

High strength connectivity in fuzzy graphs

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Abstract

A minimum vertex-cut set of a connected fuzzy graph $T = (V, E, \kappa, \lambda)$, abbreviated MVC, is a vertex-cut set with minimum cardinality. Let W be an MVC set of T . The strength of W is defined as the $S(W) = \max\{\min\{\lambda(p, q) \mid q \in V \setminus W\} \mid p \in W\}$. The maximum value of the strength of all MVC sets of $T = (V, E, \kappa, \lambda)$ is called high strength connectivity of graph T and represented by $HS(T)$. In this article, the high strength connectivity of the vertex-cut set in famous fuzzy graphs was determined. Also, an algorithm was introduced to calculate high strength connectivity in these graphs by using Sage software. Examples of the high strength connectivity applications in economic and power plant were given.

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Keywords: Vertex-cut, Minimum vertex-cut, Strengths, High strength connectivity.

1. Introduction

Fuzzy logic, developed in the 1960s [37], revolutionized the way we handle uncertainty and imprecision in decision-making processes. Zadeh's ground breaking work laid the foundation for fuzzy systems, which have since found applications across different domains, such as control systems, artificial intelligence, and pattern identification. Fuzzy logic allows for the representation of vague or ambiguous concepts using degrees of truth, enabling a more human-like approach to reasoning. Rosenfeld [29] expanded the idea of fuzzy logic to graph theory by presenting fuzzy graphs, which integrate uncertainty into graph structures.

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In fuzzy graphs, degrees of membership are assigned to edges reflecting the strength of the relationship between their incident vertices [21]. This fuzzy representation of graph edges allows for a more detailed examination of intricate systems where connections are not strictly binary. Fuzzy graphs have been applied in multiple areas such as social network analysis, image processing, and optimization problems [18]. The study of fuzzy graphs has sparked advancements in network analysis, clustering algorithms, and decision-making processes [32, 17, 19]. Mordeson and Chang-Shyh [24] conducted research on the functions and processes related to fuzzy graphs. Bhattacharya conducted an examination of certain observations related to fuzzy graphs [6]. Applications of graph theory in computer science was studied by Shrinivas, Vertrivel, and Elango [31]. The development of fuzzy graph algorithms has enhanced our ability to analyze and interpret fuzzy relationships in graph-based data [25].

Recent research has further expanded the understanding of fuzzy graphs. For instance, Al-Hawary [2] investigated density results for perfectly regular and perfectly edge-regular fuzzy graphs, providing deeper insights into their structural properties. Additionally, Prabavathy and Preethika [26] introduced twin proper neutrosophic fuzzy graphs, broadening the scope of fuzzy graph theory to include neutrosophic concepts, which handle indeterminacy more effectively. Yousif, Ameer and Sulaiman [36] studied cutoff fuzzy dominating sets in strong fuzzy graphs, contributing to the understanding of domination and control within fuzzy systems. These developments have enhanced our ability to analyze complex systems with inherent uncertainty, especially in applications like network reliability and decision-making.

The idea of vertex-cut sets in graphs has its origins in the early developments of graph theory during the 19th century. Pioneering mathematicians like Leonhard Euler and Gustav Kirchhoff laid the groundwork for understanding graph connectivity. The notion of a vertex-cut set gained prominence with the formation of network theory. Edmonds and Karp have made important advancements in understanding vertex-cut sets and the connectivity of graphs [10].

The concept of vertex-cut sets has been applied in various fields, including computer science, telecommunications, and social network analysis. Bauer et al. [3] explored challenges associated with connectivity problems and the advancement of reliable networks. Biedl and Murali investigated the vertex connectivity of 1-plane graphs [5]. Sadati and Talebi studied connectivity in a cubic fuzzy graph [30]. Conditional connectivity, the connectivity of a type of products of graphs and Circulants graph, connections in combinatorial optimization and directed vertex-connectivity augmentation were studied in [7, 9, 12, 11, 13]. Huang and Zhang [15] explored the concept of cyclic vertex-connectivity within the context of Cartesian product digraphs. Jackson and Jordán [16] conducted a study on free graphs and the enhancement of vertex connectivity. Manzano et al. [20] studied the connectivity of data center networks. Meng and Ji [23] proposed a novel concept related to restricted edge connectivity in graphs. Mathew and Sunitha explored the concepts of node and arc

connectivity within the context of fuzzy graphs [22]. In the last decade, a lot of research has been done on graph connectivity and many articles have been published in this field, for example [1, 14, 27, 28, 33, 34].

In this paper, we will explore the idea of high strength connectivity in fuzzy graphs, focusing on the role of minimum vertex-cut (*MVC*) sets, and then, we will calculate the high strength connectivity in some fuzzy graphs. We will express an example of high strength connectivity application in World Transit Crossroads. In a fuzzy graph, corresponding to a company with several *MVC* sets, using the concept of the high strength connectivity, communication routes can be equipped in the shortest possible time, and thus, increase the efficiency of the company. Also, we will explain an application of high strength connectivity in power plants. By increasing the number of graph vertices, we can calculate the high strength connectivity in the fuzzy graph through using a mathematical software Sage[4].

2. Definitions and Notations

In this section, we review needed notions of graph theory and fuzzy graph theory [8, 35].

Let V represent a finite, non-empty set of elements. A graph $T = (V, E)$ consists of this collection of elements, referred to as vertices, along with a set E of edges. Each edge is indicated by an unordered pair of different vertices. An edge that connects the vertices f and e is represented as (f, e) . If (f, e) belongs to E , then we say that f and e are adjacent to one another.

A graph T is termed complete if every possible pair of vertices shares an edge. A path is defined as a sequence of vertices in which each pair of consecutive vertices is connected by an edge. A cycle is a specific type of path that begins and ends at the same vertex. A graph is deemed connected if there is a route that links every pair of vertices. An edge is classified as a bridge if its removal results in an increase in the number of disconnected components in the graph.

Definition 2.1: The union of two graphs, $T_1 = (V_1, E_1)$ and $T_2 = (V_2, E_2)$, results in a new graph $T_3 = (V_3, E_3)$, where $V_3 = V_1 \cup V_2$ and $E_3 = E_1 \cup E_2$. The intersection of two graphs $T_4 = (V_4, E_4)$ and $T_5 = (V_5, E_5)$ forms a graph $T_6 = (V_6, E_6)$, where $V_6 = V_4 \cap V_5$ and $E_6 = E_4 \cap E_5$.

Definition 2.2: A bipartite graph consists of two distinct sets of vertices, Q and P , such that there are no edges connecting vertices within the same set. A complete bipartite graph is a type of graph where all vertices in one partition are linked to every vertex in the other partition. This type of graph is represented as $K_{m,n}$, where m and n indicate the number of vertices in each set. Specifically, $K_{1,n}$ is referred to as a star graph.

Definition 2.3: A cycle graph, or circular graph, is a type of graph that features a single loop where vertices are interconnected by edges.

Definition 2.4: A path graph is a graph that consists of a single straight line of vertices connected by edges.

A fuzzy graph is a quartuple $T = (V, E, \kappa, \lambda)$ with V a vertex set, E an edge set, $\kappa: V \rightarrow [0,1]$ the fuzzy vertex function which measures the fuzziness of each vertex, and $\lambda: V \times V \rightarrow [0,1]$, a symmetric function with $\lambda(p, q) \leq \min\{\kappa(p), \kappa(q)\}$, $\forall p, q \in V$, which measures the fuzziness of edges. If $\kappa(q) = 1$ and $\lambda(q_1, q_2) = 1$ for all $q, q_1, q_2 \in V$, we have an ordinary non fuzzy graph, $G = (V, E)$. Moreover, for each fuzzy graph T , we define its crisp graph $T^* = (V^*, E^*)$ with $T^* = \{q \in V | \kappa(q) > 0\}$ and $E^* = \{(q, w) \in E | \lambda(q, w) > 0\}$.

