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Euler Sombor energy of a graph

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Abstract

This paper defines and calculates the Euler Sombor energy of a graph by summing the absolute values of the eigenvalues derived from its Euler Sombor matrix. This matrix, constructed based on vertex degrees, is linked to the Euler Sombor index. We establish mathematical properties for the Euler Sombor energy. Furthermore, we obtain bounds for this energy measure in graphs. Our analysis connects this new energy measure to other graph invariants, providing a comprehensive understanding of its characteristics and implications within the context of graph theory.

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1. Introduction

This work focuses on a simple graph, denoted as Γ. This graph Γ is defined by its set of vertices, $V_\Gamma = \{\omega_1, \dots, \omega_n\}$, and its set of edges, E_Γ. The number of vertices, referred to as the order of Γ, while the number of edges, known as its size.

We use the notation $\omega_i\omega_j$ to represent two adjacent vertices ω_i and ω_j in graph Γ, meaning that $\omega_i\omega_j \in E_\Gamma$. The vertex ω in V_Γ has a degree represented

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by $\zeta(\omega)$, which corresponds to the number of vertices adjacent to ω. If all vertices in Γ have the same degree k, we refer to it as a k-regular graph. Additionally, the complement graph $\bar{\Gamma}$ is defined such that $V_{\bar{\Gamma}} = V_\Gamma$, and an edge exists between two vertices in $\bar{\Gamma}$ if and only if there is no edge between them in Γ. Graphs Γ_1 and Γ_2 are considered isomorphic, denoted as $\Gamma_1; \Gamma_2$, if there is a bijective correspondence between their vertices and edges [1].

A degree-based topological index is a numerical parameter that characterizes the topology of a graph. For more details on the degree-based topological indices, one can explore the survey [2]. The topological index is generally defined as follows

$$TI = TI(\Gamma) = \sum_{\omega_i\omega_j \in E_\Gamma} f(\zeta(\omega_i), \zeta(\omega_j)),$$

where f is a suitable function of vertex degree such that $f(x,y) = f(y,x)$.

Das et al. [3] presented an adjacency-type matrix based on topological indices which is a square symmetric matrix, denoted by $A_{TI}(\Gamma) = (a_{ij})$ and is defined as

$$a_{ij} = \begin{cases} f(\zeta(\omega_i), \zeta(\omega_j)) & \text{if } \omega_i\omega_j \in E_\Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

The eigenvalues of the general extended matrix $A_{TI}(\Gamma)$, are the eigenvalues corresponding to the topological index TI of graph Γ [3]. Also, the energy of the matrix $A_{TI}(\Gamma)$ is defined as the summation of the absolute values of its eigenvalues. For more details and information about the general extended matrix $A_{TI}(\Gamma)$ and its eigenvalues, we refer to [4, 5]. There are many graph energies based on topological indices such as the Zagreb energies [6], the harmonic energy [7], the Randic$'$ energy [8, 9] and Sombor energy [10]. Other kinds of graph energies and their mathematical properties can be found in [11-18].

More recently, a new topological index called the Euler Sombor index, denoted by $EU(\Gamma)$, by applying a novel geometric method based on an ellipse, is introduced [19, 20] and is defined as follows

$$EU(\Gamma) = \sum_{\omega_i\omega_j \in E_\Gamma} \sqrt{\zeta(\omega_i)^2 + \zeta(\omega_j)^2 + \zeta(\omega_i)\zeta(\omega_j)}.$$

Motivated by the definition of the general extended matrix $A_{TI}(\Gamma)$ and the presented matrices based on degree-based topological index, we define the Euler Sombor matrix, denoted by $A_{EU}(\Gamma) = (a_{ij})_{n\times n}$ where

$$a_{ij} = \begin{cases} \sqrt{\zeta(\omega_i)^2 + \zeta(\omega_j)^2 + \zeta(\omega_i)\zeta(\omega_j)} & \text{if } \omega_i\omega_j \in E_\Gamma, \\ 0 & \text{otherwise.} \end{cases}$$

We denote the eigenvalues of the Euler Sombor matrix by $\rho_1 \geq \cdots \geq \rho_n$ as a non-increasing sequence.

