2(\mathcal{G}) = \sum_{\mathfrak{g}_a\mathfrak{g}_b \in E(\mathcal{G})} \deg(\mathfrak{g}_a)\deg(\mathfrak{g}_b).$$

Example 2.5: Consider $\mathcal{G}$ in Figure 1.



$\mathcal{G}$

Figure 1
Graph $\mathcal{G}$ with $\kappa = 5$ and $\lambda = 5$.

$D(\mathcal{G})$ together with vertex degrees is shown in Table 1.

Table 1
$D(\mathcal{G})$ and vertex degrees of $\mathcal{G}$.

	$\mathfrak{g}_1$	$\mathfrak{g}_2$	$\mathfrak{g}_3$	$\mathfrak{g}_4$	$\mathfrak{g}_5$	
$\mathfrak{g}_1$	0	2	1	1	2	$\deg(\mathfrak{g}_1) = 2$
$\mathfrak{g}_2$	2	0	1	2	3	$\deg(\mathfrak{g}_2) = 1$
$\mathfrak{g}_3$	1	1	0	1	2	$\deg(\mathfrak{g}_3) = 3$
$\mathfrak{g}_4$	1	2	1	0	1	$\deg(\mathfrak{g}_4) = 3$
$\mathfrak{g}_5$	2	3	2	1	0	$\deg(\mathfrak{g}_5) = 1$

Since $D(\mathcal{G})$ is symmetric, we only need the entries in its lower triangle when computing $H(\mathcal{G})$ and $W(\mathcal{G})$, which eliminates the factor $\frac{1}{2}$ from their formulas. In the lower triangle, there are five entries equal to 1, four entries equal to 2, and one entry equal to 3. Thus,

$$H(\mathcal{G}) = 5 \cdot 1 + 4 \cdot \tfrac{1}{2} + 1 \cdot \tfrac{1}{3} = \tfrac{22}{3} \approx 7.33$$
$$W(\mathcal{G}) = 5 \cdot 1 + 4 \cdot 2 + 1 \cdot 3 = 16.$$

For the $M_1(\mathcal{G})$, we have

$$M_1(\mathcal{G}) = 2^2 + 1^2 + 3^2 + 3^2 + 1^2 = 24.$$

To compute $M_2(\mathcal{G})$, consider the edges $\mathfrak{g}_1\mathfrak{g}_3$, $\mathfrak{g}_1\mathfrak{g}_4$, $\mathfrak{g}_2\mathfrak{g}_3$, $\mathfrak{g}_3\mathfrak{g}_4$, and $\mathfrak{g}_4\mathfrak{g}_5$. Hence,

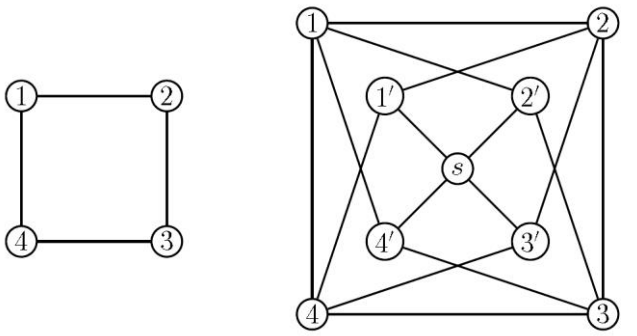
$$M_2(\mathcal{G}) = (2 \cdot 3) + (2 \cdot 3) + (1 \cdot 3) + (3 \cdot 3) + (3 \cdot 1) = 27.$$

3. Star Shadow Graph

Definition 3.1: (Star Shadow Graph). **[2]** defined the Star Shadow Graph as $S^*(\mathcal{G}) = (\mathcal{V}, \mathcal{E})$, where,

$$\mathcal{V} = V(\mathcal{G}) \cup V'(\mathcal{G}) \cup \{s\}, \text{where } V'(\mathcal{G}) = \{\mathfrak{g}' | \mathfrak{g} \in V(\mathcal{G})\}$$
$$\mathcal{E} = E(\mathcal{G}) \cup \{\mathfrak{g}\mathfrak{h}' | \mathfrak{g}\mathfrak{h} \in E(\mathcal{G})\} \cup \{s\mathfrak{g}' | \mathfrak{g}' \in V'(\mathcal{G})\}$$

Here, $V(\mathcal{G})$ denotes the original vertices, $V'(\mathcal{G})$ the shadow vertices, and s is the star vertex. Originally, this is called the Mycielskian of $\mathcal{G}$, denoted by $\mu(\mathcal{G})$ (see [1, 2]).



C_4 $S^*(C_4)$

Figure 2
C_4 and its Star Shadow Graph $S^*(C_4)$.

Mycielski showed in [1] that if $\mathcal{G}$ is triangle-free, then so is $\mu(\mathcal{G})$. Also, $\chi(\mathcal{G}) = \varepsilon$ implies $\chi(\mu(\mathcal{G})) = \varepsilon + 1$ (Figure 2).

Proposition 3.2:

(1) $S^*(\mathcal{G})$ has $2\kappa + 1$ vertices and $3\lambda + \kappa$ edges.

(2) For $\mathfrak{g} \in V$, $\mathfrak{g}' \in V'$, and the star vertex s:
   (a) $\deg_{S^*(\mathcal{G})}(\mathfrak{g}) = 2\deg_{\mathcal{G}}(\mathfrak{g})$,
   (b) $\deg_{S^*(\mathcal{G})}(\mathfrak{g}') = \deg_{\mathcal{G}}(\mathfrak{g}) + 1$,
   (c) $\deg_{S^*(\mathcal{G})}(s) = \kappa$.

(3) $S^*(\mathcal{G})$ contains $4t$ triangles.

Proof: (1) and (2) follow directly from the construction of $S^*(\mathcal{G})$. For (3), let $\mathcal{G}$ contain t triangles. If $\mathfrak{g}_i\mathfrak{g}_j\mathfrak{g}_k$ is a triangle in $\mathcal{G}$, then $\mathfrak{g}_i\mathfrak{g}_j\mathfrak{g}_k, \mathfrak{g}'_i\mathfrak{g}_j\mathfrak{g}_k, \mathfrak{g}_i\mathfrak{g}'_j\mathfrak{g}_k, \mathfrak{g}_i\mathfrak{g}_j\mathfrak{g}'_{\mathfrak{g}}$ are triangles in $S^*(\mathcal{G})$. Thus, each triangle in $\mathcal{G}$ corresponds to four in $S^*(\mathcal{G})$, giving a total of $4t$.

[4] studied the degree distance index of the Mycielskian and observed certain patterns in its distance values. They also noted that the largest distance in the Mycielskian of any graph does not exceed 4. In this paper, we prove these distance properties in the context of the Star Shadow Graph.

Proposition 3.3: For distinct vertices $\mathfrak{g}, \mathfrak{h} \in V(S^*(\mathcal{G}))$, the distances satisfy:

(1) $d_{S^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = \begin{cases} 1 & \text{if } \mathfrak{g} = s \text{ and } \mathfrak{h} \in V', \\ 2 & \text{if } \mathfrak{g} = s \text{ and } \mathfrak{h} \in V. \end{cases}$

(2) If $\mathfrak{g}, \mathfrak{h} \in V'$, then $d_{S^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = 2$.

