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Edge δ– graceful labeling of some snake and chain related networks

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Abstract

Consider a collection of charges $\{\delta, 2\delta, 3\delta, ..., \beta\delta\}$ for an integer $\delta \geq 1$ and a network Γ with $\alpha = |V(\Gamma)|$ and $\beta = |E(\Gamma)|$. The challenge is to create a variety of electric network designs that allow each wire in the network to employ a single charge from the set $\{\delta, 2\delta, 3\delta, ..., \beta\delta\}$ to each wire in the network, ensuring that the amount of accumulation charges at each nodes (connection points) must vary. In this work, we developed and expanded a form of labeling of Γ entitled an edge δ– graceful labeling ($E\delta - GL$) which is a bijection Ψ from the edge set $E(\Gamma)$ to the set $\{\delta, 2\delta, 3\delta, ..., \beta\delta\}$ so that the produced transformation $\Psi^*: V(\Gamma) \to \{0, \delta, 2\delta, ..., (\beta - 1)\delta\}$, given by: $\Psi^*(\lambda) = (\sum_{\lambda\omega \in E(\Gamma)} \Psi(\lambda\omega)) \bmod (\delta\Pi)$, where $\Pi = \max(\alpha, \beta)$, is an injective. We provide eight designs for electric networks associated with snake graphs and four designs for electric networks associated with chain graphs. Every single one of these designs matches our required features and specifications.

Subject Classification (2020): *05C78, 05C90.*

Keywords: *Edge δ-graceful labeling, Triangular quadrilateral snake, Wheel snake, Linear quadrilateral chain, Diagonal linear hexagonal chain.*

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1. Introduction

This paper will go over finite, undirected, and simple graphs (networks), where the vertex and edge sets of a graph Γ are denoted by $V(\Gamma)$ and $E(\Gamma)$, respectively, with $\alpha = |V(\Gamma)|$ and $\beta = |E(\Gamma)|$[1].

The area of graph labeling is becoming more and more popular due to its broad range of applications. Labelled graphs have proven indispensable in several application domains [2, 3]. For example, identifying various coding strategies for database administration and communication network security. Numerous works structure coding through specific types of graphs with various labelling schemes. Uma Maheswari et al. presented coding with Fibonacci web graph utilising super mean labeling,[4]. Sun flower graphs, SF_n, are a method of secret message coding devised by Prasad et al. Moreover, GMJ coding techniques can be used to transform any labeling graph into a code (see [5] and the references therein).

A transformation that, with some restrictions, conveys a graph's elements (edges, vertices, or both) to positive integers is called a labeling of the graph. Rosa's work [6] in 1967 is cited by the majority of references in the literature as being the origin of graph labeling.

A variety of graph labeling strategies have been researched. If the vertex set is the domain of the labeling transformation, then the labeling is classified as vertex labeling (see [7-9]). If the edge set is the transformation's domain, then labeling a graph is known as edge labeling (see [10-12]). If the union of the vertices and edges is the domain of the transformation, then the labeling of a graph referred to as total labelling (see [13]). Gallian provides a vibrant analysis of graph labeling in which the reader may learn about all the many types of labeling [14].

Edge graceful labeling introduced by Lo [15] in 1985 as a bijective Ψ from the set of edges to the set $\{1,2,\cdots,\beta\}$ such that the produced map Ψ^* from $V(\Gamma)$ to $\{0,1,2,\cdots,\alpha-1\}$ given by $\Psi^*(\lambda) = (\sum_{\lambda\omega\in E(\Gamma)} \Psi(\lambda\omega)) mod(\alpha)$ is a bijective. The graph that allows a graceful labeling is called a graceful graph. Elsonbaty and Daoud [16] have created an exciting type of labeling called edge even graceful labeling. In [17] the authors extending the edge even graceful labeling of graphs to strong r-edge even graceful labeling.

A generalization of the edge graceful labeling, the edge $\delta-$ graceful labeling of graphs was presented by Zeen El Deen [18]. This technique employs the numbers $\{\delta, 2\delta, 3\delta, \cdots, \beta\delta\}$, for any positive integer δ, to identify labels to the edges of the graph.