The Euler Sombor energy of graph Γ is defined as follows

$$E_{EU}(\Gamma) = \sum_{i=1}^{n} |\rho_i|.$$

Tang et al. [19] analyzed the chemical applicability of the Euler Sombor index, and determined the extremal values for this index among all trees with the corresponding extremal graphs. Gutman in [20] obtained the relations between the Euler Sombor index and the Sombor index of the graph. Movahedi et al. [21] investigated the EU index on chemical graphs and determined sharp bounds for this index on these graphs. Also, they calculated the index for selected nanostructures.

The definition of the second Zagreb index, first presented by Gutman et al. [22], is as follows

$$M_2(\Gamma) = \sum_{\omega_i\omega_j\in E_\Gamma} \zeta(\omega_i)\zeta(\omega_j).$$

Furtula and Gutman [23] introduced the following forgotten index and obtained some interesting results. The index is expressed as follows

$$F(\Gamma) = \sum_{\omega_i\omega_j\in E_\Gamma} \left(\zeta(\omega_i)^2 + \zeta(\omega_j)^2\right) = \sum_{u\in V_\Gamma} \zeta(\omega_i)^3.$$

For more details on the second Zagreb index and the Forgotten index, we refer to [24-28].

In this study, we investigate mathematical properties of the Euler Sombor energy. Moreover, we present the results on the E_{EU} of graphs and study the relationship between this invariant and other graph invariants.

2. Some properties of Euler Sombor energy

We present here the mathematical properties of the Euler Sombor energy.

Lemma 2.1: *Take a simple graph Γ with n vertices. If $\rho_1, \dots, \rho_n$ represent the eigenvalues of the Euler Sombor matrix $A_{EU}(\Gamma)$, then*

(i) $\sum_{i=1}^{n} \rho_i = 0$,

(ii) $\sum_{i=1}^{n} \rho_i^2 = 2\alpha$,

where $\alpha = F(\Gamma) + M_2(\Gamma)$ such that $F(\Gamma)$ and $M_2(\Gamma)$ are the forgotten and the second Zagreb indices, respectively.

Proof: Suppose that $\rho_1, \dots, \rho_n$ represent the eigenvalues of the Euler Sombor matrix $A_{EU}(\Gamma)$.

i) The notation $Tr(A_{EU}(\Gamma))$ is used to denote the trace of the trace of $A_{EU}(\Gamma)$. Therefore, for the Euler Sombor matrix $A_{EU}(\Gamma)$, we have

$$\sum_{i=1}^{n} \rho_i = Tr(A_{EU}(\Gamma)) = \sum_{i=1}^{n} a_{ii} = 0.$$

ii) The sum of ρ_i^2s is equal to $Tr[A_{EU}(\Gamma)]^2$. Let $\omega_1, \dots, \omega_n$ denotes the vertices of Γ. Thus, we get

$$\begin{aligned}
\sum_{i=1}^{n} \rho_i^2 &= Tr(A_{EU}(\Gamma))^2 \\
&= \sum_{i=1}^{n}\sum_{j=1}^{n} a_{ij}a_{ji} \\
&= \sum_{i=1}^{n} a_{ii}^2 + 2\sum_{i\neq j} a_{ij}^2 \\
&= 2\sum_{\omega_i\omega_j\in E_\Gamma} \left(\zeta(\omega_i)^2 + \zeta(\omega_j)^2 + \zeta(\omega_i)\zeta(\omega_j)\right) \\
&= 2\left(\sum_{\omega_i\omega_j\in E_\Gamma} \left(\zeta(\omega_i)^2 + \zeta(\omega_j)^2\right) + \sum_{\omega_i\omega_j\in E_\Gamma} \zeta(\omega_i)\zeta(\omega_j)\right) \\
&= 2(F(\Gamma) + M_2(\Gamma)).
\end{aligned}$$

With putting $\alpha = F(\Gamma) + M_2(\Gamma)$, the result is complete.

Theorem 2.2: *Take a simple graph Γ with n vertices and ρ_1 is the largest eigenvalue of the Euler Sombor matrix $A_{EU}(\Gamma)$. Then*

$$\rho_1 \geq \frac{2EU(\Gamma)}{n}.$$

Equality is valid if and only if Γ is regular graph.