(3) $d_{S^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = \begin{cases} d_{\mathcal{G}}(\mathfrak{g}, \mathfrak{h}) & \text{if } \mathfrak{g}, \mathfrak{h} \in V \text{ and } d_{\mathcal{G}}(\mathfrak{g}, \mathfrak{h}) \leq 4, \\ 4 & \text{if } \mathfrak{g}, \mathfrak{h} \in V \text{ and } d_{\mathcal{G}}(\mathfrak{g}, \mathfrak{h}) > 4. \end{cases}$

(4) $d_{S^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) =$

$$\begin{cases} 2 & \text{if } \mathfrak{g} = v_a \in V, \mathfrak{h} = v_{a'} \in V', \\ d_{\mathcal{G}}(v_a, v_b) & \text{if } \mathfrak{g} = v_a \in V, \mathfrak{h} = v_{b'} \in V', a \neq b, d_{\mathcal{G}}(v_a, v_b) \leq 3, \\ 3 & \text{if } \mathfrak{g} = v_a \in V, \mathfrak{h} = v_{b'} \in V', a \neq b, d_{\mathcal{G}}(v_a, v_b) > 3. \end{cases}$$

Proof: Let s be the star vertex of $S^*(\mathcal{G})$. Then,

(1) By construction, s is adjacent to every shadow vertex $v_{a'}$, so $d_{S^*(\mathcal{G})}(s, v_{a'}) = 1$. If $\mathfrak{h} = v_a \in V$, then v_a connects to s via $(v_a, v_{b'}, s)$ for some neighbor v_b of v_a, giving distance 2.

(2) For two shadow vertices $v_{a'}$ and $v_{b'}$, the path $(v_{a'}, s, v_{b'})$ has length 2, and since no direct edge joins them, this is the exact distance.

(3) If $\mathfrak{g}, \mathfrak{h} \in V$, then the path $(\mathfrak{g}, v_{a'}, s, v_{b'}, \mathfrak{h})$ has length 4, so $d_{S^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) \leq 4$. When $d_{\mathcal{G}}(\mathfrak{g}, \mathfrak{h}) \leq 4$, this distance is preserved; otherwise the constructed path ensures $d_{S^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = 4$.

(4) If $\mathfrak{g} = v_a$ and $\mathfrak{h} = v_{a'}$, take a neighbor v_b of v_a; then $(v_a, v_b, v_{a'})$ gives a 2-path. If $\mathfrak{g} = v_a$ and $\mathfrak{h} = v_{b'}$ with $a \neq b$, the path $(v_a, v_{c'}, s, v_{b'})$ shows $d_{S^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) \leq 3$. When $d_{\mathcal{G}}(v_a, v_b) \leq 3$, the distance agrees with $d_{\mathcal{G}}(v_a, v_b)$; otherwise it equals 3.

Example 3.4: Consider $\mathcal{G}$ and $S^*(\mathcal{G})$ in Figure 3.

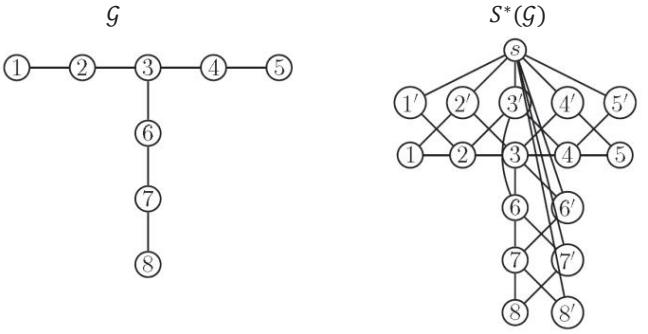


Figure 3
$\mathcal{G}$ and $S^*(\mathcal{G})$.

Table 2

Distance matrix $D(\mathcal{G})$.

	1	2	3	4	5	6	7	8
1	0	1	2	3	4	3	4	5
2	1	0	1	2	3	2	3	4
3	2	1	0	1	2	1	2	3
4	3	2	1	0	1	2	3	4
5	4	3	2	1	0	3	4	5
6	3	2	1	2	3	0	1	2
7	4	3	2	3	4	1	0	1
8	5	4	3	4	5	2	1	0

Table 3

Distance matrix $D(S^*(\mathcal{G}))$.

	1	2	3	4	5	6	7	8	1′	2′	3′	4′	5′	6′	7′	8′	s
1	0	1	2	3	4	3	4	4	2	1	2	3	3	3	3	3	2
2	1	0	1	2	3	2	3	4	1	2	1	2	3	2	3	3	2
3	2	1	0	1	2	1	2	3	2	1	2	1	2	1	2	3	2
4	3	2	1	0	1	2	3	4	3	2	1	2	1	2	3	3	2
5	4	3	2	1	0	3	4	4	3	3	2	1	2	2	3	3	2
6	3	2	1	2	3	0	1	2	3	2	1	2	3	2	1	2	2
7	4	3	2	3	4	1	0	1	3	3	2	3	3	2	2	1	2
8	4	4	3	4	4	2	1	0	3	3	3	3	3	2	1	2	2
1′	2	1	2	3	3	3	3	3	0	2	2	2	2	2	2	2	1
2′	1	2	1	2	3	2	3	3	2	0	2	2	2	2	2	2	1
3′	2	1	2	1	2	1	2	3	2	2	0	2	2	2	2	2	1
4′	3	2	1	2	1	2	3	3	2	2	2	0	2	2	2	2	1
5′	3	3	2	1	2	3	3	3	2	2	2	2	0	2	2	2	1
6′	3	2	1	2	3	2	1	2	2	2	2	2	2	0	2	2	1
7′	3	3	2	3	3	1	2	1	2	2	2	2	2	2	0	2	1
8′	3	3	3	3	3	2	1	2	2	2	2	2	2	2	2	0	1
s	2	2	2	2	2	2	2	2	1	1	1	1	1	1	1	1	0

From Table 2 and Table 3, one observes that $\mathrm{diam}(\mathcal{G}) = 5$, while $\mathrm{diam}(S^*(\mathcal{G})) = 4$.

The main difficulty in constructing the distance matrix of a Star Shadow Graph lies in identifying how many pairs of vertices have distance at least 4. To handle this, we introduce the following notions.

Definition 3.5: (α-distance matrix). For a nonnegative integer $\alpha \leq d$, $\tau_\alpha(\mathcal{G}) = [t_\alpha(a,b)]$ is a $\kappa \times \kappa$ matrix, where

$$t_\alpha(a,b) = \begin{cases} 1 & \text{if } d_{\mathcal{G}}(v_a, v_b) = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

Remark 1: $A(\mathcal{G}) = \tau_1(\mathcal{G})$, and $I_\kappa = \tau_0(\mathcal{G})$.

With this concept, $D(\mathcal{G})$ can be written as

$$D(\mathcal{G}) = \sum_{\alpha=1}^{d} \alpha \cdot \tau_\alpha(\mathcal{G}).$$

Observe that each nonzero entry of $\tau_\alpha(\mathcal{G})$ corresponds to a pair of vertices in $\mathcal{G}$ at distance α. Hence, the number of such pairs can be obtained by summing all entries of $\tau_\alpha(\mathcal{G})$ and dividing the total by 2. We formally define this as follows.

Definition 3.6: (α-distance number). Let $t_\alpha(a,b)$ be the entries of $\tau_\alpha(\mathcal{G})$. The α-distance number of $\mathcal{G}$ is given by

$$\phi_\alpha(\mathcal{G}) = \frac{1}{2}\sum_{a=1}^{\kappa} \sum_{b=1}^{\kappa} t_\alpha(a,b), t_\alpha(a,b) = \begin{cases} 1 & \text{if } d_{\mathcal{G}}(v_a, v_b) = \alpha, \\ 0 & \text{otherwise.} \end{cases}$$

The α-distance number counts the vertex pairs of $\mathcal{G}$ that are exactly α units apart. Such pairs contribute equally to the entries of $D(\mathcal{G})$ and are useful for distance-based computations.