Definition 1.1: For any positive integer δ and a graph Γ with $\alpha = |V(\Gamma)|$ and $\beta = |E(\Gamma)|$, an edge $\delta-$ graceful labeling of Γ (**E δ −GL**) is a bijection Ψ from the edge set $E(\Gamma)$ to the set $\{\delta, 2\delta, 3\delta, \cdots, \beta\delta\}$ such that the produced transformation $\Psi^*: V(\Gamma) \to \{0, \delta, 2\delta, \cdots, (\beta-1)\delta\}$, given by $\Psi^*(\lambda) = (\sum_{\lambda\omega\in E(\Gamma)} \Psi(\lambda\omega)) mod(\delta\Pi)$, where $\Pi = max(\alpha, \beta)$, is an injective. A graph is referred to as an edge $\delta-$ graceful graph if it permits an edge $\delta-$ graceful labeling.

Not every graph exhibits an edge $\delta -$ graceful labeling. For several graphs associated with cycles, Zeen El Deen [18] found an edge $\delta -$ graceful labeling. Following that Zeen El Deen and Elmahdy [19] demonstrated that there is an edge δ graceful labeling, for any positive integer δ, for the following graphs: the splitting graphs of fans, cycles, and crowns; the shadow graphs of paths, cycles, and fans; the total graphs and the middle graphs of fans, cycles, and crowns. Also, they examined twigs and snails graphs.

This paper aims to demonstrate the existence of an edge $\delta -$ graceful labeling for certain snake and chain -related graphs. For instance, not all snakes nor chains have an edge $\delta -$ graceful labeling as demonstrated by the following theorems.

2. E δ −GL of snake graphs

2.1 *E δ −GL of an alternate triangular snake graph $A(T_\ell)$*

Definition 2.1: An alternative triangular snake graph $A(T_\ell), \ell \equiv 0 \, mod \, 2$ is generated from a path $P_\ell = \{\mu_1, \mu_2, \cdots, \mu_\ell\}$ by joining the alternate edges $\mu_\kappa\mu_{\kappa+1}, \kappa \in [1, \ell-1]$ to a new vertex $\lambda_\kappa, \kappa \in [1, \ell-1], \kappa \equiv 1 \, mod \, 2$.

Theorem 2.2: *For $\ell \equiv 0 \, mod \, 2$, the alternative triangular snake graph $A(T_\ell)$ allows an edge $\delta -$ graceful labeling for $\delta \geq 1$.*

Proof: The graph $A(T_\ell)$ includes $\alpha = \frac{3\ell}{2}$ vertices and $\beta = 2\ell - 1$ edges. The labeling transformation $\Psi_1: E[A(T_\ell)] \to \{\delta, 2\delta, \cdots, (2\ell-1)\delta\}$ that we propose is:

$$\Psi_1(\mu_\kappa\mu_{\kappa+1}) = \begin{cases} \frac{\delta}{2}(4\ell - \kappa - 1) & \text{if} \quad \kappa \in [1, \ell-1] \quad \kappa \equiv 1 \, mod \, 2; \\ \frac{\kappa\delta}{2} & \text{if} \quad \kappa \in [2, \ell-2] \quad \kappa \equiv 0 \, mod \, 2, \end{cases}$$

$$\Psi_1(\lambda_\kappa\mu_\kappa) = \frac{\delta}{2}(\ell + \kappa - 1), \ \kappa \in [1, \ell-1], \ \kappa \equiv 1 \, mod \, 2,$$

$$\Psi_1(\lambda_\kappa\mu_{\kappa+1}) = \frac{\delta}{2}(2\ell + \kappa - 1), \ \kappa \in [1, \ell-1], \ \kappa \equiv 1 \, mod \, 2.$$