Proof: Using the Rayleigh-Ritz variational principle, for any non-zero vertex X, we have

$$\rho_1 = \rho_1(A_{EU}(\Gamma)) = \max_{X \neq 0}\left\{\frac{X^T A_{EU}(\Gamma) X}{X^T X}\right\},$$

in which X^T is the transpose of vector X.

With putting $X = I = (1, \dots, 1)^T$, we obtain

$$\begin{aligned}\rho_1 &\geq \frac{I^T A_{EU}(\Gamma) I}{I^T I}\\ &= \frac{\sum_{i=1}^n \sum_{j=1}^n a_{ij}}{n}\\ &= \frac{2}{n}\left(\sum_{\omega_i\omega_j \in E_\Gamma} a_{ij}\right)\\ &= \frac{2}{n}\left(\sum_{\omega_i\omega_j \in E_\Gamma} \sqrt{\zeta(\omega_i)^2 + \zeta(\omega_j)^2 + \zeta(\omega_i)\zeta(\omega_j)}\right)\\ &= \frac{2EU(\Gamma)}{n}.\end{aligned}$$

Recall that for a k-regular graph Γ, $I = (1, \dots, 1)^T$ is an eigenvector of $A(\Gamma)$ corresponding to eigenvalue k.

Also, for k-regular graph Γ, $A_{EU}(\Gamma) = \sqrt{3}kA(\Gamma)$. Therefore, the vector I is the eigenvector of $A_{EU}(\Gamma)$, associated with the eigenvalue ρ_1 if and only if Γ is regular.

For the purpose of proving the next result, the following lemma is essential.

Lemma 2.3: [29] *Let an eigenvalue of a graph be a rational number. Then this eigenvalue is an integer.*

Theorem 2.4: *Take a graph Γ of order n. If the Euler Sombor energy is a rational number, then it can not be an odd integer.*

Proof: Consider the eigenvalues $\rho_1, \dots, \rho_n$ of the Euler Sombor matrix $A_{EU}(\Gamma)$, where $\rho_1, \dots, \rho_c$ are positive and the remaining eigenvalues are non-positive. Using the definition of Euler Sombor energy and Lemma 2.1 (part (i)), we obtain

$$\begin{aligned}E_{EU}(\Gamma) &= \sum_{i=1}^n |\rho_i|\\ &= \sum_{i=1}^c \rho_i - \sum_{i=c+1}^n \rho_i\end{aligned}$$

$$= 2(\sum_{i=1}^{c} \rho_i) - \sum_{i=1}^{n} \rho_i$$
$$= 2\sum_{i=1}^{c} \rho_i.$$

Since $E_{EU}(\Gamma)$ is a rational number, the eigenvalues of the Euler Sombor matrix of Γ are rational and consequently, using Lemma 2.3, they are integers. Therefore, if $E_{EU}(\Gamma)$ is rational, then it must be an even integer.

3. Bounds for the Euler Sombor energy

We obtain here some bounds for the Euler Sombor energy.

Theorem 3.1: *For a graph Γ of order n,*

$$\sqrt{2\alpha + n(n-1)P^{\frac{2}{n}}} \leq E_{EU}(\Gamma) \leq \sqrt{2n\alpha},$$

where $P = |det(A_{EU}(\Gamma))|$ and $\alpha = F(\Gamma) + M_2(\Gamma)$ such that $F(\Gamma)$ and $M_2(\Gamma)$ are the forgotten and second Zagreb indices, respectively. The equality is valid if and only if $\Gamma; nK_1$.

Proof: Consider the eigenvalues $\rho_1, \dots, \rho_n$ of the Euler Sombor matrix $A_{EU}(\Gamma)$. For the upper bound, we clearly have

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(|\rho_i| - |\rho_j|\right)^2 \geq 0.$$

Hence,

$$\sum_{i=1}^{n} \sum_{j=1}^{n} \left(|\rho_i|^2 + |\rho_j|^2\right) \geq 2\sum_{i=1}^{n} \sum_{j=1}^{n} |\rho_i||\rho_j|.$$

Using the definition of the Euler Sombor energy and Lemma 2.1(i), we have

$$4n(F(\Gamma) + M_2(\Gamma)) \geq 2(E_{EU}(\Gamma))^2.$$

Therefore, we get

$$E_{EU}(\Gamma) \leq \sqrt{2n(F(\Gamma) + M_2(\Gamma))}.$$

With setting $\alpha = F(\Gamma) + M_2(\Gamma)$, the result for the upper bound is complete.