3. High strength connectivity of fuzzy graphs

We show below several illustrative graphs and propositions giving a feel for the concepts just introduced. By using Sage software, we present a sample program for one of the graphs in this section, the other programs being similar.

Definition 3.1: A vertex-cut of a connected fuzzy graph, abbreviated *CFG*, $T = (V, E, \kappa, \lambda)$ refers to a subset of vertices that, when removed, causes the graph to become either disconnected or reduced to a single vertex. This means that the removal of these vertices significantly impacts the connectivity of the graph.

Definition 3.2: A minimum vertex-cut set of a *CFG* $T = (V, E, \kappa, \lambda)$, abbreviated *MVC*, is a vertex-cut set with minimum cardinality.

Definition 3.3: Let $T = (V, E, \kappa, \lambda)$ be a *CFG* and W be an *MVC* set of T . We consider $ST(d) = \min\{\lambda(d, p) | p \in V \setminus W\}$, $d \in W$. The strength of W is determined as the $S(W) = \max\{\min\{\lambda(d, p) | p \in V \setminus W\} | d \in W\}$.

Definition 3.4: The highest value of the strength among all *MVC* sets of $T = (V, E, \kappa, \lambda)$ is called high strength connectivity of graph T and denoted by $HS(T)$.

Note: An *MVC* set of W with a high strength connectivity is the strongest if $\sum_{x \in W} \kappa(x)$ is the greatest across all *MVC* sets with the maximum value of the strength. This *MVC* set is referred to as *SHS*(T).

Proposition 3.5: Assume that $T = (V, E, \kappa, \lambda)$ is a *CFG* of the order $p = \sum_{v \in V(T)} \sigma(v)$. Then, $HS(T) \leq p$.

Proof: Let W be a *SHS*(T). For all $x \in W$ and $u \in V \setminus W$, we have $\lambda(x, u) \leq \kappa(x) \wedge \kappa(u) \leq \kappa(u)$ where $\wedge$ stands for minimum, and hence $HS(T) \leq \kappa(u) \leq \kappa(u) + \sum_{w \neq u} \kappa(w) = p$, where $w \in V$. Therefore $HS(T) \leq p$.

Throughout each vertex v is labeled as a pair, the first component being its label and its second component $\kappa(v)$. Each edge (x, y) is labeled $\lambda(x, y)$.

Throughout, $W_i, i = 1, 2, \dots$ are subsets of vertices. The calculations of high strength connectivity, $HS(T)$, were facilitated by use of Sage software.

Example 3.6: Assume that the CFG $T = (V, E, \kappa, \lambda)$ given in Figure 1 with $V = \{(0, 0.2), (1, 0.3), (2, 0.5), (3, 0.9), (4, 0.8), (5, 0.6), (6, 0.8), (7, 0.7)\}$. For this fuzzy graph, the subsets $W_1 = \{(2, 0.5)\}$ and $W_2 = \{(3, 0.9)\}$ are MVC sets. In this fuzzy graph, $HS(T) = 0.2$ and $SHS(T) = W_2$.

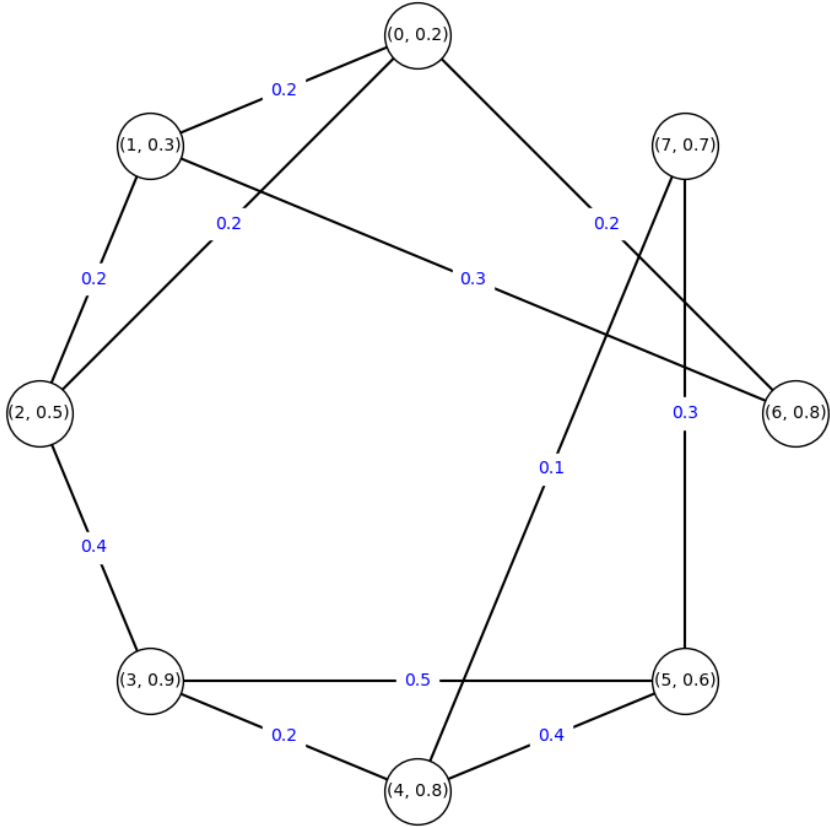


Figure 1
CFG T with MVC sets $W_1 = \{(2, 0.5)\}$ and $W_2 = \{(3, 0.9)\}$ and $HS(T) = 0.2$

Proposition 3.7: Suppose that $T = (V, E, \kappa, \lambda)$ is a path fuzzy graph with a vertex set $\{v_r | 0 \leq r \leq n - 1\}$. Then, $HS(T) =$

$$\max\{\min\{\lambda(v_r, v_{r+1}), \lambda(v_r, v_{r-1})\} | r = 1, 2, \dots, n - 2\}.$$

Proof: Clearly, by removing each of the internal vertices, the fuzzy graph T becomes disconnected. Hence $W_r = \{v_r\}$, $r = 1, 2, \dots, n - 2$ are all MVC sets in T . For any r , $1 \leq r \leq n - 2$, $ST(v_r) = \min\{\lambda(v_r, v_{r-1}), \lambda(v_r, v_{r+1})\}$. Therefore, $S(W_r) = ST(v_r) =$

$$\min\{\lambda(v_r, v_{r-1}), \lambda(v_r, v_{r+1})\}. \text{ It follows that } HS(T) = \max\{\min\{\lambda(v_r, v_{r-1}), \lambda(v_r, v_{r+1})\} | r = 1, 2, \dots, n - 2\}.$$

By using Sage software, we calculated the high strength connectivity in the path fuzzy graph. The commands for calculating the high strength connectivity in this graph are given as below.

Example 3.8: Assume that the path fuzzy graph with 20 vertices, whose vertices and edges weights are shown in the Figure 2. The high strength connectivity is 0.79.