With these, $D(S^*(\mathcal{G}))$ can be constructed as follows:

$$D(S^*(\mathcal{G})) = \begin{array}{c} \\ V \\ V' \\ s \end{array} \begin{array}{c} \begin{array}{ccc} V & V' & s \end{array} \\ \begin{pmatrix} D^*_{\mathcal{G}} & D^*_{\mathcal{G}}(D'_{\mathcal{G}})^T + 2I_\kappa & 2J_{1\times\kappa} \\ D'_{\mathcal{G}} + 2I_\kappa & 2(J_{\kappa\times\kappa} - I_\kappa) & J_{1\times\kappa} \\ 2J_{\kappa\times 1} & J_{\kappa\times 1} & 0 \end{pmatrix} \end{array}$$

Where,

- $D^*_{\mathcal{G}} = \sum_{\alpha=1}^{3} \alpha \cdot \tau_\alpha(\mathcal{G}) + \sum_{\alpha=4}^{d} 4 \cdot \tau_\alpha(\mathcal{G})$
- $D'_{\mathcal{G}} = \sum_{\alpha=1}^{3} \alpha \cdot \tau_\alpha(\mathcal{G}) + \sum_{\alpha=4}^{d} 3 \cdot \tau_\alpha(\mathcal{G})$
- $J_{\kappa\times m}$ is a $\kappa \times m$ matrix where the entries are entirely 1.
- I_κ is the $\kappa \times \kappa$ identity matrix.

3.1 *Star Double Graphs*

Many researchers have studied the classical Mycielskian construction (also known as the Star Shadow Graph), since it preserves the triangle-free property of a graph while enabling arbitrarily large chromatic numbers. Motivated by this idea, we propose a new variation, which we call the **Star Double Graph**.

In this subsection, we examine whether the basic properties of the original Mycielskian remain valid for this new construction. Where they do not, we aim to establish analogous results.

Definition 3.7: (Star Double Graph). The Star Double of $\mathcal{G}$ is the graph $D^*(\mathcal{G}) = (\mathcal{V}, \mathcal{E})$, where

$$\mathcal{V} = V(\mathcal{G}) \cup V'(\mathcal{G}) \cup \{s^*\}, \text{where } V'(\mathcal{G}) = \{\mathfrak{g}' | \mathfrak{g} \in V(\mathcal{G})\}$$

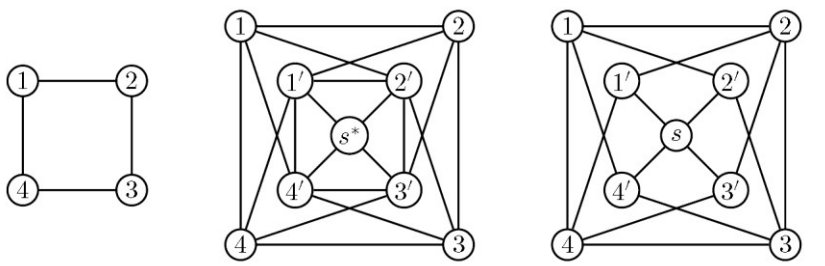
$$\mathcal{E} = E(\mathcal{G}) \cup E'(\mathcal{G}) \cup \{\mathfrak{g}\mathfrak{h}' | \mathfrak{g}\mathfrak{h} \in E(\mathcal{G})\} \cup \{s^*\mathfrak{g}' | \mathfrak{g}' \in V'(\mathcal{G})\},$$

$$\text{where, } E'(\mathcal{G}) = \{\mathfrak{g}'\mathfrak{h}' | \mathfrak{g}\mathfrak{h} \in E(\mathcal{G})\}$$

Here, $V(\mathcal{G})$ denotes the original vertices, $V'(\mathcal{G})$ the duplicate vertices, $E'(\mathcal{G})$ the duplicate edges, and s^* is the star vertex.

Remark 2: $D^*(\mathcal{G})$ contains a copy of $\mathcal{G}$, denoted $\mathcal{G}'$, since $\mathfrak{g}'\mathfrak{h}'$ is included in $D^*(\mathcal{G})$ whenever $\mathfrak{g}\mathfrak{h}$ is an edge in $\mathcal{G}$.

Example 3.8: Consider the graphs shown in Figure 4.



C_4 $D^*(C_4)$ $S^*(C_4)$;

Figure 4
C_4, $D^*(C_4)$, and $S^*(C_4)$.

It follows directly that $S^*(\mathcal{G})$ is a subgraph of $D^*(\mathcal{G})$. Moreover, $\mathcal{V}_{S^*(\mathcal{G})} = \mathcal{V}_{D^*(\mathcal{G})}$ and $\mathcal{E}_{D^*(\mathcal{G})} = \mathcal{E}_{S^*(\mathcal{G})} \cup E$

The corresponding result of Proposition 3.2 for Star Double Graphs is stated below.

Proposition 3.9:

(1) In $D^*(\mathcal{G})$, $|\mathcal{V}| = 2\kappa + 1$, and $|\mathcal{E}| = 4\lambda + \kappa$.

(2) For each $\mathfrak{g} \in V$, its duplicate $\mathfrak{g}' \in V'$, and the star vertex s^*:

(a) $\deg_{D^*(\mathcal{G})}(\mathfrak{g}) = 2\deg_{\mathcal{G}}(\mathfrak{g})$,

(b) $\deg_{D^*(\mathcal{G})}(\mathfrak{g}') = 2\deg_{\mathcal{G}}(\mathfrak{g}) + 1$,

(c) $\deg_{D^*(\mathcal{G})}(s^*) = \kappa$.

(3) $D^*(\mathcal{G})$ contains exactly $5t + \lambda$ triangles.

(4) If $\chi(\mathcal{G}) = \varepsilon$, then $\chi(D^*(\mathcal{G})) = \varepsilon + 1$.

Proof:

(1) Since the $\mathcal{V}_{D^*(\mathcal{G})}$ and $\mathcal{V}_{S^*(\mathcal{G})}$ have the same elements, and $\mathcal{E}_{D^*(\mathcal{G})}$ is obtained from the union of $\mathcal{E}_{S^*(\mathcal{G})}$ and E, the claimed vertex and edge counts follow.

(2) Consider $\mathfrak{g} \in V(\mathcal{G})$, its copy $\mathfrak{g}' \in V'(\mathcal{G})$, and the star vertex s^*.

(a) Each neighbor u of $\mathfrak{g}$ in $\mathcal{G}$ contributes both u and u' as neighbors of $\mathfrak{g}$ in $D^*(\mathcal{G})$. Thus, $\deg_{D^*(\mathcal{G})}(\mathfrak{g}) = 2\deg_{\mathcal{G}}(\mathfrak{g})$.

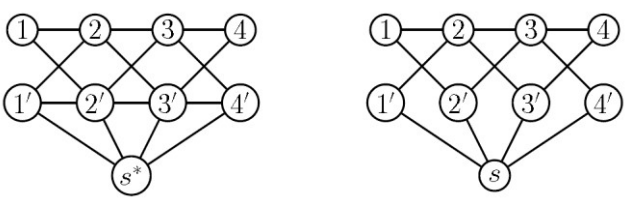
(b) For every $u \in N_{\mathcal{G}}(\mathfrak{g})$, the vertices u, u', and s^* are adjacent to $\mathfrak{g}'$. Hence $\deg_{D^*(\mathcal{G})}(\mathfrak{g}') = 2\deg_{\mathcal{G}}(\mathfrak{g}) + 1$.

(c) Every duplicate vertex $\mathfrak{g}' \in V'(\mathcal{G})$ is linked to s^*, giving $\deg_{D^*(\mathcal{G})}(s^*) = \kappa$.