For the lower bound, we apply the arithmetic-geometric mean inequality and consequently

$$\sum_{i \neq j} |\rho_i||\rho_j| \geq n(n-1)\prod_{i \neq j} \left(|\rho_i||\rho_j|\right)^{\frac{1}{n(n-1)}}$$
$$= n(n-1)\left(\prod_{i=1}^{n} |\rho_i|^{2(n-1)}\right)^{\frac{1}{n(n-1)}}$$
$$= n(n-1)(\prod_{i=1}^{n} |\rho_i|)^{\frac{2}{n}}$$
$$= n(n-1)|\prod_{i=1}^{n} \rho_i|^{\frac{2}{n}}$$
$$= n(n-1)|det(A_{EU}(\Gamma))|^{\frac{2}{n}},$$

since $|det(A_{EU}(\Gamma))| = \prod_{i=1}^{n} \rho_i$.

By setting $P = |det(A_{EU}(\Gamma))|$, we have

$$\sum_{i\neq j} |\rho_i||\rho_j| \geq n(n-1)P^{\frac{2}{n}}. \tag{1}$$

Hence

$$\begin{aligned}(E_{EU}(\Gamma))^2 &= (\textstyle\sum_{i=1}^{n} |\rho_i|)^2 \\ &= \textstyle\sum_{i=1}^{n} \rho_i^2 + \sum_{i\neq j} |\rho_i||\rho_j|.\end{aligned}$$

Using Lemma 2.1(i) and inequality (1), we get

$$E_{EU}(\Gamma) \geq \sqrt{2\alpha + n(n-1)P^{\frac{2}{n}}},$$

where $\alpha = F(\Gamma) + M_2(\Gamma)$.

The equality in the above relation is valid if and only if $|\rho_i| = |\rho_j|$ for any $i \neq j$, that is, the graph Γ is an edgeless graph and consequently, Γ is a graph with n isolated vertices.

The upper bound in Theorem 3.1 is similar to the classical McClelland bound for graph energy, namely, $E_\Gamma \leq \sqrt{2nm}$ [30].

Theorem 3.2: *Take a simple graph Γ of order n. Then*

$$E_{EU}(\Gamma) \leq \frac{2EU(\Gamma)}{n} + \sqrt{(n-1)\left(2\alpha - \left(\frac{2EU(\Gamma)}{n}\right)^2\right)},$$

where $\alpha = F(\Gamma) + M_2(\Gamma)$.

Proof: Consider the eigenvalues $\rho_1, \dots, \rho_n$ of the Euler Sombor matrix $A_{EU}(\Gamma)$. Since $\sum_{i=2}^{n} \sum_{j=2}^{n} \left(|\rho_i| - |\rho_j|\right)^2 \geq 0$, we can have

$$2(E_{EU}(\Gamma) - |\rho_1|)^2 \leq 2(n-1)(\textstyle\sum_{i=1}^{n} |\rho_i|^2 - \rho_1^2).$$

Using Lemma 2.1(i) and since $\rho_1 \geq 0$, we have

$$E_{EU}(\Gamma) \leq \rho_1 + \sqrt{(n-1)(2\alpha - \rho_1^2)},$$

where $\alpha = F(\Gamma) + M_2(\Gamma)$.