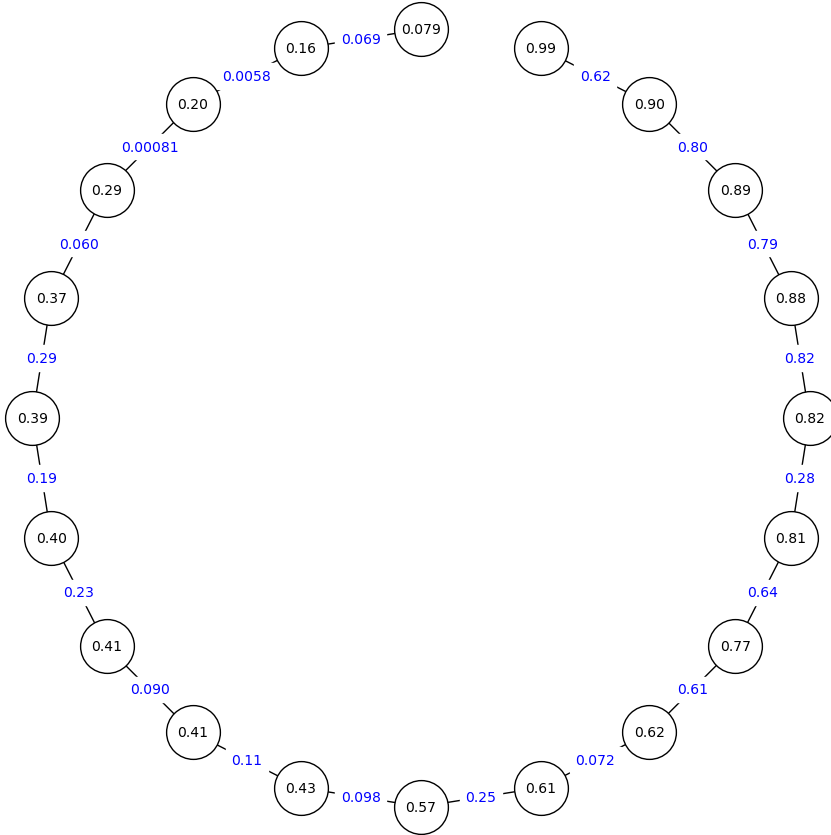


Figure 2
The path fuzzy graph T with $HS(T) = 0.79$

```

T = graphs.PathGraph(20)
k=T.order()
List=[]
for vertex in T.vertices(sort=True):
    T.delete_vertices([vertex])
for t in range(k):
    T.add_vertex(n(random(),digits=3))
N= T.vertices()
for r in range(0,k-1):
    T.add_edge(N[r],N[r+1])
    T.set_edge_label(N[r], N[r+1], RR.random_element(0,min(N[r],N[r+1])))
for r in range(k):
    List.append(min(T.edge_label(N[r],N[r+1]), T.edge_label(N[r-1],N[r])))
print("HS(T)=",max(List))

```

Proposition 3.9: Assume that $T = (V, E, \kappa, \lambda)$ is a cycle fuzzy graph. Then, $W = \{w_1, w_2\}$, in that w_1 and w_2 are not adjacent in an *MVC* set with a high strength connectivity if $\max\{\rho_{w_i} | i = 1, 2\} \geq \lambda(p, q)$, $p, q \in V \setminus W$, where

$$\rho_{w_i} = \min\{\lambda(w_i, p) | p \text{ and } w_i \text{ are adjacent}\}, \text{ and then,}$$

$$HS(T) = \max\{\rho_{w_i} | i = 1, 2\}.$$

Proof: In order to find the *MVC* sets in the cycle fuzzy graph, we have to choose two non-adjacent vertices. First, we choose a vertex w_1 with maximum ρ_w . Then, by selecting any other vertex w_2 , so that w_1 and w_2 are not adjacent *MVC* = $\{w_1, w_2\}$ with a high strength connectivity obtained.

Example 3.10: Assume that the cycle fuzzy graph $T = (V, E, \kappa, \lambda)$ given in Figure 3 with $V = \{(0, 0.9), (1, 0.7), (2, 0.7), (3, 0.3), (4, 0.5), (5, 0.4)\}$. For this fuzzy graph, $HS(G) = 0.5$ and $SHS(T) = \{(1, 0.7), (4, 0.5)\}$.

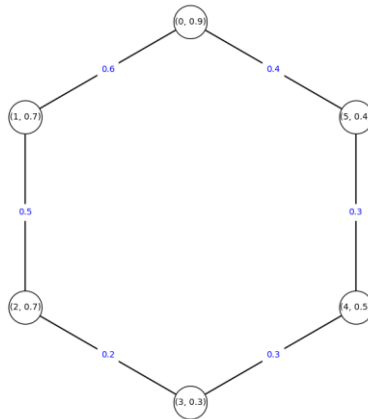


Figure 3

The cycle fuzzy graph T with $HS(T) = 0.5$ and $SHS(T) = \{(1, 0.7), (4, 0.5)\}$

Example 3.11: In the following figure 4, the Sage software has introduced an arbitrary cycle fuzzy graph with 20 vertices whose weights are known in the figure. The high strength connectivity is 0.37.

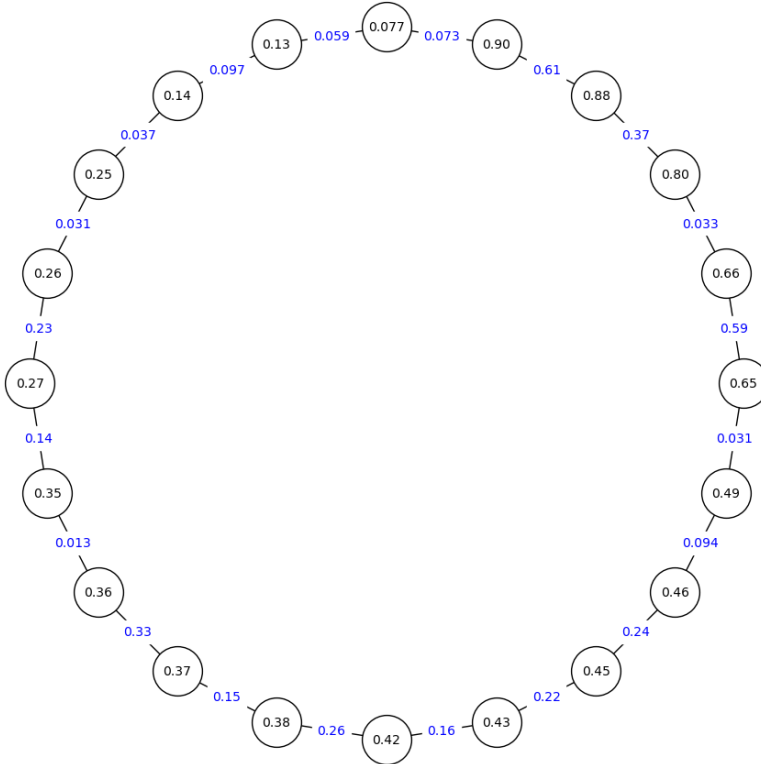


Figure 4
The cycle fuzzy graph with $HS(T) = 0.37$

Proposition 3.12: Let $T = (V, E, \kappa, \lambda)$ be a star fuzzy graph. Then, $\{u\}$ is a $SHS(T)$ where u is central vertex of T and the $HS(T)$ is $\min\{\lambda(u, z) | z \in V \setminus \{u\}\}$.

Proof: Clearly, by removing u that is adjacent to all the other vertices, we obtained disconnected fuzzy graph and hence the MVC set is obtained. Now, we selected the edges with the lowest weight among the weight of adjacent vertices u . This value is $HS(T)$.