(4) The graph $\mathcal{G}$ contributes t triangles, and the copy $\mathcal{G}'$ contributes another t. By Proposition 3.2, attaching each $\mathfrak{g}'$ to the neighbors of $\mathfrak{g}$ produces $4t$ more. Finally, for every edge $\mathfrak{g}_1\mathfrak{g}_2 \in E$, the vertices $\mathfrak{g}_1', \mathfrak{g}_2'$, and s^* form a triangle, yielding λ additional ones. Altogether, this gives $5t + \lambda$ triangles.

(5) Suppose $\mathcal{G}$ is ε-colorable. Assign to each $\mathfrak{g}' \in V'(\mathcal{G})$ the same color as its original vertex $\mathfrak{g}$. Since s^* is adjacent to all of $V'(\mathcal{G})$, an extra color is needed for s^*. Therefore $\chi(D^*(\mathcal{G})) = \varepsilon + 1$.

Example 3.10: As an illustration, consider the Star Double and Star Shadow graphs of the path P_4, shown in Figure 5.



$D^*(P_4)$ $S^*(P_4)$

Figure 5
$D^*(P_4)$ and $S^*(P_4)$.

Some distances in $D^*(P_4)$ coincide with those in $S^*(P_4)$. For instance, the distance between vertices 1 and 4 is 3 in both graphs. On the other hand,

certain distances in $D^*(P_4)$ are shorter. For example, whenever $ab \in E(P_4)$, we have $d_{D^*(P_4)}(a', b') = 1$,while this is not the case in $S^*(P_4)$.

Proposition 3.11: For distinct vertices $\mathfrak{g}, \mathfrak{h} \in \mathcal{V}$, the distances satisfy:

(1) $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = \begin{cases} 1 & \text{if } \mathfrak{g} = s^*, \mathfrak{h} \in V', \\ 2 & \text{if } \mathfrak{g} = s^*, \mathfrak{h} \in V \end{cases}$

(2) $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = \begin{cases} 1 & \text{if } \mathfrak{g} = v_{a'}, \mathfrak{h} = v_{b'}, d_{\mathcal{G}}(v_a, v_b) = 1, \\ 2 & \text{if } \mathfrak{g} = v_{a'}, \mathfrak{h} = v_{b'}, d_{\mathcal{G}}(v_a, v_b) \neq 1 \end{cases}$

(3) $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = \begin{cases} d_{\mathcal{G}}(v_a, v_b) & \text{if} \mathfrak{g} = v_a, \mathfrak{h} = v_b, d_{\mathcal{G}}(v_a, v_b) \leq 4, \\ 4 & \text{if } \mathfrak{g} = v_a, \mathfrak{h} = v_b, d_{\mathcal{G}}(v_a, v_b) > 4 \end{cases}$

(4) $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = \begin{cases} 2 & \text{if } \mathfrak{g} = v_a, \mathfrak{h} = v_{a'}, \\ d_{\mathcal{G}}(v_a, v_b) & \text{if } \mathfrak{g} = v_a, \mathfrak{h} = v_{b'}, a \neq b, d_{\mathcal{G}}(v_a, v_b) \leq 3, \\ 3 & \text{if } \mathfrak{g} = v_a, \mathfrak{h} = v_{b'}, a \neq b, d_{\mathcal{G}}(v_a, v_b) > 3 \end{cases}$

Proof: Let $v_a, v_b \in V$ and let s^* be the star vertex of $D^*(\mathcal{G})$. For distinct vertices $\mathfrak{g}, \mathfrak{h} \in \mathcal{V}$, we have:

(1) If $\mathfrak{g} = s^*$ and $\mathfrak{h} = v_{a'}$, then by construction $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = 1$. If $\mathfrak{h} = v_a \in V$, choose a neighbor v_c of v_a in $\mathcal{G}$. Then $(v_a, v_{c'}, s^*)$ is a 2-path, and since v_a is not adjacent to s^*, $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = 2$.

(2) If $\mathfrak{g} = v_{a'}$ and $\mathfrak{h} = v_{b'}$, then:
- If $v_a v_b \in \mathcal{E}$, $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = 1$.
- Otherwise, $(\mathfrak{g}, s^*, \mathfrak{h})$ is a 2-path, so $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = 2$.

(3) If $\mathfrak{g} = v_a$ and $\mathfrak{h} = v_b$, choose neighbors v_c of v_a and v_d of v_b. Then $(v_a, v_{c'}, s^*, v_{d'}, v_b)$ is a 4-path, giving $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) \leq 4$. If $d_{\mathcal{G}}(v_a, v_b) \leq 4$, the distance is preserved, that is, $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = d_{\mathcal{G}}(v_a, v_b)$. Otherwise, the constructed path ensures $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = 4$.

(4) If $\mathfrak{g} = v_a$ and $\mathfrak{h} = v_{b'}$, let v_c be a neighbor of v_a in $\mathcal{G}$:
- If $a = b$, $(v_a, v_c, v_{b'})$ is a 2-path, so $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = 2$.
- If $a \neq b$, $(v_a, v_{c'}, s^*, v_{b'})$ is a 3-path, hence $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) \leq 3$. If $d_{\mathcal{G}}(v_a, v_b) \leq 3$, then $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = d_{\mathcal{G}}(v_a, v_b)$; otherwise, $d_{D^*(\mathcal{G})}(\mathfrak{g}, \mathfrak{h}) = 3$.

From these observations, we see that the diameter of $D^*(\mathcal{G})$ does not exceed 4. The only entries that differ from those in $S^*(\mathcal{G})$ occur for pairs of duplicate vertices, specifically when $v_a v_b \in \mathcal{E}$, in which case the distance between $v_{a'}$ and $v_{b'}$ changes. With this information, one can build the distance matrix of the Star Double Graph directly.

$$D(D^*(\mathcal{G})) = \begin{matrix} \\ V \\ V' \\ s \end{matrix} \begin{matrix} \begin{matrix} V & V' & s \end{matrix} \\ \begin{pmatrix} D^*_{\mathcal{G}} & (D'_{\mathcal{G}})^T + 2I_\kappa & 2J_{1\times\kappa} \\ D'_{\mathcal{G}} + 2I_\kappa & W_{\mathcal{G}} - 2I_\kappa & J_{1\times\kappa} \\ 2J_{\kappa\times 1} & J_{\kappa\times 1} & 0 \end{pmatrix} \end{matrix}$$

Where,

- $D^*_{\mathcal{G}} = \sum_{i=1}^{3} i \cdot \tau_i(\mathcal{G}) + \sum_{i=4}^{d} 4 \cdot \tau_i(\mathcal{G})$
- $D'_{\mathcal{G}} = \sum_{i=1}^{3} i \cdot \tau_i(\mathcal{G}) + \sum_{i=4}^{d} 3 \cdot \tau_i(\mathcal{G})$
- $J_{\kappa\times\kappa}$ is a $\kappa \times \kappa$ matrix whose entries are entirely 1 (and similarly $J_{1\times\kappa}, J_{\kappa\times 1}$).
- I_κ is the $\kappa \times \kappa$ identity matrix.
- $W_{\mathcal{G}} = [w_{ab}], w_{a,b} = \begin{cases} 1 & \text{if } \mathfrak{g}_a\mathfrak{g}_b \in \mathcal{E}, \\ 2 & \text{if } \mathfrak{g}_a\mathfrak{g}_b \notin \mathcal{E}. \end{cases}$

3.2 *Some Topological Indices of Star Shadow Graph and Star Double Graph*

Topological indices are numerical parameters derived from graphs that capture structural characteristics while remaining invariant under graph isomorphisms. They are fundamental tools in Graph Theory and have significant applications in related areas, particularly in Chemical Graph Theory, where they are frequently employed to model molecular properties.

In this work, we focus on two constructions introduced earlier, the Star Shadow Graph and the Star Double Graph, and analyze several indices that reflect their structure. In particular, we consider distance-based measures such as the Wiener Index and the Harary Index, as well as degree-based measures including the Zagreb Co-indices.