Suppose that $f(x) = x + \sqrt{(n-1)(2\alpha - x^2)}$. The function $f(x)$ is a decreasing function for $x \geq \sqrt{\frac{2\alpha}{n}} \geq \frac{2\alpha}{n}$. Using Theorem 2.2 and since $\rho_1 \geq \frac{2EU(\Gamma)}{n}$, then $f(\rho_1) \leq f(\frac{2EU(\Gamma)}{n})$. Therefore,

$$E_{EU}(\Gamma) \leq \frac{2EU(\Gamma)}{n} + \sqrt{(n-1)\left(2\alpha - \left(\frac{2EU(\Gamma)}{n}\right)^2\right)}.$$

Since in bipartite graphs, $|\rho_1| = |\rho_n|$ with a similar method from Theorem 3.2 and with considering this fact that $\sum_{i=2}^{n-1} \sum_{j=2}^{n-1} \left(|\rho_i| - |\rho_j|\right)^2 \geq 0$, we get the following result.

Corollary 3.3: *For the bipartite graph $K_{p,q}$ of order $n = p + q$,*

$$E_{EU}(\Gamma) \leq \frac{4EU(\Gamma)}{n} + \sqrt{2(n-2)\left(\alpha - \left(\frac{2EU(\Gamma)}{n}\right)^2\right)},$$

where $\alpha = F(\Gamma) + M_2(\Gamma)$.

The obtained upper bound in Corollary 3.3 is similar to the Koolen-Moulton bound in [31]

Lemma 3.4: [32] *Assume that* $0 < \lambda_1 \leq t_i \leq \Lambda_1$ *and* $0 < \lambda_2 \leq s_i \leq \Lambda_2$, *for* $1 \leq i \leq k$. *Then,*

$$\left(\sum_{i=1}^{k} t_i^2\right)\left(\sum_{i=1}^{k} s_i^2\right) \leq \frac{1}{4}\left(\sqrt{\frac{\Lambda_1\Lambda_2}{\lambda_1\lambda_2}} + \sqrt{\frac{\lambda_1\lambda_2}{\Lambda_1\Lambda_2}}\right)^2 \left(\sum_{i=1}^{k} t_i s_i\right)^2.$$

Lemma 3.5: [33] *Assume that* $(t_i)_{i=1}^{n}$ *and* $(s_i)_{i=1}^{n}$ *are non-negative real numbers. Then*

$$\left(\sum_{i=1}^{n} t_i^2\right)\left(\sum_{i=1}^{n} s_i^2\right) - \left(\sum_{i=1}^{n} t_i s_i\right)^2 \leq \frac{n^2}{4}(\Lambda_1\Lambda_2 - \lambda_1\lambda_2)^2,$$

with $\lambda_1 = min\{t_i\}$, $\lambda_2 = min\{s_i\}$, $\Lambda_1 = max\{t_i\}$, *and* $\Lambda_2 = max\{s_i\}$, *where* $1 \leq i \leq n$.

Lagrange identity [34] Let $(t_i)_{i=1}^{n}$ and $(s_i)_{i=1}^{n}$ be the real sequences. Then

$$\left(\sum_{i=1}^{n} t_i s_i\right)^2 = \left(\sum_{i=1}^{n} t_i^2\right)\left(\sum_{i=1}^{n} s_i^2\right) - \sum_{1\leq i<j\leq n} \left(t_i s_j - t_j s_i\right)^2.$$

Jensen's identity [35] Consider a convex function ψ on the interval I and $t_1, t_2, \ldots, t_n \in I$. Then

$$\psi\left(\frac{t_1+\cdots+t_n}{n}\right) \leq \frac{\psi(t_1)+\cdots+\psi(t_n)}{n}. \tag{2}$$

The equality is valid if and only if $t_1 = t_2 = \cdots = t_n$.

Theorem 3.6: *Take a graph* Γ *of order* n *and zero is not an eigenvalue of* $A_{EU}(\Gamma)$. *Let* $|\rho_1^*| = \max_{1\leq i\leq n}\{|\rho_i|\}$ *and* $|\rho_n^*| = \min_{1\leq i\leq n}\{|\rho_i|\}$ *in which* ρ_i *is the eigenvalue of* $A_{EU}(\Gamma)$. *Then*

$$E_{EU}(\Gamma) \geq 2\sqrt{2}n\alpha\left(\frac{\sqrt{|\rho_1^*||\rho_n^*|}}{|\rho_1^*| + |\rho_n^*|}\right),$$

where $\alpha = F(\Gamma) + M_2(\Gamma)$.