Example 3.13: Consider the star fuzzy graph $T = (V, E, \kappa, \lambda)$ given in Figure 5 with $V = \{(0,0.6), (1,0.9), (2,0.5), (3,0.7), (4,0.8), (5,0.4)\}$. For this fuzzy graph, by using Sage as following, we have $SHS(T) = \{(0,0.6)\}$.

```
from sage.graphs.connectivity import
vertex_connectivity
vertex_connectivity(T,False)
```

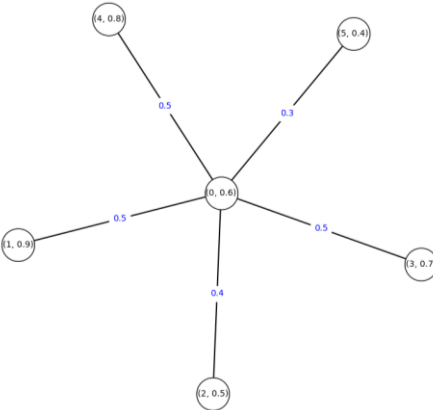


Figure 5

The star fuzzy graph with $W = \{(0, 0.6)\}$ and $HS(T) = 0.3$

Example 3.14: Consider the star fuzzy graph T with 21 vertices in figure 6. In this fuzzy graph, $W = \{(0,0.38)\}$ is MVC set and $HS(T) = 0.0058$.

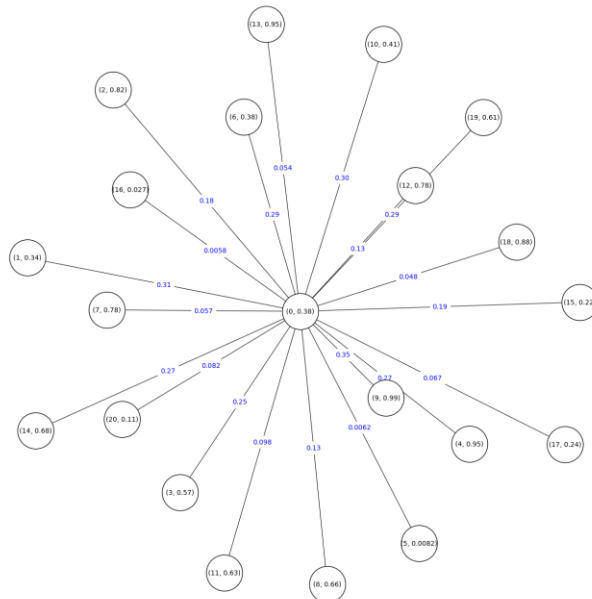


Figure 6

The star fuzzy graph with $W = \{(0, 0.38)\}$ and $HS(T) = 0.0058$

Proposition 3.15: Let $T = (V, E, \kappa, \lambda)$ be a complete fuzzy graph with n vertices. Then, a subset of V with $n - 1$ vertices is a *MVC* set of T and the $HS(T)$ is $\max\{\lambda(u, v) | u, v \in V\}$. Suppose that $e = (u, v)$ is an edge with the maximum weight and $\kappa(u) > \kappa(v)$, then each $SHS(T)$ contains u and $n - 2$ vertices of $V \setminus \{u, v\}$.

Proof: Suppose that $T = (V, E, \kappa, \lambda)$ is a complete fuzzy graph. By removing $n - 1$ vertices of T , we have an *MVC* set. Let $W = \{u_1, u_2, \dots, u_{n-1}\}$ be an *MVC* set. For calculating $S(W)$, we have to consider all edges (u_i, u_n) where $1 \leq i \leq n - 1$. By the definition of high strength connectivity, $HS(T)$ is $\max\{\lambda(u, v) | u, v \in V(T)\}$. For finding an $W \in SHS(T)$, first, we have to select an edge with the maximum weight. Let (u, v) be the edge satisfied in the above condition and $\kappa(u) > \kappa(v)$. Then, X contains u and $(n - 2)$ vertices of $V \setminus \{u, v\}$, because in this case, the sum weight of the vertices of an *MVC* set will be more than the other cases.

Example 3.16: Consider the *CFG* $T = (V, E, \kappa, \lambda)$ given in Figure 7 with $V = \{(0, 0.7), (1, 0.7), (2, 0.4), (3, 0.8)\}$. For this fuzzy graph, we have $HS(T) = 0.7$.

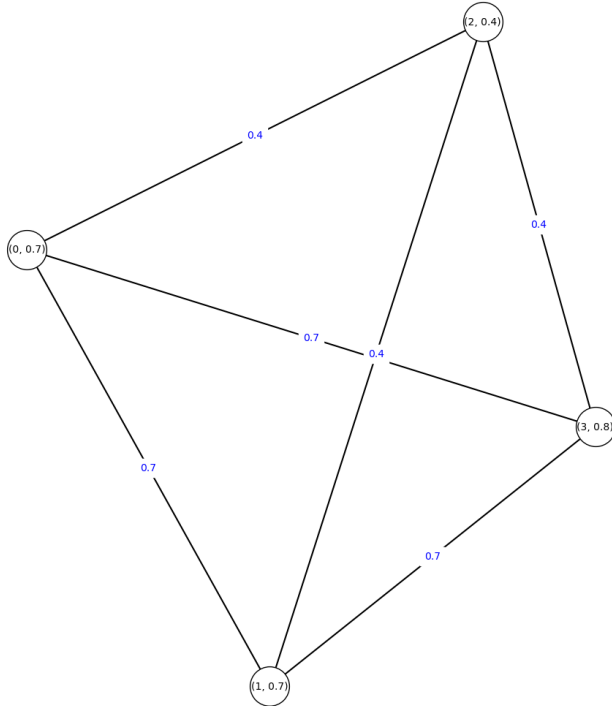


Figure 7
The complete fuzzy graph with $HS(T) = 0.7$

Example 3.17: Consider the complete fuzzy graph with random weights for vertices in Figure 8.

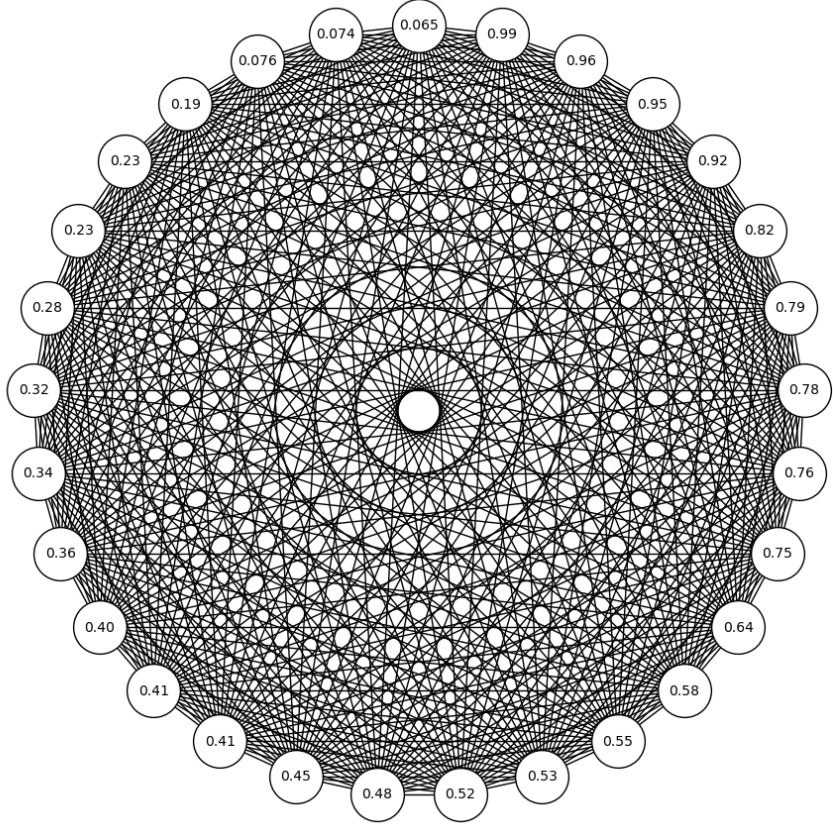


Figure 8
The complete fuzzy graph with $HS(T) = 0.96$

Proposition 3.18: Consider a complete bipartite fuzzy graph denoted as $K_{n,m} = (V, E, \kappa, \lambda)$, which consists of two parts $P = \{p_i | 1 \leq i \leq n\}$ and $Q = \{q_j | 1 \leq j \leq m\}$ with n and m vertices, respectively where $m > n$. Then, P , is an MVC set and $HS(K_{n,m}) = \min\{\max\{\kappa(p_i)\}, \min\{\kappa(q_j)\}\}$.