Considering the labeling pattern above, then the produced vertex labels are;

$$\Psi_1^*(\mu_\kappa) = [\Psi_1(\mu_\kappa\mu_{\kappa+1}) + \Psi_1(\mu_\kappa\mu_{\kappa-1}) + \Psi_1(\mu_\kappa\lambda_\kappa)] \, mod \, [(2\ell-1)\delta],$$

$$\Psi_1^*(\mu_\kappa) = \begin{cases} \frac{\ell\delta}{2} & \text{if} \quad \kappa = 1, \\ \frac{\delta}{2}[2\ell + \kappa] & \text{if} \quad \kappa \in [2, \ell-2], \ \kappa \equiv 0 \, mod \, 2; \\ \frac{\delta}{2}[\ell + \kappa - 1] & \text{if} \quad \kappa \in [3, \ell-1], \ \kappa \equiv 1 \, mod \, 2; \\ \delta[(\frac{3\ell}{2}) + (\frac{3\ell-2}{2})] mod[(2\ell-1)\delta] = \delta\ell, & \text{if} \quad \kappa = \ell. \end{cases}$$

Thus, $\left(\frac{\ell}{2}+1\right)\delta, \left(\frac{\ell}{2}+2\right)\delta, \cdots, (\ell-2)\delta, \delta(\ell-1)$ are the labels of the vertices $\mu_3, \mu_5, \cdots, \mu_{\ell-3}, \mu_{\ell-1}$, and the labels of the vertices $\mu_2, \mu_4, \cdots, \mu_{\ell-4}, \mu_{\ell-2}$ are $\delta(\ell+1), \delta(\ell+2), \cdots, \delta\left(\frac{3\ell}{2}-2\right), \delta\left(\frac{3\ell}{2}-1\right)$, respectively.

$$\Psi_1^*(\lambda_\kappa) = [\Psi_1(\lambda_\kappa\mu_\kappa) + \Psi_1(\mu_{\kappa+1}\lambda_\kappa)] mod[(2\ell-1)\delta]$$
$$= \delta\left(\frac{3\ell}{2}+\kappa-1\right) \, mod \, [(2\ell-1)\delta], \ \kappa \in [1, \ell-1], \ \kappa \equiv 1 \, mod \, 2.$$

Hence, the labels of the vertices $\lambda_1, \lambda_3, \lambda_5, \cdots, \lambda_{\ell-3}, \lambda_{\ell-1}$ are $\left(\frac{3\ell}{2}\right)\delta, \left(\frac{3\ell}{2}+2\right)\delta, \left(\frac{3\ell}{2}+4\right)\delta, \cdots, \left(\frac{\ell}{2}-3\right)\delta, \delta\left(\frac{\ell}{2}-1\right)$, respectively. The proof has been achieved because there are no identical vertex labels.

Example 1: In Fig. 1, we propose the alternative triangular snake AT_{12} with an edge $3-$ graceful labeling according to the labeling transformation defined above.

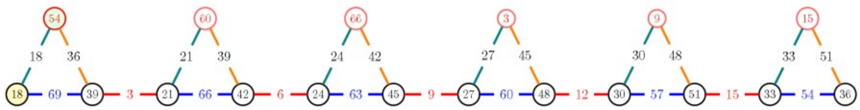

Figure 1

The alternative triangular snake graph AT_{12} with an E3 − GL.

2.2 E δ −GL of a triangular quadrilateral snake graph TQ_ℓ

Definition 2.3: From a path $P_\ell = \{\mu_1, \mu_2, \cdots, \mu_{\ell+1}\}$ a quadrilateral snake is obtained by substituting a cycle C_4 for each edge of the path. Stated differently, by connecting μ_κ and $\mu_{\kappa+1}$ to new vertices λ_κ and ω_κ, correspondingly, and connecting the vertices λ_κ and ω_κ for $\kappa \in [1, \ell]$. By adding more edges $\lambda_\kappa\mu_{\kappa+1}, \kappa \in [1, \ell]$, we have a triangular quadrilateral snake graph with ℓ blocks, represented by TQ_ℓ. The vertex set of TQ_ℓ is $V[TQ_\ell] = \{\mu_\kappa, \kappa \in [1, \ell+1]\} \cup \{\lambda_\kappa, w_\kappa, \kappa \in [1, \ell]\}$ and the edge set is $E[TQ_\ell] = \{\lambda_\kappa\mu_{\kappa+1}, \mu_\kappa\lambda_\kappa, \lambda_\kappa w_\kappa, w_\kappa\mu_{\kappa+1}, \mu_\kappa\mu_{\kappa+1}, \kappa \in [1, \ell]\}$.