Proof: Suppose that $\rho_1, \ldots, \rho_n$ are the eigenvalues of $A_{EU}(\Gamma)$. With considering $t_i = |\rho_i|$ and $s_i = 1$ for $1 \leq i \leq n$ in Lemma 3.4, we obtain

$$\left(\sum_{i=1}^{n} |\rho_i|^2\right)\left(\sum_{i=1}^{n} 1\right) \leq \frac{1}{4}\left(\sqrt{\frac{|\rho_1^*|}{|\rho_n^*|}} + \sqrt{\frac{|\rho_n^*|}{|\rho_1^*|}}\right)^2 \left(\sum_{i=1}^{n} |\rho_i|\right)^2,$$

where $|\rho_1^*| = \max_{1\leq i\leq n}\{|\rho_i|\}$ and $|\rho_n^*| = \min_{1\leq i\leq n}\{|\rho_i|\}$.

Using Lemma 2.1 (ii) and considering $\alpha = F(\Gamma) + M_2(\Gamma)$, we have

$$2n\alpha \leq \frac{1}{4}\left(\frac{|\rho_1^*|+|\rho_n^*|}{\sqrt{|\rho_1^*||\rho_n^*|}}\right)^2 (E_{EU}(\Gamma))^2.$$

Therefore, we have

$$\sqrt{8n\alpha} \leq \frac{|\rho_1^*|+|\rho_n^*|}{\sqrt{|\rho_1^*||\rho_n^*|}}(E_{EU}(\Gamma)).$$

With rearranging, the result is complete.

Theorem 3.7: *Take a simple graph Γ with n vertices and zero is not an eigenvalue of $A_{EU}(\Gamma)$. Let $|\rho_1^*| = \max_{1\leq i\leq n}\{|\rho_i|\}$ and $|\rho_n^*| = \min_{1\leq i\leq n}\{|\rho_i|\}$ in which ρ_i is the eigenvalue of $A_{EU}(\Gamma)$. Then*

$$\sqrt{2n\alpha - \frac{n^2}{4}(|\rho_1^*| - |\rho_n^*|)^2} \leq E_{EU}(\Gamma) \leq \sqrt{2n\alpha - \frac{n}{2}(|\rho_1^*| - |\rho_n^*|)^2},$$

where $\alpha = F(\Gamma) + M_2(\Gamma)$. The equality is valid if and only if $\Gamma; \overline{K}_n$ or $\Gamma; \frac{n}{2}K_2$.

Proof: Suppose that $\rho_1, \dots, \rho_n$ are the eigenvalues of $A_{EU}(\Gamma)$. For the lower bound, by setting $a_i = |\rho_i|$ and $b_i = 1$ for $1 \leq i \leq n$ in Lemma 3.5,

$$(\sum_{i=1}^n |\rho_i|^2)(\sum_{i=1}^n 1) - (\sum_{i=1}^n |\rho_i|)^2 \leq \frac{n^2}{4}(|\rho_1^*| - |\rho_n^*|)^2,$$

in which $|\rho_1^*| = \max_{1\leq i\leq n}\{|\rho_i|\}$ and $|\rho_n^*| = \min_{1\leq i\leq n}\{|\rho_i|\}$. Using Lemma 2.1($ii$) and setting $\alpha = F(\Gamma) + M_2(\Gamma)$,

$$2n\alpha - (E_{EU}(\Gamma))^2 \leq \frac{n^2}{4}(|\rho_1^*| - |\rho_n^*|)^2.$$

Therefore, by rearranging the above inequality, the result is complete.

The equality is valid if and only if $\rho_1 = \cdots = \rho_n$, that is Γ is a union of n isolated vertices. That is to say, the equality is valid if and only if $G; nK_1$.