Proof: It is obvious that P is a unique MVC set of $K_{n,m}$. For any i , $1 \leq i \leq n$, we have:

$$\begin{aligned} ST(p_i) &= \min\{\lambda(p_i, q_j) | 1 \leq j \leq m\} \\ &= \min\{\kappa(p_i) \wedge \kappa(q_j) | 1 \leq j \leq m\} \\ &= \kappa(p_i) \wedge \min\{\kappa(q_j) | 1 \leq j \leq m\}. \end{aligned}$$

Hence,

$$\begin{aligned}
 S(U) &= \max\{ST(p_i) | 1 \leq i \leq n\} \\
 &= \max\{\kappa(p_i) \wedge \min\{\kappa(q_j) | 1 \leq j \leq m\} | 1 \leq i \leq n\} \\
 &= \min\{\max\{\kappa(p_i) | 1 \leq i \leq n\}, \min\{\kappa(v_j) | 1 \leq j \leq m\}\}.
 \end{aligned}$$

Since P is a unique MVC set of $K_{n,m}$, the proof of the proposition is finished.

In the examples provided, the high strength connectivity was calculated.

Example 3.19: Consider the CFGs $T = (V, E, \kappa, \lambda)$ given in Figure 9. We calculate the high strength connectivity in these graphs.

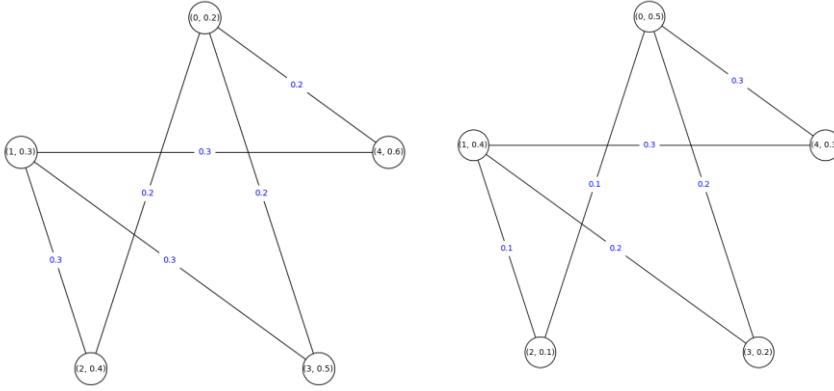


Figure 9(I)

The complete bipartite fuzzy graph $K_{2,3}$ with MVC sets $W_1 = \{(0, 0.2), (1, 0.3)\}$ and $W_2 = \{(0, 0.5), (1, 0.4)\}$ and $HS(T) = 0.3$ and 0.1 , respectively from left to right

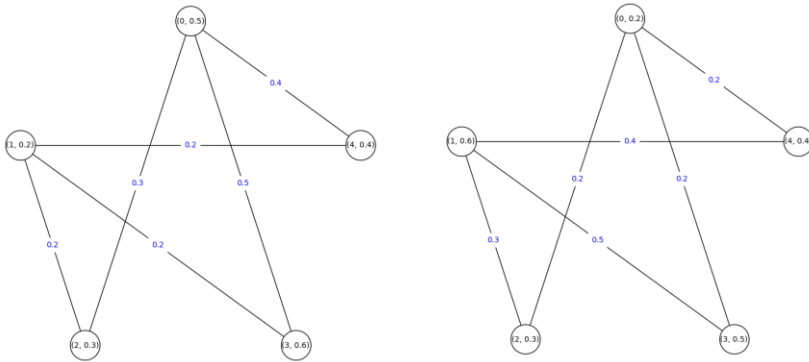


Figure 9(II)

The complete bipartite fuzzy graph $K_{2,3}$ with MVC sets $W_1 = \{(1, 0.2), (0, 0.5)\}$ and $W_2 = \{(1, 0.6), (0, 0.2)\}$ and $HS(T) = 0.3$ and 0.1 , respectively from left to right

Example 3.20: Consider the complete bipartite fuzzy graph $K_{10,12}$, whose vertices and edges weights are shown in the Figure 10. The high strength connectivity is 0.11.

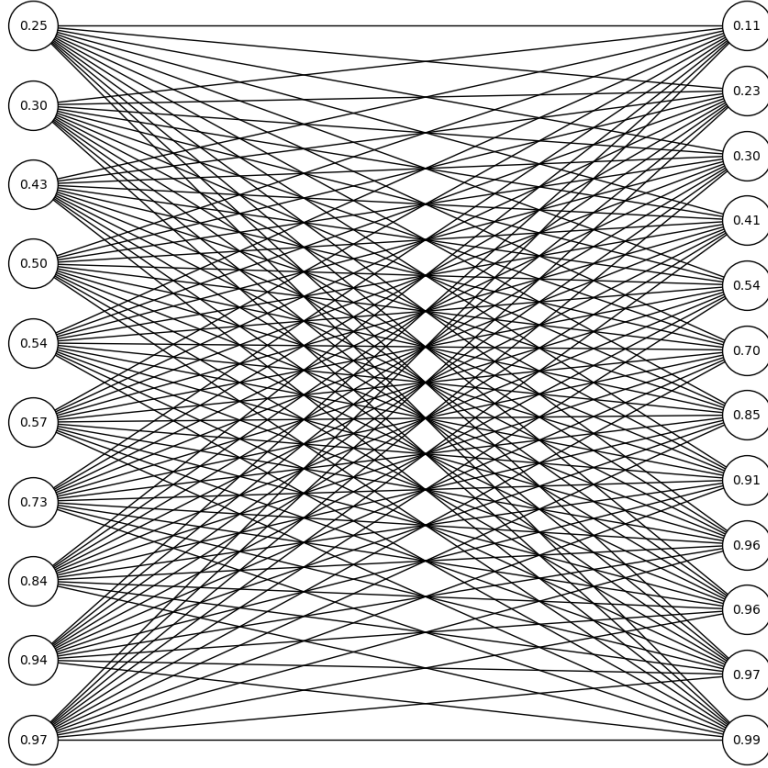


Figure 10
The complete bipartite fuzzy graph $K_{10,12}$ with $HS(K_{10,12}) = 0.11$

Proposition 3.21: Let $T = (V, E, \kappa, \lambda)$ be a fuzzy graph and $m = \max\{\lambda(z, k) \mid z, k \in V\}$. Then, $HS(T) \leq m$.

Proof: Let W be an MVC set of T . For any $z \in W$, we have $ST(z) = \min\{\lambda(z, k) \mid k \in V \setminus W\} \leq m$, hence $S(W) \leq m$. It follows that $HS(T) \leq m$.

Proposition 3.22: Let $T = (V, E, \kappa, \lambda)$ represent a cycle fuzzy graph. If we take away any edge f from T , then it follows that $HS(T \setminus f) \leq HS(T)$.