Proposition 3.12: (Harary Index of Star Shadow Graph).

$$H(S^*(\mathcal{G})) = 3\left(\sum_{i=1}^{3} \frac{\phi_i(\mathcal{G})}{i}\right) + \frac{11}{12}\left(\sum_{i=4}^{d} \phi_i(\mathcal{G})\right) + \frac{1}{4}(\kappa^2 + 7\kappa).$$

Proof: $D(S^*(\mathcal{G}))$ may be written as

$$D(S^*(\mathcal{G})) = \begin{matrix} \\ V \\ V' \\ s \end{matrix} \begin{matrix} \begin{matrix} V & V' & s \end{matrix} \\ \begin{pmatrix} D^*_{\mathcal{G}} & (D'_{\mathcal{G}})^T + 2I_\kappa & 2J_{1\times\kappa} \\ D'_{\mathcal{G}} + 2I_\kappa & 2(J_{\kappa\times\kappa} - I_\kappa) & J_{1\times\kappa} \\ 2J_{\kappa\times 1} & J_{\kappa\times 1} & 0 \end{pmatrix} \end{matrix}$$

Where,

- $D^*_{\mathcal{G}} = \sum_{i=1}^{3} i \cdot \tau_i(\mathcal{G}) + \sum_{i=4}^{d} 4 \cdot \tau_i(\mathcal{G})$,
- $D'_{\mathcal{G}} = \sum_{i=1}^{3} i \cdot \tau_i(\mathcal{G}) + \sum_{i=4}^{d} 3 \cdot \tau_i(\mathcal{G})$,
- $J_{\kappa\times m}$ denotes the $\kappa \times m$ all-ones matrix,
- I_κ is the $\kappa \times \kappa$ identity matrix.

Since $D(S^*(\mathcal{G}))$ is symmetric, it suffices to sum the reciprocals of the entries in its lower triangle. The same applies to $D^*_{\mathcal{G}}$, $J_{\kappa\times m}$, and I_κ.

- From $D^*_{\mathcal{G}}$, the contribution is $\sum_{i=1}^{3}\frac{\phi_i(\mathcal{G})}{i}+\sum_{i=4}^{d}\frac{1}{4}\phi_i(\mathcal{G})$.
- The block $2(J_{\kappa\times\kappa}-I_\kappa)$ contains $\frac{\kappa(\kappa-1)}{2}$ entries equal to 2 in the lower triangle. Their reciprocal sum is $\frac{\kappa(\kappa-1)}{2}\cdot\frac{1}{2}$.
- For $D'_{\mathcal{G}}$, the lower and upper triangles coincide, so the reciprocals sum to $\sum_{i=1}^{3}\frac{2\phi_i(\mathcal{G})}{i}+\sum_{i=4}^{d}\frac{2}{3}\phi_i(\mathcal{G})$.
- The diagonal block $2I_\kappa$ contributes κ entries of 2, giving $\frac{\kappa}{2}$.
- The block $2J_{\kappa\times1}$ has κ entries equal to 2, contributing another $\frac{\kappa}{2}$. Similarly, $J_{\kappa\times1}$ contributes κ entries of 1.

Adding everything together, we obtain

$$\begin{aligned}H(S^*(\mathcal{G})) &= \tfrac{1}{2}\sum_{a=1}^{2\kappa+1}\sum_{b=1}^{2\kappa+1}(d_{S^*(\mathcal{G})}(v_a,v_b))^{-1}, a\neq b\\ &= (\sum_{i=1}^{3}\tfrac{\phi_i(\mathcal{G})}{i})+(\sum_{i=4}^{d}\tfrac{1}{4}\phi_i(\mathcal{G}))+\tfrac{\kappa(\kappa-1)}{2}\cdot\tfrac{1}{2}\\ &\quad+(\sum_{i=1}^{3}\tfrac{2\phi_i(\mathcal{G})}{i})+(\sum_{i=4}^{d}\tfrac{2}{3}\phi_i(\mathcal{G}))+\tfrac{\kappa}{2}+\tfrac{\kappa}{2}+\kappa\\ &= 3\left(\sum_{i=1}^{3}\tfrac{\phi_i(\mathcal{G})}{i}\right)+\tfrac{11}{12}\left(\sum_{i=4}^{d}\phi_i(\mathcal{G})\right)+\tfrac{1}{4}(\kappa^2+7\kappa).\end{aligned}$$

Proposition 3.13: (Harary Index of Star Double Graphs).

$$H(D^*(\mathcal{G}))=3\left(\sum_{i=1}^{3}\tfrac{\phi_i(\mathcal{G})}{i}\right)+\tfrac{11}{12}\left(\sum_{i=4}^{d}\phi_i(\mathcal{G})\right)+\tfrac{1}{4}(\kappa^2+7\kappa+2\lambda).$$

Proof: $D(D^*(\mathcal{G}))$, can be arranged as

$$D(S^*(\mathcal{G}))=\begin{array}{c}\\ V\\ V'\\ S\end{array}\begin{array}{c}\begin{array}{ccc}V & V' & S\end{array}\\ \begin{pmatrix}D^*_{\mathcal{G}} & (D'_{\mathcal{G}})^T+2I_\kappa & 2J_{1\times\kappa}\\ D'_{\mathcal{G}}+2I_\kappa & W_{\mathcal{G}}-2I_\kappa & J_{1\times\kappa}\\ 2J_{\kappa\times1} & J_{\kappa\times1} & 0\end{pmatrix}\end{array}$$

with the following blocks:

- $D^*_{\mathcal{G}}=\sum_{i=1}^{3}i\cdot\tau_i(\mathcal{G})+\sum_{i=4}^{d}4\cdot\tau_i(\mathcal{G})$,
- $D'_{\mathcal{G}}=\sum_{i=1}^{3}i\cdot\tau_i(\mathcal{G})+\sum_{i=4}^{d}3\cdot\tau_i(\mathcal{G})$,
- $J_{\kappa\times m}$ is the all-ones $\kappa\times m$ matrix,
- I_κ denotes the $\kappa\times\kappa$ identity matrix,
- $W_{\mathcal{G}}=[w_{ab}], w_{a,b}=\begin{cases}1 & \text{if } \mathfrak{g}_a\mathfrak{g}_b\in\mathcal{E},\\ 2 & \text{if } \mathfrak{g}_a\mathfrak{g}_b\notin\mathcal{E}.\end{cases}$

Since the matrix is symmetric, it is sufficient to focus on its lower triangular part when computing $H(D^*(\mathcal{G}))$. The individual contributions are as follows:

- From $D_{\mathcal{G}}^{*}$: $\sum_{i=1}^{3} \frac{\phi_i(\mathcal{G})}{i} + \sum_{i=4}^{d} \frac{1}{4}\phi_i(\mathcal{G})$.
- From $W_{\mathcal{G}} - 2I_{\kappa}$: among its $\frac{\kappa(\kappa-1)}{2}$ lower-triangular entries, λ are equal to 1 and the rest equal to 2. Their reciprocal sum is $\frac{1}{2}\left(\frac{\kappa(\kappa-1)}{2} - \lambda\right) + \lambda$.
- From $D'_{\mathcal{G}}$: symmetry ensures the sum is $\sum_{i=1}^{3} \frac{2\phi_i(\mathcal{G})}{i} + \sum_{i=4}^{d} \frac{2}{3}\phi_i(\mathcal{G})$.
- From $2I_{\kappa}$: $\kappa/2$.
- From $2J_{\kappa\times 1}$: $\kappa/2$.
- From $J_{\kappa\times 1}$: κ.