For the upper bound, using theorem 3.1,

$$E_{EU}(\Gamma)^2 \leq 2n(F(\Gamma) + M_2(\Gamma)).$$

Hence $0 \leq 2n(F(\Gamma) + M_2(\Gamma)) - E_{EU}(\Gamma)^2$. By putting $t_i = 1$ and $s_i = |\rho_i|$ for $1 \leq i \leq n$ in Lagrange identity and using Lemma 2.1(ii), and the definition of the Euler Sombor energy, we get

$$0 \leq (\sum_{i=1}^n 1)(\sum_{i=1}^n |\rho_i|^2) - (\sum_{i=1}^n |\rho_i|)^2 = \sum_{1\leq i<j\leq n} \left(|\rho_i| - |\rho_j|\right)^2.$$

Also, we have

$$\sum_{1\leq i<j\leq n} \left(|\rho_i| - |\rho_j|\right)^2 \geq \sum_{i=2}^{n-1} [(|\rho_1^*| - |\rho_i|)^2 + (|\rho_i| - |\rho_n^*|)^2] + (|\rho_1^*| - |\rho_n^*|)^2.$$

From (2), we have

$$\sum_{1\leq i<j\leq n} \left(|\rho_i| - |\rho_j|\right)^2 \geq \frac{n-2}{2}(|\rho_1^*| - |\rho_n^*|)^2 + (|\rho_1^*| - |\rho_n^*|)^2 = \frac{n}{2}(|\rho_1^*| - |\rho_n^*|)^2.$$

Therefore,

$$2n\alpha - (E_{EU}(\Gamma))^2 \geq \frac{n}{2}(|\rho_1^*| - |\rho_n^*|)^2.$$

Consequently, we get

$$E_{EU}(\Gamma) \leq \sqrt{2n\alpha - \frac{n}{2}(|\rho_1^*| - |\rho_n^*|)^2}.$$

Consider a class of real polynomials denoted as $\mathcal{P}(r_1, r_2)$, where $P_n(x) = x^n + r_1x^{n-1} + r_2x^{n-2} + t_3x^{n-3} + \cdots + t_n$ in which r_1 and r_2 are given real numbers are defined in [35]. We suppose that $x_1 \geq x_2 \geq \cdots \geq x_n$ are roots of $P_n(x)$ in this class. In the next lemma, bounds for x_i's are obtained [35].

Lemma 3.8: *Let* $P_n(x) = x^n + r_1x^{n-1} + r_2x^{n-2} + t_3x^{n-3} + \cdots + t_n$ *be a polynomial of class* $\mathcal{P}(r_1, r_2)$ *in which* r_1 *and* r_2 *are given real numbers. If* $x_1 \geq x_2 \geq \cdots \geq x_n$ *are roots of* $P_n(x)$*, then*

$$\bar{x} + \frac{1}{n}\sqrt{\frac{\Delta}{n-1}} \leq x_1 \leq \bar{x} + \frac{1}{n}\sqrt{(n-1)\Delta},$$

$$\bar{x} - \frac{1}{n}\sqrt{\frac{i-1}{n-i+1}\Delta} \leq x_i \leq \bar{x} + \frac{1}{n}\sqrt{\frac{n-i}{i}\Delta}, 2 \leq i \leq n-1$$

$$\bar{x} - \frac{1}{n}\sqrt{(n-1)\Delta} \leq x_n \leq \bar{x} - \frac{1}{n}\sqrt{\frac{\Delta}{n-1}},$$

in which

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} x_i, \Delta = n\sum_{i=1}^{n} x_i^2 - \left(\sum_{i=1}^{n} x_i\right)^2.$$

Also,

$$r_1 = -\sum_{i=1}^{n} x_i, r_2 = \frac{1}{2}\left[\left(\sum_{i=1}^{n} x_i\right)^2 - \sum_{i=1}^{n} x_i^2\right].$$

Theorem 3.9: *Take a simple graph* Γ *with* n *vertices. Let* $\rho_1, \ldots, \rho_n$ *be the eigenvalues of the Euler Sombor matrix of the graph* Γ*. Then*

$$\sqrt{\frac{2\alpha}{n(n-1)}} \leq \rho_1 \leq \sqrt{\frac{2(n-1)\alpha}{n}},$$

$$-\sqrt{\frac{n\alpha(i-1)}{n(n-i+1)}} \leq \rho_i \leq \sqrt{\frac{2\alpha(n-i)}{in}}, 2 \leq i \leq n-1$$

$$-\sqrt{\frac{2\alpha(n-1)}{n}} \leq \rho_n \leq -\sqrt{\frac{2\alpha}{n(n-1)}},$$