Proof: Let $T = (V, E, \kappa, \lambda)$ be a cycle fuzzy graph and $f = (z, k)$ an edge in T . Then, $T \setminus f$ is a path fuzzy graph. Considering propositions 3.7 and 3.9, we have $HS(T \setminus f) = \max\{\rho_x \mid x \in V \setminus \{z, k\}\} \leq \max\{\rho_x \mid x \in V\} = HS(T)$, where $\rho_x = \min\{\lambda(x, y) \mid y \in N(x)\}$.

Example 3.23: Consider the cycle fuzzy graph $T = (V, E, \kappa, \lambda)$ given in Figure 11, $HS(T) = 0.4$. By removing the edge $e = ((1, 0.7), (2, 0.8))$, the high strength connectivity in the path graph $T \setminus e$ does not change, but by removing an edge $e = ((0, 0.6), (1, 0.7))$, $HS(T \setminus e) = 0.3$, it decreases.

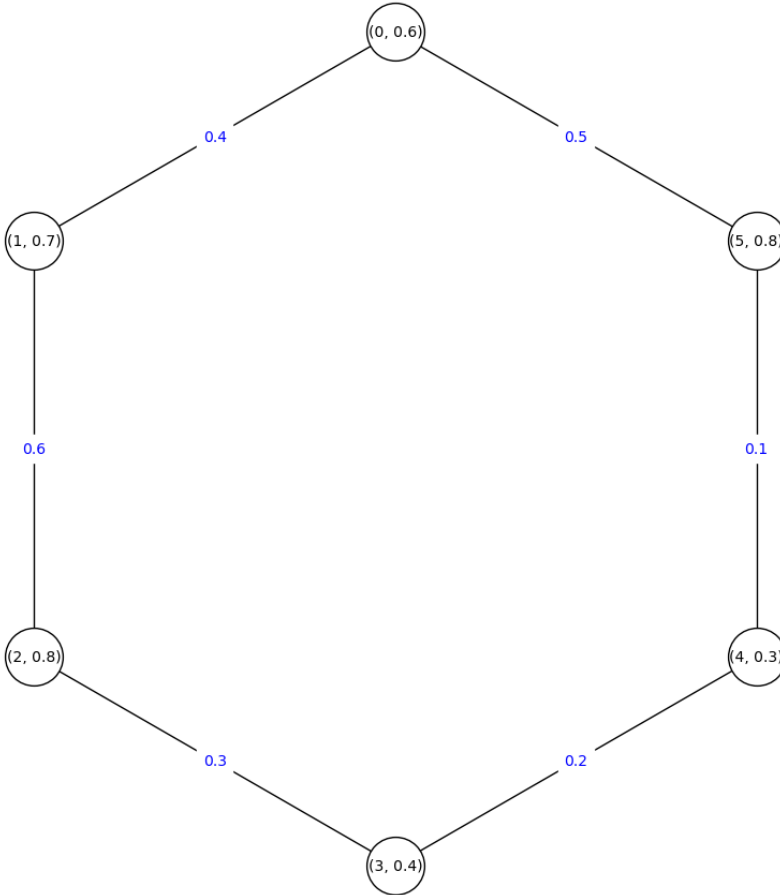


Figure 11
The cycle fuzzy graph with $HS(T) = 0.4$

Proposition 3.24: Let $T = (V, E, \kappa, \lambda)$ be a complete fuzzy graph. Then, for any edge e , $HS(T \setminus e) \leq HS(T)$.

Proof: Suppose that $m = \max\{\lambda(x, y) | x, y \in V\}$. By proposition 3.15, $HS(T) = m$. Now from proposition 3.21, we obtain

$$\begin{aligned} HS(T \setminus e) &\leq \max\{\lambda(x, y) | x, y \in V, (x, y) \neq e\} \\ &\leq \max\{\lambda(x, y) | x, y \in V\} = HS(T). \end{aligned}$$

Example 3.25: Consider the complete fuzzy graph $T = (V, E, \kappa, \lambda)$ given in Figure 7, $HS(T) = 0.7$. By removing an edge $\{(0, 0.7), (1, 0.7)\}$, the high strength connectivity does not change, but by removing an edge $e = \{(0, 0.7), (2, 0.4)\}$, $HS(T \setminus e) = 0.4$, it decreases.

Proposition 3.26: Let $T = (V, E, \kappa, \lambda)$ be a complete fuzzy graph and $e = (u, v)$ be an edge with maximum weight. Then, $HS(T \setminus e) = \min\{\kappa(x) | x \in V \setminus \{u, v\}\}$.

Proof: The set $W = V \setminus \{u, v\}$ is a unique MVC set in $T \setminus e$. Because $\mu(e)$ is the maximum, for any $x \in V(T) \setminus \{u, v\}$, $\kappa(x) \leq \kappa(u) \wedge \kappa(v)$. Hence, $ST(u) = ST(v) = \min\{\kappa(x) | x \in V \setminus \{u, v\}\}$. It follows that $HS(T \setminus e) = S(V \setminus \{u, v\}) = \max\{ST(u), ST(v)\} = \min\{\kappa(x) | x \in V \setminus \{u, v\}\}$. From proposition 3.24, we can conclude that, if e is an edge with a maximum weight in a complete fuzzy graph T , then, $HS(T \setminus e) = HS(T)$, if and only if, λ is a constant function.

Proposition 3.27: Let $K_{n,m} = (V, E, \kappa, \lambda)$ be a complete bipartite fuzzy graph containing two parts U and V with n and m vertices, respectively where $n < m$. If we remove an edge e in $K_{n,m}$, the $HS(K_{n,m} \setminus e) \geq HS(K_{n,m})$.

Proof: Let $K_{n,m} = (V, E, \kappa, \lambda)$ be a complete bipartite fuzzy graph containing two parts U and V with n and m vertices, respectively, where $n < m - 1$. Clearly, $U = \{u_i | 1 \leq i \leq n\}$ is a unique MVC set of $K_{n,m}$. Suppose that we remove an edge $e = (u_i, v_j)$. Clearly, $U = \{u_i | 1 \leq i \leq n\}$ is a unique MVC set of $K_{n,m} \setminus e$. If $\lambda(e) \neq HS(K_{n,m})$, then, $HS(K_{n,m} \setminus e) = HS(K_{n,m})$. Assume that e is a unique edge that $\kappa(e) = HS(K_{n,m})$. Then, by removing e , $\min\{\lambda(u_i, x) | x \in N(u_i)\} > HS(K_{n,m})$ and hence $HS(K_{n,m} \setminus e) > HS(K_{n,m})$. Now, suppose that $n = m - 1$. We remove an arbitrary edge $e = (u_i, v_j)$. It is obvious that $\{u_1, u_2, \dots, u_n\}$ and $\{v_1, v_2, \dots, v_{j-1}, v_{j+1}, \dots, v_m\}$ are MVC sets of $K_{n,m} \setminus e$. If $\lambda(e) = HS(K_{n,m})$, then, $\min\{\lambda(u_i, x) | x \in N(u_i)\} \geq HS(K_{n,m})$. Hence MVC set $\{u_1, u_2, \dots, u_n\}$ has a high strength connectivity $HS(K_{n,m})$. Suppose that MVC set $\{v_1, v_2, \dots, v_{j-1}, v_{j+1}, \dots, v_m\}$ has a high strength connectivity α . Therefore $HS(K_{n,m} \setminus e) = \max\{\alpha, HS(K_{n,m})\} \geq HS(K_{n,m})$. If $\lambda(e) \neq HS(K_{n,m})$, then, MVC set $\{u_1, u_2, \dots, u_n\}$ has a high strength connectivity $\alpha \geq HS(K_{n,m})$. Suppose that MVC set $\{v_1, v_2, \dots, v_{j-1}, v_{j+1}, \dots, v_m\}$ has a high strength connectivity β . Therefore, $HS(K_{n,m} \setminus e) = \max\{\alpha, \beta\} \geq HS(K_{n,m})$.