Bringing these contributions together, we find

$$\begin{aligned} H(D^{*}(\mathcal{G})) &= \tfrac{1}{2}\sum_{a=1}^{2\kappa+1}\sum_{b=1}^{2\kappa+1} \frac{1}{d_{D^{*}(\mathcal{G})}(v_a, v_b)}, a \neq b \\ &= \left(\sum_{i=1}^{3} \frac{\phi_i(\mathcal{G})}{i}\right) + \left(\sum_{i=4}^{d} \frac{1}{4}\phi_i(\mathcal{G})\right) + \frac{1}{2}\left(\frac{\kappa(\kappa-1)}{2} - \lambda\right) + \lambda \\ &\quad + \left(\sum_{i=1}^{3} \frac{2\phi_i(\mathcal{G})}{i}\right) + \left(\sum_{i=4}^{d} \frac{2}{3}\phi_i(\mathcal{G})\right) + \frac{\kappa}{2} + \frac{\kappa}{2} + \kappa \\ &= 3\left(\sum_{i=1}^{3} \frac{\phi_i(\mathcal{G})}{i}\right) + \frac{11}{12}\left(\sum_{i=4}^{d} \phi_i(\mathcal{G})\right) + \frac{1}{4}(\kappa^2 + 7\kappa + 2\lambda). \end{aligned}$$

In the Star Shadow Graph, any two distinct vertices $v'_a, v'_b \in V'(\mathcal{G})$ are always separated by a distance of 2. In the Star Double Graph, where $\mathcal{G}'$ is a copy of $\mathcal{G}$, the situation changes slightly: if $v_a v_b \in \mathcal{E}$, the corresponding vertices $v_{a'}$ and $v_{b'}$ in $\mathcal{G}'$ are at distance 1, while all other pairs of duplicate vertices remain at distance 2. This adjustment causes a difference of $\frac{\lambda}{2}$ in the Harary Index between the two graphs.

Proposition 3.14: (Wiener Index of Star Shadow Graphs).

$$W(S^{*}(\mathcal{G})) = 3\left(\sum_{i=1}^{3} i\phi_i(\mathcal{G})\right) + 10\left(\sum_{i=4}^{d} \phi_i(\mathcal{G})\right) + \kappa^2 + 4\kappa.$$

Proof: $D(S^{*}(\mathcal{G}))$ is organized as follows

$$D(S^{*}(\mathcal{G})) = \begin{array}{c} \\ V \\ V' \\ S \end{array} \begin{array}{c} \begin{array}{ccc} V & V' & S \end{array} \\ \begin{pmatrix} D_{\mathcal{G}}^{*} & (D'_{\mathcal{G}})^{T} + 2I_{\kappa} & 2J_{1\times\kappa} \\ D'_{\mathcal{G}} + 2I_{\kappa} & 2(J_{\kappa\times\kappa} - I_{\kappa}) & J_{1\times\kappa} \\ 2J_{\kappa\times 1} & J_{\kappa\times 1} & 0 \end{pmatrix} \end{array}$$

with the blocks defined as:

- $D_{\mathcal{G}}^{*} = \sum_{i=1}^{3} i \cdot \tau_i(\mathcal{G}) + \sum_{i=4}^{d} 4 \cdot \tau_i(\mathcal{G})$,
- $D'_{\mathcal{G}} = \sum_{i=1}^{3} i \cdot \tau_i(\mathcal{G}) + \sum_{i=4}^{d} 3 \cdot \tau_i(\mathcal{G})$,
- $J_{\kappa\times m}$ is the $\kappa \times m$ all-ones matrix,
- I_{κ} is the identity matrix of order κ.

Since $D(S^*(\mathcal{G}))$ is symmetric, we only need to sum over its lower triangular part when evaluating $W(S^*(\mathcal{G}))$. The individual contributions are:

- From $D^*_{\mathcal{G}}$: $\sum_{i=1}^{3} i\phi_i(\mathcal{G}) + \sum_{i=4}^{d} 4\phi_i(\mathcal{G})$.
- From $2(J_{\kappa\times\kappa} - I_\kappa)$: there are $\frac{\kappa(\kappa-1)}{2}$ entries equal to 2, giving $\kappa(\kappa-1)$.
- From $D'_{\mathcal{G}}$: symmetry implies its lower and upper parts coincide, contributing $\sum_{i=1}^{3} 2\cdot i\phi_i(\mathcal{G}) + \sum_{i=4}^{d} 2\cdot 3\phi_i(\mathcal{G})$.
- From $2I_\kappa$: 2κ.
- From $2J_{\kappa\times 1}$: 2κ.
- From $J_{\kappa\times 1}$: κ.

Adding all parts yields

$$\begin{aligned} W(S^*(\mathcal{G})) &= \tfrac{1}{2}\sum_{a=1}^{2\kappa+1}\sum_{b=1}^{2\kappa+1} d_{S^*(\mathcal{G})}(v_a, v_b) \\ &= \left(\sum_{i=1}^{3} i\phi_i(\mathcal{G})\right) + \left(\sum_{i=4}^{d} 4\phi_i(\mathcal{G})\right) + \kappa(\kappa-1) \\ &+ \left(\sum_{i=1}^{3} 2\cdot i\phi_i(\mathcal{G})\right) + \left(\sum_{i=4}^{d} 2\cdot 3\phi_i(\mathcal{G})\right) + 2\kappa + 2\kappa + \kappa \\ &= 3\left(\sum_{i=1}^{3} i\phi_i(\mathcal{G})\right) + 10\left(\sum_{i=4}^{d} \phi_i(\mathcal{G})\right) + \kappa^2 + 4\kappa. \end{aligned}$$

Proposition 3.5: (Wiener Index of Star Double Graphs).

$$W(D^*(\mathcal{G})) = 3\left(\sum_{i=1}^{3} i\phi_i(\mathcal{G})\right) + 10\left(\sum_{i=4}^{d} \phi_i(\mathcal{G})\right) + \kappa^2 + 4\kappa - \lambda.$$

Proof: $D(D^*(\mathcal{G}))$ is organized as follows

$$D(D^*(\mathcal{G})) = \begin{array}{c} \\ V \\ V' \\ S \end{array} \begin{array}{c} \begin{array}{ccc} V & V' & S \end{array} \\ \begin{pmatrix} D^*_{\mathcal{G}} & (D'_{\mathcal{G}})^T + 2I_\kappa & 2J_{1\times\kappa} \\ D'_{\mathcal{G}} + 2I_\kappa & W_{\mathcal{G}} - 2I_\kappa & J_{1\times\kappa} \\ 2J_{\kappa\times 1} & J_{\kappa\times 1} & 0 \end{pmatrix} \end{array}$$

where

- $D^*_{\mathcal{G}} = \sum_{i=1}^{3} i\cdot\tau_i(\mathcal{G}) + \sum_{i=4}^{d} 4\cdot\tau_i(\mathcal{G})$,
- $D'_{\mathcal{G}} = \sum_{i=1}^{3} i\cdot\tau_i(\mathcal{G}) + \sum_{i=4}^{d} 3\cdot\tau_i(\mathcal{G})$,
- $J_{\kappa\times m}$ is the $\kappa\times m$ all-ones matrix,
- I_κ is the $\kappa\times\kappa$ identity matrix,
- $W_{\mathcal{G}} = [w_{ab}]$, where

$$w_{ab} = \begin{cases} 1, & \mathfrak{g}_a\mathfrak{g}_b \in \mathcal{E}, \\ 2, & \mathfrak{g}_a\mathfrak{g}_b \notin \mathcal{E}. \end{cases}$$

Again, by symmetry it suffices to sum over the lower triangular entries:

- From $D^*_{\mathcal{G}}$: $\sum_{i=1}^{3} i\phi_i(\mathcal{G}) + \sum_{i=4}^{d} 4\phi_i(\mathcal{G})$.
- From $W_{\mathcal{G}} - 2I_\kappa$: out of $\frac{\kappa(\kappa-1)}{2}$ entries, λ are 1 and the rest are 2. Their sum is $2(\frac{\kappa(\kappa-1)}{2} - \lambda) + \lambda = \kappa^2 - \kappa - \lambda$.
- From $D'_{\mathcal{G}}$: $\sum_{i=1}^{3} 2\cdot i\phi_i(\mathcal{G}) + \sum_{i=4}^{d} 2\cdot 3\phi_i(\mathcal{G})$.