Proof: Let $P_n(x) = x^n + r_1x^{n-1} + r_2x^{n-2} + t_3x^{n-3} + \cdots + t_n$ be the characteristic polynomial of $A_{EU}(\Gamma)$. From Lemmas 3.8 and 2.1(i), we get

$$r_1 = -\sum_{i=1}^{n} \rho_i = 0,$$

$$r_2 = \frac{1}{2}\left[\left(\sum_{i=1}^{n} \rho_i\right)^2 - \sum_{i=1}^{n} \rho_i^2\right] = -\alpha,$$

where $\alpha = F(\Gamma) + M_2(\Gamma)$.

Therefore, $r_1 = 0$ and $r_2 = -\alpha$ and consequently, the characteristic polynomial $P_n(\rho) \in \mathcal{P}(0, -\alpha)$. Also

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} \rho_i = 0,$$
$$\Delta = n\sum_{i=1}^{n} \rho_i^2 - \left(\sum_{i=1}^{n} \rho_i\right)^2 = 2n\alpha.$$

By setting $\bar{x} = 0$ and $\Delta = 2n\alpha$ in Lemma 3.8, the result follows.

Theorem 3.10: *Take a simple graph Γ with n vertices and the largest eigenvalue of the Euler Sombor matrix of graph Γ, denoted by ρ_1 with the property $\rho_1 \geq R \geq \sqrt{\frac{2\alpha}{n}}$ for any real number R. Then*

$$E_{EU}(\Gamma) \leq R + \sqrt{(n-1)(2\alpha - R)^2},$$

where $\alpha = F(\Gamma) + M_2(\Gamma)$ in which $F(\Gamma)$ and $M_2(\Gamma)$ are the forgotten and second Zagreb indices, respectively.

Proof: Suppose that $\rho_1, \ldots, \rho_n$ are the eigenvalues of $A_{EU}(\Gamma)$. Suppose that

$$f(x) = \prod_{i=1}^{n} (x - |\rho_i|) = x^n + r_1 x^{n-1} + r_2 x^{n-2} + r_3 x^{n-3} + \cdots + r_n.$$

From Lemmas 2.1 and 3.8, we get

$$r_1 = -\sum_{i=1}^{n} |\rho_i| = -E_{EU}(\Gamma),$$
$$r_2 = \frac{1}{2}\left[\left(\sum_{i=1}^{n} |\rho_i|\right)^2 - \sum_{i=1}^{n} \rho_i^2\right] = \frac{1}{2}\left[(E_{EU}(\Gamma))^2 - 2\alpha\right],$$

where $\alpha = F(\Gamma) + M_2(\Gamma)$.

Therefore, $f(x) \in \mathcal{P}\left(-E_{EU}(\Gamma), \frac{1}{2}((E_{EU}(\Gamma))^2 - 2\alpha)\right)$. On the other hand, we have

$$\bar{x} = \frac{1}{n}\sum_{i=1}^{n} |\rho_i| = \frac{E_{EU}(\Gamma)}{n},$$
$$\Delta = n\sum_{i=1}^{n} |\rho_i|^2 - \left(\sum_{i=1}^{n} |\rho_i|\right)^2 = 2n\alpha - (E_{EU}(\Gamma))^2.$$

For the largest Euler Sombor eigenvalue of $A_{EU}(\Gamma)$, we get

$$\frac{E_{EU}(\Gamma)}{n} + \frac{1}{n}\sqrt{\frac{2n\alpha + (E_{EU}(\Gamma))^2}{n-1}} \leq \rho_1 \leq \frac{E_{EU}(\Gamma)}{n} + \frac{1}{n}\sqrt{(n-1)(2n\alpha - (E_{EU}(\Gamma))^2)}. \quad (3)$$

For each real number R with the property $\rho_1 \geq R \geq \sqrt{\frac{2\alpha}{n}}$ and the right side of the relation (3), we get

$$R \leq \frac{E_{EU}(\Gamma)}{n} + \frac{1}{n}\sqrt{(n-1)(2n\alpha - (E_{EU}(\Gamma))^2)}.$$

With simple calculate and rearranging of the above relation, the result follows.

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