Proposition 3.28: Let $T = (V, E, \kappa, \lambda)$ be a CFG containing only a bridge $e = (u, v)$. The sets $\{u\}$ or $\{v\}$ is a SHS(T) if it meets any of the following conditions:

Case I: for $x \in V$, if $\min\{\lambda(u, x) | x \in N(u)\} > \min\{\lambda(v, x) | x \in N(v)\}$ then, $\{u\}$ is SHS(T) set.

Case II: for $x \in V$, $\min\{\lambda(v, x) | x \in N(v)\} > \min\{\lambda(u, x) | x \in N(v)\}$ then, $\{v\}$ is $SHS(T)$ set.

Case III: otherwise, each of u , v that has more vertex weight is considered as $SHS(T)$ set. If $\kappa(u) = \kappa(v)$, both $\{u\}$ and $\{v\}$ are $SHS(T)$ sets.

Proof: It is trivial.

Proposition 3.29: In a star fuzzy graph $T = (V, E, \kappa, \lambda)$ with a central vertex u , by removing an edge $e = (u, t)$, the graph is disconnected and $HS(T \setminus e) = 0$. But by removing vertex t , $HS(T \setminus t) \geq HS(T)$.

Proof: Suppose that $T = (V, E, \kappa, \lambda)$ is a star fuzzy graph. By removing an edge e , T become disconnected and hence $HS(T \setminus e) = 0$. But by removing vertex t , we have the subsequent situations:

Case I: If $\lambda(e) = HS(T)$, then, $\lambda(E(T) \setminus e) \geq \lambda(e)$ and hence $HS(T \setminus t) \geq HS(T)$.

Case II: If $\lambda(e) > HS(T)$, then, by removing e , $HS(T \setminus t)$ does not change because there exists an edge $f \in E(T)$ so that $\lambda(f) = HS(T)$.

Proposition 3.30: Let $T_1 = (V_1, E_1, \kappa_1, \lambda_1)$ and $T_2 = (V_2, E_2, \kappa_2, \lambda_2)$ be two complete fuzzy graphs with high strength connectivity ξ_1 and ξ_2 , respectively. Then, $HS(T_1 \cap T_2) \leq \min\{\xi_1, \xi_2\}$.

Proof: Assume that T_1 and T_2 is two complete fuzzy graphs with high strength connectivity ξ_1 and ξ_2 , respectively. Let $e \in E(T_1 \cap T_2)$. Since $(\lambda_1 \cap \lambda_2)(e) = \min\{\lambda_1(e), \lambda_2(e)\}$ and by proposition 3.15, $\mu_1(e) \leq \xi_1$ and $\mu_2(e) \leq \xi_2$, we have $HS(T_1 \cap T_2) \leq \min\{\xi_1, \xi_2\}$.

Proposition 3.31: Let $T_1 = (V_1, E_1, \kappa_1, \lambda_1)$ and $T_2 = (V_2, E_2, \kappa_2, \lambda_2)$ be two complete fuzzy graph with the high strength connectivity ξ_1 and ξ_2 , respectively. Then, $HS(T_1 \cup T_2) \leq \max\{\xi_1, \xi_2\}$.

Proof: Suppose that T_1 and T_2 is two complete fuzzy graphs with the high strength connectivity of ξ_1 and ξ_2 , respectively, and $e \in E(T_1 \cup T_2)$. Since $(\lambda_1 \cup \lambda_2)(e) = \max\{\lambda_1(e), \lambda_2(e)\}$ and by proposition 3.15, $\lambda_1(e) \leq \xi_1$ and $\lambda_2(e) \leq \xi_2$, we have $HS(T_1 \cup T_2) \leq \max\{\xi_1, \xi_2\}$.

4. Application

Example of the high strength connectivity application in World Transit Crossroads:

The international transport company World Transit Crossroads(WTC) includes 10 representatives in different countries. Economic power is the capacity of individuals, organizations, or nations to sway and manage economic activities,

resources, and results. It can be measured by factors such as Gross Domestic Product, (GDP), wealth distribution, trade balance, and market competitiveness. According to TradeStat Database site (WITS) and World Development Indicators/ DataBank (WDI), we define a graph structure on WTC by considering each country as a graph vertex, the economic power (GDP) of each country as the vertex weight, and the amount of export between two countries as the edge weights.

In 2022, after reviewing the performance of one year and the level of economic productivity of the company, the CEO of the WTC company wants to equip the countries that have the most transport and export connections with the rest of the countries and are located on the communication path in order to increase the economic profit of the company. For minimize the cost of this project, the CEO of the WTC company selected a subset of the introduced countries that have the most exports with other countries.

Suppose that WTC includes 10 representatives the countries

Bangladesh(B), China (C), France (F), Germany (G), Hong Kong (HK), India (I), Japan (J), Netherlands (N), Poland (P), United States (US), have the GDP of 0.0046, 0.72, 0.16, 0.0036, 0.14, 0.18, 0.009, 0.6, 0.08 and 0.95, respectively. The relationship between the countries is given in the Figure 12 and Table 1.

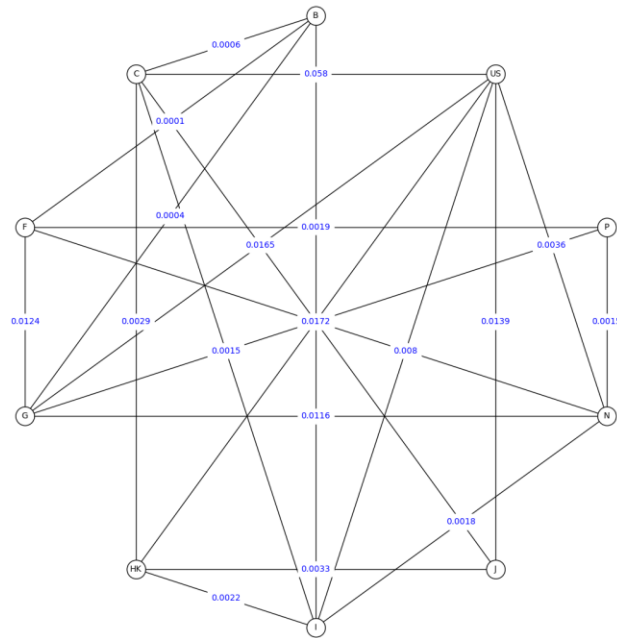


Figure 12
The graph relationship between different countries

Table 1
Trade statistics corresponding to Figure 12

	B	C	F	G	HK	I	J	N	P	US
B	0	0.0006	0.0001	0.0004	0	0.0014	0	0	0	0
C	0.0006	0	0	0	0.0029	0.0015	0.0172	0	0	0.058
F	0.0001	0	0	0.0124	0	0	0	0.0062	0.0019	0
G	0.0004	0	0.0124	0	0	0	0	0.0116	0.0097	0.0165
HK	0	0.0029	0	0	0	0.0022	0.0033	0	0	0.0028
I	0.0014	0.0015	0	0	0.0022	0	0	0.0018		0.008
J	0	0.0172	0	0	0.0033	0	0	0	0	0.0139
N	0	0	0.0062	0.0116	0	0.0018	0	0	0.0015	0.0036
P	0	0	0.0019	0.0097	0	0	0	0.0015	0	0
US	0	0.058	0	0.0165	0.0028	0.008	0.0139	0.0036	0	0