Theorem 2.4: *For an integer $\ell \geq 2$, the triangular quadrilateral snake graph TQ_ℓ is an edge $\delta -$ graceful graph for $\delta \geq 1$.*

Proof: The triangular quadrilateral snake graph $TQ_\ell, \ell \geq 2$ has $\alpha = 3\ell + 1$ vertices and $\beta = 5\ell$ edges.

The labeling function $\Psi_2: E(TQ_\ell) \to \{\delta, 2\delta, 3\delta, \cdots, 5\ell\delta\}$ that we suggest is.

$\Psi_2(\lambda_\kappa\mu_{\kappa+1}) = \delta\kappa$ for $\kappa \in [1, \ell]$,

$\Psi_2(\lambda_\kappa\mu_\kappa) = \delta(\ell+\kappa)$ for $\kappa \in [1, \ell]$,

$$\Psi_2(\mu_\kappa\mu_{\kappa+1}) = \begin{cases} \delta(3\ell-\kappa) & \text{if} \quad \kappa \in [1, \ell-1]; \\ 5\ell\,\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

$\Psi_2(\lambda_\kappa\omega_\kappa) = \delta(4\ell-\kappa)$ for $\kappa \in [1, \ell]$,

$\Psi_2(\omega_\kappa\mu_{\kappa+1}) = \delta(5\ell-\kappa)$ for $\kappa \in [1, \ell]$.

Following the labeling pattern mentioned above, the induced vertex labels $\Psi_2^*$are;

$\Psi_2^*(\lambda_\kappa) = \Psi_2(\lambda_\kappa\mu_{\kappa+1}) = \kappa\delta, \kappa \in [1,\ell]$,

$\Psi_2^*(\omega_\kappa) = [\Psi_2(\omega_\kappa\lambda_\kappa) + \Psi_2(\omega_\kappa\mu_{\kappa+1})] \, mod \, [5\ell\,\delta] = (4\ell - 2\kappa)\delta, \kappa \in [1,\ell]$,

$$\Psi_2^*(\mu_\kappa) = [\Psi(\mu_\kappa\mu_{\kappa+1}) + \Psi(\mu_\kappa\mu_{\kappa-1}) + \Psi(\mu_\kappa\lambda_\kappa) + \Psi(\mu_\kappa\lambda_{\kappa-1}) + \Psi(\mu_\kappa\omega_{\kappa-1})] \, mod \, [5\ell\,\delta],$$

$$\Psi_2^*(\mu_\kappa) = \begin{cases} 4\ell\,\delta & \text{if} \quad \kappa = 1; \\ (12\ell - \kappa + 1)\delta \, mod \, [5\ell\delta] = (2\ell - \kappa + 1)\delta & \text{if} \quad \kappa \in [2, \ell - 1]; \\ (14\ell + 1)\delta \, mod \, [5\ell\delta] = (4\ell + 1)\delta & \text{if} \quad \kappa = \ell; \\ 0 & \text{if} \quad \kappa = \ell + 1. \end{cases}$$

It is possible to verify that Ψ_2^* has injective property since there aren't any repeated vertex labels. Thus, an edge δ graceful labeling is admitted by TQ_ℓ.

Example 2: In Fig. 2, we present the triangular quadrilateral snake TQ_7 with an edge 5− graceful labeling.

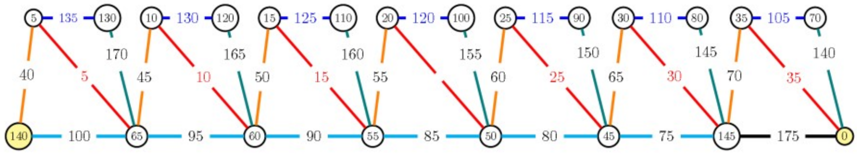

Figure 2
The graph TQ_7 with an E5 − GL.