- From $2I_\kappa$: 2κ.
- From $2J_{\kappa\times 1}$: 2κ.
- From $J_{\kappa\times 1}$: κ.

Adding these contributions we obtain

$$\begin{aligned} W(D^*(\mathcal{G})) &= \tfrac{1}{2}\textstyle\sum_{a=1}^{2\kappa+1}\sum_{b=1}^{2\kappa+1} d_{D^*(\mathcal{G})}(v_a, v_b) \\ &= (\textstyle\sum_{i=1}^{3} i\phi_i(\mathcal{G})) + (\sum_{i=4}^{d} 4\phi_i(\mathcal{G})) + (\kappa^2 - \kappa - \lambda) \\ &\quad + (\textstyle\sum_{i=1}^{3} 2\cdot i\phi_i(\mathcal{G})) + (\sum_{i=4}^{d} 2\cdot 3\phi_i(\mathcal{G})) + 2\kappa + 2\kappa + \kappa \\ &= 3(\textstyle\sum_{i=1}^{3} i\phi_i(\mathcal{G})) + 10(\sum_{i=4}^{d} \phi_i(\mathcal{G})) + \kappa^2 + 4\kappa - \lambda. \end{aligned}$$

The difference of $-\lambda$ between $W(S^*(\mathcal{G}))$ and $W(D^*(\mathcal{G}))$ arises because in $D^*(\mathcal{G})$ the duplicate copy $\mathcal{G}'$ contributes λ additional pairs of vertices $v_{a'}, v_{b'}$ at distance 1.

Proposition 3.16: (First Zagreb Index of Star Shadow Graph)

$$M_1(S^*(\mathcal{G})) = 5M_1(\mathcal{G}) + 4\lambda + \kappa(\kappa+1).$$

Proof: From Proposition 3.2 (2), the degrees in $S^*(\mathcal{G})$ satisfy

$$\deg_{S^*(\mathcal{G})}(\mathfrak{g}_a) = \begin{cases} 2\deg_{\mathcal{G}}(\mathfrak{g}_a), & \mathfrak{g}_a \in V(\mathcal{G}), a = 1,\dots,\kappa, \\ \deg_{\mathcal{G}}(\mathfrak{g}_a) + 1, & \mathfrak{g}_a \in V'(\mathcal{G}), a = \kappa+1,\dots,2\kappa, \\ \kappa, & \mathfrak{g}_a = s, a = 2\kappa+1. \end{cases}$$

Hence, $M_1(S^*(\mathcal{G}))$ is

$$\begin{aligned} M_1(S^*(\mathcal{G})) &= \textstyle\sum_{a=1}^{2\kappa+1} (\deg_{S^*(\mathcal{G})}(\mathfrak{g}_a))^2 \\ &= \textstyle\sum_{a=1}^{\kappa} (2\deg_{\mathcal{G}}(\mathfrak{g}_a))^2 + \sum_{a=1}^{\kappa} (\deg_{\mathcal{G}}(\mathfrak{g}_a)+1)^2 + \kappa^2 \\ &= 4M_1(\mathcal{G}) + M_1(\mathcal{G}) + 2(2\lambda) + \kappa + \kappa^2 \\ &= 5M_1(\mathcal{G}) + 4\lambda + \kappa(\kappa+1). \end{aligned}$$

Proposition 3.17: (First Zagreb Index of Star Double Graph).

$$M_1(D^*(\mathcal{G})) = 8M_1(\mathcal{G}) + 8\lambda + \kappa(\kappa+1).$$

Proof: From Proposition 3.9 (2), the degrees in $D^*(\mathcal{G})$ are given by

$$\deg_{D^*(\mathcal{G})}(\mathfrak{g}_a) = \begin{cases} 2\deg_{\mathcal{G}}(\mathfrak{g}_a), & \mathfrak{g}_a \in V(\mathcal{G}), a = 1,\dots,\kappa, \\ 2\deg_{\mathcal{G}}(\mathfrak{g}_a) + 1, & \mathfrak{g}_a \in V'(\mathcal{G}), a = \kappa+1,\dots,2\kappa, \\ \kappa, & \mathfrak{g}_a = s^*, a = 2\kappa+1. \end{cases}$$

Therefore,

$$\begin{aligned} M_1(D^*(\mathcal{G})) &= \textstyle\sum_{a=1}^{2\kappa+1} (\deg_{D^*(\mathcal{G})}(\mathfrak{g}_a))^2 \\ &= \textstyle\sum_{a=1}^{\kappa} (2\deg_{\mathcal{G}}(\mathfrak{g}_a))^2 + \sum_{a=1}^{\kappa} (2\deg_{\mathcal{G}}(\mathfrak{g}_a)+1)^2 + \kappa^2 \\ &= 4M_1(\mathcal{G}) + 4M_1(\mathcal{G}) + 4(2\lambda) + \kappa + \kappa^2 \\ &= 8M_1(\mathcal{G}) + 8\lambda + \kappa(\kappa+1). \end{aligned}$$

Proposition 3.18: (Second Zagreb Index of Star Shadow Graph)

$$M_2(S^*(\mathcal{G})) = 8M_2(\mathcal{G}) + 2M_1(\mathcal{G}) + \kappa(2\lambda + \kappa).$$

Proof: By definition,

$$M_2(S^*(\mathcal{G})) = \sum_{uv\in\mathcal{E}_{S^*(\mathcal{G})}} \deg(u)\deg(v).$$

Each edge $uv \in \mathcal{E}$ contributes five edges in $S^*(\mathcal{G})$: uv, $u'v$, uv', us, and vs, where s is the star vertex. By Proposition 3.2(2), the degrees in $S^*(\mathcal{G})$ are

$$\deg_{S^*(\mathcal{G})}(x) = \begin{cases} 2\deg_{\mathcal{G}}(\mathfrak{g}), & \mathfrak{g} \in V(\mathcal{G}), \\ \deg_{\mathcal{G}}(\mathfrak{g}) + 1, & \mathfrak{g} \in V'(\mathcal{G}), \\ \kappa, & \mathfrak{g} = s. \end{cases}$$

Substituting these into the definition,

$$\begin{aligned} M_2(S^*(\mathcal{G})) &= \sum_{ab\in\mathcal{E}} 2\deg_{\mathcal{G}}(\mathfrak{g}_a)\cdot 2\deg_{\mathcal{G}}(\mathfrak{g}_b) \\ &\quad + \sum_{ab\in\mathcal{E}} (\deg_{\mathcal{G}}(\mathfrak{g}_a) + 1)(2\deg_{\mathcal{G}}(\mathfrak{g}_b)) \\ &\quad + \sum_{ab\in\mathcal{E}} (2\deg_{\mathcal{G}}(\mathfrak{g}_a))(\deg_{\mathcal{G}}(\mathfrak{g}_b) + 1) + \sum_{a=1}^{\kappa} \kappa(\deg_{\mathcal{G}}(\mathfrak{g}_a) + 1) \\ &= 4M_2(\mathcal{G}) + 4M_2(\mathcal{G}) + \kappa(2\lambda + \kappa) + \sum_{ab\in\mathcal{E}} 2(\deg_{\mathcal{G}}(\mathfrak{g}_a) + \deg_{\mathcal{G}}(\mathfrak{g}_b)) \\ &= 8M_2(\mathcal{G}) + 2M_1(\mathcal{G}) + \kappa(2\lambda + \kappa). \end{aligned}$$