In the WTC company,

$W_1 = \{\text{India, United States, Bangladesh}\},$

$W_2 = \{\text{United States, Bangladesh, Netherlands}\},$

$W_3 = \{\text{United States, China, HongKong}\},$

$W_4 = \{\text{United States, China, India}\},$

$W_5 = \{\text{Germany, Bangladesh, Netherlands}\}$ and

$W_6 = \{\text{Germany, France, Netherlands}\}$

are six subsets so that if the WTC company cuts its connection with these six subsets, then, the transit connection of the countries will be cut off and so we can consider these six sets as *MVC* sets. Considering the main goal of the company to equip the countries that have the most connections and exports with other countries, it is enough to choose *SHS* set among the *MVC* sets. By investigation strength of six *MVC* sets $W_i, 1 \leq i \leq 6$, the strength $S(W_i), 1 \leq i \leq 6$ are 0.0097, 0.0028, 0.0036, 0.0028, 0.0015 and 0.0028, respectively. Then the high strength connectivity of this graph is 0.0097 and hence *SHS* is W_1 . This means that if a company wants to equip its representatives, it must choose representatives who are influential in the communication path, and by removing them, the entire communication network will be disconnected. As we have seen, the number of these sets is more than one. Certainly, among these *MVC* sets, the company will choose a set that has the highest exports to other countries. For example, among sets W_1 and W_2 , the highest exports are related to set W_1 , meaning that this set is profitable for the company, and if we can equip the countries present in this set, the profitability compared to set W_2 will increase. Therefore, if the company equips its representatives in these three countries, it will benefit economically. As seen in this example, the high strength connectivity number greatly helped the company in finding the best subset of countries to achieve higher profitability.

Example of the high strength connectivity application in the power plant:

One of the applications of the high strength connectivity is in the electricity connection between the cities of a province. In Mazandaran province of Iran, there are 2 power plants, Neka and HassanKif. The electricity from Neka power plant enters directly into Narivaran Babol. If the receiving route from Neka or the power plant is disturbed, Narivaran Babol can use Hasankif power plant for its electricity supply in some hours.

The electricity distribution model from the Neka power plant is shown as a graph in Figure 13 and Table 2. In this graph, the vertices are the cities with the amount of electricity consumption as the weight of the vertices. We consider the electricity communication path from Neka power plant to Narivaran and from there to different cities as edges of the graph. The amount of economic connection due to the existence of industrial factories is considered as the weight of the edges of the graph.

The communication route from Neka power plant to Hasankif power plant passing through the city of Narivaran Babol was shown in red color. As we explained above, if the communication path between Neka and Narivaran is disturbed, Narivaran can use Hasankif during some hours of the day. But if the electricity is cut off in Narioran Babol due to excessive use and overload, then, the electricity will be cut off in all the cities of the province. In this case, industrial factories in the city will have problems and economic communication will be cut off until the problem is solved and hence it will cause a lot of damage to the economy of the region. To solve this problem, a high strength can help us. We can connect a city whose minimum economic connection with other cities is greater than the economic connection of other cities with a direct route to Hasankif or Neka. In this case, if the electricity communication path to reach this city is disrupted, we will have the least economic loss because this city directly uses electricity from Hasankif or Neka. In Figure 13, $W_1 = \{Amol\}$, $W_2 = \{Ghaemshahr\}$, $W_3 = \{Royan\}$, $W_4 = \{Sari\}$ and $W_5 = \{Savadkooh\}$ are five MVC sets that receive their electricity from the Narivaran. The strength number for these five sets are 0.85, 0.5, 0.7, 0.6 and 0.4. The lowest economic connection between Amol and neighboring cities is more than the economic connection between other cities, so we must have a direct route to Hasankif or Neka for the industrial city of Amol. In other words, in this electricity distribution network, Amol has a stronger economic connection compared to other cities. Therefore, in the event of a power outage in Narivaran, if we can establish a direct route between Amol and Hasankif or Neka, the damages caused by the power outage will be somewhat mitigated.

Table 2
Electricity consumption corresponding to Figure 13

u	Neka(N)	Narivaran(NA)	Hasankif(H)	Daryasar(D)	Amol(A)
$\kappa(u)$	0.95	1	1	0.7	0.95
u	Ghaemshahr(G)	Sari(S)	Savadkooh(SK)	Firoozkooh(F)	Mahmoodabad(M)
$\kappa(u)$	0.7	0.8	0.8	0.9	0.85
u	Royan(R)	Nour(NO)	Noshahr(NSH)	Joybar(J)	-
$\kappa(u)$	0.9	0.85	0.8	0.7	-

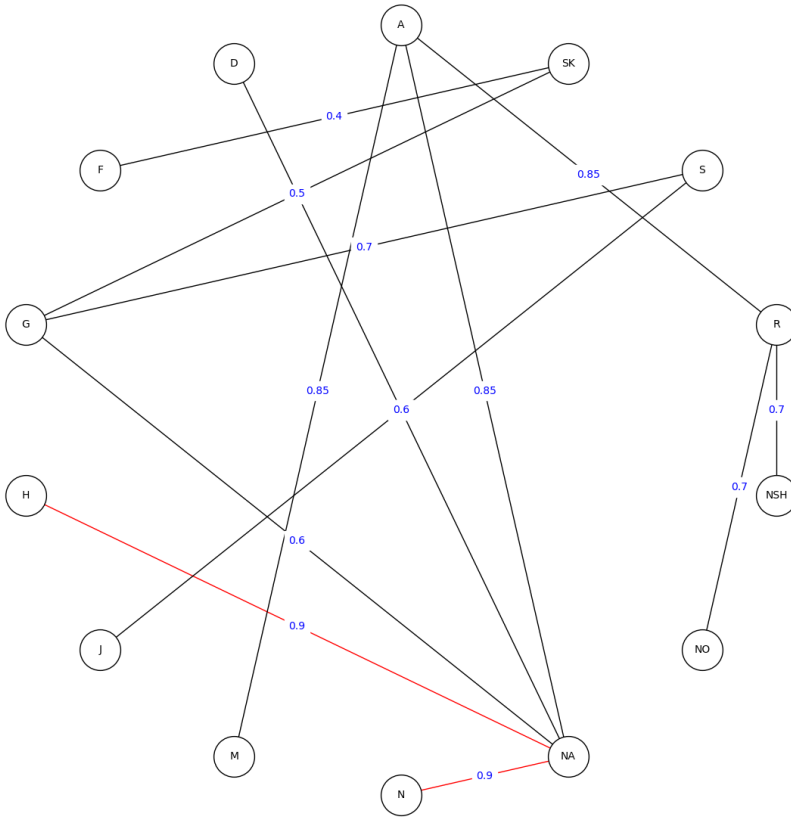


Figure 13
Electricity distribution route from Neka power plant to Narivaran and other cities of Mazandaran province

5. Conclusion

This article presents an exploration of the robust connectivity features found within *CFGs*. The high strength connectivity was calculated in famous *CFGs*. For *CFGs* with a lot of vertices, the high strength connectivity was calculated by using coding in the Sage software environment. The concept of the high strength connectivity in fuzzy graphs has an important impact in enhancing the efficiency of communication paths and optimizing vertices in the economic and communication problems. By utilizing the high strength connectivity, it becomes possible to improve the connectivity and reliability of networks, leading to more effective communication and better performance in various applications. This highlights the importance of leveraging advanced mathematical concepts like strength in addressing complex challenges in economic and communication systems. In future papers, the authors plan to obtain additional software capable of optimizing strength in arbitrary *CFGs*.

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