2.3 *E δ −GL of double triangular quadrilateral snake graph DTQ_ℓ*

Definition 2.5: Double triangular quadrilateral snake graph with ℓ blocks, denoted by DTQ_ℓ, is made up of two triangular quadrilateral snake that have shared path. The graph DTQ_ℓ consists of the vertex set

$V[DTQ_\ell] = \{\lambda_\kappa, w_\kappa, \tau_\kappa, \vartheta_\kappa, \kappa \in [1,\ell]\} \cup \{\mu_\kappa, \kappa \in [1, \ell+1]\}$ and the edge set

$E[DTQ_\ell] = \{\mu_\kappa\lambda_\kappa, \lambda_\kappa w_\kappa, w_\kappa\mu_{\kappa+1}, \mu_\kappa\mu_{\kappa+1}, \lambda_\kappa\mu_{\kappa+1}, \mu_\kappa\tau_\kappa, \tau_\kappa\vartheta_\kappa, \vartheta_\kappa\mu_{\kappa+1}, \tau_\kappa\mu_{\kappa+1}, \kappa \in [1,\ell]\}$, see Fig. 3.

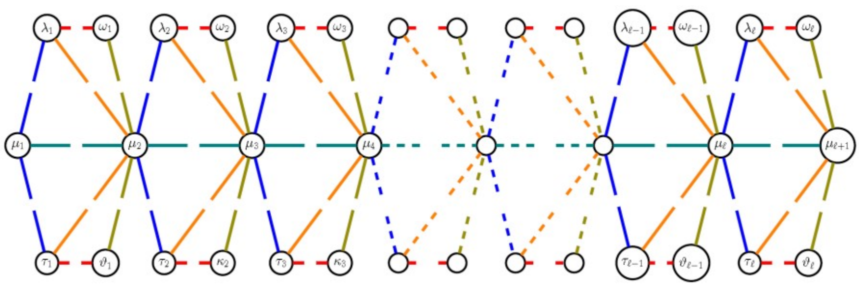

Figure 3
The double triangular quadrilateral snake graph DTQ_ℓ with ordinary labeling

Theorem 2.6: *For an integer $\ell \geq 2$, the double triangular quadrilateral snake graph DTQ_ℓ is an edge $\delta-$ graceful graph for $\delta \geq 1$.*

Proof: The double triangular quadrilateral snake graph $TQ_\ell, \ell \geq 2$ includes $\alpha = 5\ell + 1$ vertices and $\beta = 9\ell$ edges.

Case 1: When $\ell \equiv 1 \, mod \, 2$. The labeling transformation $\Psi_3 : E(DTQ_\ell) \to \{\delta, 2\delta, \cdots, 9\ell\delta\}$ constructed by

$$\Psi_3(\lambda_\kappa \mu_{\kappa+1}) = \delta\kappa \text{ for } \kappa \in [1, \ell],$$

$$\Psi_3(\lambda_\kappa \mu_\kappa) = \delta(\ell + \kappa) \text{ for } \kappa \in [1, \ell],$$

$$\Psi_3(\tau_\kappa \mu_{\kappa+1}) = \begin{cases} \delta(2\ell + \kappa) & \text{if} \quad \kappa \in [1, \ell - 1]; \\ 6\ell\,\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_3(\mu_\kappa \mu_{\kappa+1}) = \delta(3\ell + \kappa) \text{ for } \kappa \in [1, \ell],$$

$$\Psi_3(\tau_\kappa \vartheta_\kappa) = \begin{cases} \delta(5\ell - \kappa) & \text{if} \quad \kappa \in [1, \ell - 1]; \\ 9\ell\,\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_3(\tau_\kappa \mu_\kappa) = \delta(6\ell - \kappa) \text{ for } \kappa \in [1, \ell],$$

$$\Psi_3(\vartheta_\kappa \mu_{\kappa+1}) = \begin{cases} \delta(7\ell - \kappa) & \text{if} \quad \kappa \in [1, \ell - 1]; \\ 3\ell\,\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_3(\lambda_\kappa \omega_\kappa) = \delta(8\ell - \kappa) \text{ for } \kappa \in [1, \ell],$$

$$\Psi_3(\omega_\kappa \mu_{\kappa+1}) = \delta(9\ell - \kappa) \text{ for } \kappa \in [1, \ell].$$