Proposition 3.19: (Second Zagreb Index of Star Double Graph)

$$M_2(D^*(\mathcal{G})) = 16M_2(\mathcal{G}) + 4M_1(\mathcal{G}) + \kappa^2 + 4\kappa\lambda + \lambda.$$

Proof: If $\mathfrak{g}\mathfrak{h} \in E$, then $D^*(\mathcal{G})$ includes the edges $\mathfrak{g}\mathfrak{h}, \mathfrak{g}'\mathfrak{h}, \mathfrak{g}\mathfrak{h}', \mathfrak{g}'\mathfrak{h}', \mathfrak{g}'s^*, \mathfrak{h}'s^*$. From Proposition 3.9(2), the degrees are

$$\deg_{D^*(\mathcal{G})}(\mathfrak{g}) = \begin{cases} 2\deg_{\mathcal{G}}(\mathfrak{g}), & \mathfrak{g} \in V(\mathcal{G}), \\ 2\deg_{\mathcal{G}}(\mathfrak{g}) + 1, & \mathfrak{g} \in V'(\mathcal{G}), \\ \kappa, & \mathfrak{g} = s^*. \end{cases}$$

Hence

$$\begin{aligned} M_2(D^*(\mathcal{G})) &= \sum_{ab\in\mathcal{E}} (2\deg_{\mathcal{G}}(\mathfrak{g}_a)\cdot 2\deg_{\mathcal{G}}(\mathfrak{g}_b)) \\ &\quad + \sum_{ab\in\mathcal{E}} (2\deg_{\mathcal{G}}(\mathfrak{g}_a) + 1)(2\deg_{\mathcal{G}}(\mathfrak{g}_b) + 1) \\ &\quad + \sum_{ab\in\mathcal{E}} (2\deg_{\mathcal{G}}(\mathfrak{g}_a) + 1)(2\deg_{\mathcal{G}}(\mathfrak{g}_b)) \\ &\quad + \sum_{ab\in\mathcal{E}} (2\deg_{\mathcal{G}}(\mathfrak{g}_a))(2\deg_{\mathcal{G}}(\mathfrak{g}_b) + 1) \\ &\quad + \sum_{a=1}^{\kappa} \kappa(2\deg_{\mathcal{G}}(\mathfrak{g}_a) + 1). \end{aligned}$$

Simplifying gives

$$M_2(D^*(\mathcal{G})) = 16M_2(\mathcal{G}) + 4M_1(\mathcal{G}) + \kappa^2 + 4\kappa\lambda + \lambda.$$

3.3 *Summary and Further Directions*

Inspired by the classical Mycielskian construction, this paper introduced a new variant, called the Star Double Graph, and explored its structural and combinatorial features. The key findings may be summarized as follows:

(1) For a given graph G, we defined the Star Double Graph, denoted $D^*(G)$. It was shown that, like the Mycielskian, $D^*(G)$ increases the chromatic

number by one; however, in contrast to the classical construction, the triangle-free property is not preserved.

(2) Closed formulas were obtained for several degree- and distance-based topological indices for both the Star Shadow Graph and the Star Double Graph. These are summarized as follows:

- $H(S^*(\mathcal{G})) = 3\left(\sum_{i=1}^{3} \frac{\phi_i(\mathcal{G})}{i}\right) + \frac{11}{12}\left(\sum_{i=4}^{d} \phi_i(\mathcal{G})\right) + \frac{1}{4}(\kappa^2 + 7\kappa)$
- $H(D^*(\mathcal{G})) = 3\left(\sum_{i=1}^{3} \frac{\phi_i(\mathcal{G})}{i}\right) + \frac{11}{12}\left(\sum_{i=4}^{d} \phi_i(\mathcal{G})\right) + \frac{1}{4}(\kappa^2 + 7\kappa + 2\lambda$
- $W(S^*(\mathcal{G})) = 3(\sum_{i=1}^{3} i\phi_i(\mathcal{G})) + 10\left(\sum_{i=4}^{d} \phi_i(\mathcal{G})\right) + \kappa^2 + 4\kappa$
- $W(D^*(\mathcal{G})) = 3(\sum_{i=1}^{3} i\phi_i(\mathcal{G})) + 10\left(\sum_{i=4}^{d} \phi_i(\mathcal{G})\right) + \kappa^2 + 4\kappa - \lambda$
- $M_1(S^*(\mathcal{G})) = 5M_1(\mathcal{G}) + 4\lambda + \kappa(\kappa + 1)$
- $M_1(D^*(\mathcal{G})) = 8M_1(\mathcal{G}) + 8\lambda + \kappa(\kappa + 1)$
- $M_2(S^*(\mathcal{G})) = 8M_2(\mathcal{G}) + 2M_1(\mathcal{G}) + \kappa(2\lambda + \kappa)$
- $M_2(D^*(\mathcal{G})) = 16M_2(\mathcal{G}) + 4M_1(\mathcal{G}) + \kappa^2 + 4\kappa\lambda + \lambda$

A number of other generalizations of the Mycielskian have already been studied. For instance, [10] proposed the m-Mycielskian, later referred to as "Cones Over Graphs" by Tardif [9], while the t-Mycielskian for $t \in \mathbb{N}$ was examined in [5]. Another natural extension involves iterated constructions, such as $\mu^2(G) = \mu(\mu(G))$ and, more generally, $\mu^k(G)$ for $k \geq 3$.

In the classical setting, vertex degrees and distance relationships in the Mycielskian are already well understood, making degree- and distance-based indices straightforward to compute. For the Star Shadow Graph, the Star Double Graph, and other possible variations, it would be worthwhile to investigate additional indices. In particular, eigenvalue-based indices could reveal further insight into their structural behavior.

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References

[1] J. Mycielski, "Sur le coloriage des graphes," *Colloq. Math.*, vol. 3, pp. 161-162 (1955).

[2] G. Garza and N. Shinkel, "Which graphs have planar shadow graphs?," *Pi Mu Epsilon J.*, pp. 11-20 (1999).

[3] D. B. West, Introduction to Graph Theory, 2nd ed. Upper Saddle River, NJ: Prentice Hall (2001).

[4] A. Behtoei and M. Anbarloei, "Degree distance index of the Mycielskian and its complement," *Iran. J. Math. Chem.*, vol. 7, no. 1, pp. 1-9 (2016).

[5] F. J. H. Campena and J.J. A. Casica, "A variation of the generalized Mycielskian of a graph," in *Proc. DLSU Research Congress*, Manila, Philippines (2023).

[6] M. Eliasi, G. Raeisi, and B. Taeri, "Wiener index of some graph operations," *Discrete Appl. Math.*, vol. 160, no. 9, pp. 1333-1344 (2012).

[7] A. Alqesmah, K. A. A. Alloush, A. Saleh, and G. Deepak, "Entire Harary index of graphs," *Journal of Discrete Mathematical Sciences and Cryptography,* vol. 25, no. 3, pp. 1–15 (2021), doi: 10.1080/09720529.2021.1888435.

[8] G. H. Fath-Tabar, "Old and new Zagreb indices of graphs," *MATCH Commun. Math. Comput. Chem.*, vol. 65, no. 1, pp. 79-84 (2011).

[9] C. Tardif, "Fractional chromatic numbers of cones over graphs," *J. Graph Theory*, vol. 38, no. 2, pp. 87-94 (2001).

[10] M. Stiebitz, *Beiträge zur Theorie der Färbungskritischen Graphen*, Habilitationsschrift, Technische Universität Ilmenau (1985).

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