Following the labeling pattern mentioned above, the produced vertex labels Ψ_3^* are;

$$\Psi_3^*(\lambda_\kappa) = [\Psi_3(\lambda_\kappa \omega_\kappa) + \Psi_3(\lambda_\kappa \mu_\kappa) + \Psi_3(\lambda_\kappa \mu_{\kappa+1})] \, mod \, [9\ell\delta] = \kappa\delta, \kappa \in [1, \ell],$$

$$\Psi_3^*(\omega_\kappa) = [\Psi_3(\omega_\kappa \lambda_\kappa) + \Psi_3(\omega_\kappa \mu_{\kappa+1})] \, mod \, [9\ell\delta] = (8\ell - 2\kappa)\delta, \kappa \in [1, \ell],$$

$$\Psi_3^*(\tau_\kappa) = \begin{cases} (13\ell - \kappa)\delta \, mod \, [9\ell\,\delta] = (4\ell - \kappa)\delta & \text{if} \quad \kappa \in [1, \ell - 1]; \\ 20\ell\,\delta \, mod \, [9\ell\,\delta] = 2\ell\,\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_3^*(\vartheta_\kappa) = \begin{cases} (12\ell - 2\kappa)\delta \, mod \, [9\ell\delta] = (3\ell - 2\kappa)\delta & \text{if} \quad \kappa \in [1, \ell - 1]; \\ 12\ell\,\delta \, mod \, [9\ell\,\delta] = 3\ell\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_3^*(\mu_\kappa) = \begin{cases} (\ell + 1)\delta & \text{if} \quad \kappa = 1; \\ (4\ell + 2\kappa - 1)\delta & \text{if} \quad \kappa \in [2, \ell]; \\ 13\ell\,\delta \, mod \, [9\ell\,\delta] = 4\ell\,\delta & \text{if} \quad \kappa = \ell + 1. \end{cases}$$

Case 2: When $\ell \equiv 0 \, mod \, 2$. The definition of the labeling Ψ_3 can be done in a manner identical to the preceding case, but with the substitutions mentioned below:

$$\Psi_3(\tau_\kappa \mu_{\kappa+1}) = \delta(2\ell + \kappa) \text{ for } \kappa \in [1, \ell],$$

$$\Psi_3(\tau_\kappa \mu_\kappa) = \begin{cases} \delta(6\ell - \kappa) & \text{if} \quad \kappa \in [1, \ell - 1]; \\ 6\ell\,\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_3(\vartheta_\kappa\mu_{\kappa+1}) = \begin{cases}\delta(7\ell-\kappa) & \text{if} \quad \kappa \in [1,\ell-1];\\ 5\ell\,\delta & \text{if} \quad \kappa = \ell,\end{cases}$$

The produced vertex labels Ψ_3^* are;

$\Psi_3^*(\tau_\ell) = 18\ell\,\delta \bmod [9\ell\,\delta] = 0,$

$$\Psi_3^*(\mu_\kappa) = \begin{cases}(16\ell-1)\delta \bmod [9\ell\delta] = (7\ell-1)\delta & \text{if} \quad \kappa = \ell;\\ 3\ell\,\delta & \text{if} \quad \kappa = \ell+1,\end{cases}$$

$\Psi_3^*(\vartheta_\ell) = 14\ell\,\delta \bmod [9\ell\,\delta] = 5\ell\delta.$

Since there aren't any repeated vertex labels. Consequently, DTQ_ℓ permits an edge $\delta-$ elegant labeling.

Example 3: The double triangular quadrilateral snake DTQ_7 with its explicit edge 2− graceful labeling is depicted in Fig. 4.

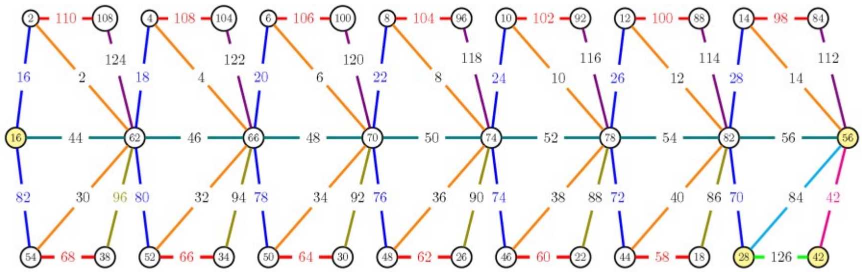

Figure 4
The graphs DTQ_7 with an E2 − GL.

2.4 *E δ −GL of the wheel snake graph $S_\ell(W_3)$*

Definition 2.7: An ℓ blocks snake graph where each block is W_3 is called wheel snake graph, indicated by $S_\ell(W_3)$. This graph consists of $V[S_\ell(W_3)] = \{\mu_\kappa, \kappa \in [1,\ell+1]\} \cup \{\lambda_\kappa, w_\kappa, \kappa \in [1,\ell]\}$ and $E[S_\ell(W_3)] = \{\mu_\kappa\lambda_\kappa, \mu_\kappa w_\kappa, w_\kappa\mu_{\kappa+1}, \mu_\kappa\mu_{\kappa+1}, \lambda_\kappa w_\kappa, \lambda_\kappa\mu_{\kappa+1}, \kappa \in [1,\ell]\}$ as in Fig. 5. The wheel snake graph $S_\ell(W_3), \ell \geq 2$ has $\alpha = 3\ell+1$ and $\beta = 6\ell$.

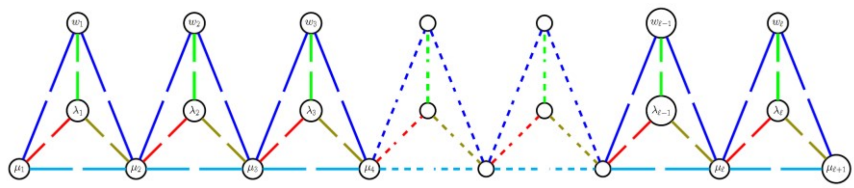

Figure 5
Normal labeling of the wheel snake graph $S_\ell(W_3)$.

Theorem 2.8: *For an integer $\ell \geq 2$, the wheel snake graph $S_\ell(W_3)$, is an edge $\delta-$ graceful graph for $\delta \geq 1$.*

Proof: The labeling transformation $\Psi_4 : E[S_\ell(W_3)] \to \{\delta, 2\delta, 3\delta, \cdots, 6\ell\delta\}$ that we suggest is:

$\Psi_4(\lambda_\kappa \mu_\kappa) = \delta\kappa$ for $\kappa \in [1, \ell]$,

$\Psi_4(\mu_\kappa \mu_{\kappa+1}) = \delta(6\ell - \kappa)$ for $\kappa \in [1, \ell]$,

$\Psi_4(\lambda_\kappa w_\kappa) = \delta(\ell + \kappa)$ for $\kappa \in [1, \ell]$,

$\Psi_4(\lambda_\kappa \mu_{\kappa+1}) = \delta(5\ell - \kappa)$ for $\kappa \in [1, \ell]$,

$$\Psi_4(\mu_\kappa w_\kappa) = \begin{cases} 6\ell\,\delta & \text{if} \quad \kappa = 1; \\ \delta(2\ell + \kappa) & \text{if} \quad \kappa \in [2, \ell] \end{cases}$$

$$\Psi_4(w_\kappa \mu_{\kappa+1}) = \begin{cases} \delta(4\ell - \kappa) & \text{if} \quad \kappa \in [1, \ell - 1]; \\ \delta(2\ell + 1) & \text{if} \quad \kappa = \ell. \end{cases}$$

With reference to the labeling pattern above, the generated vertex labels Ψ_4^* are;

$\Psi_4^*(\lambda_\kappa) = \Psi_4(\mu_\kappa \lambda_\kappa) = \kappa\delta, \ \kappa \in [1, \ell]$,

$\Psi_4^*(w_\kappa) = [\Psi_4(w_\kappa \mu_\kappa) + \Psi_4(w_\kappa \lambda_\kappa) + \Psi_4(w_\kappa \mu_{\kappa+1})] \ mod \ [6\ell\delta]$,

$$\Psi_4^*(\mu_\kappa) = \begin{cases} \delta\ 5\ell & \text{if} \quad \kappa = 1; \\ \delta(7\ell + \kappa) \ mod \ [6\ell\delta] = \delta(\ell + \kappa)\delta & \text{if} \quad \kappa \in [2, \ell]; \\ \delta(\ell + 1) & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_4^*(\mu_\kappa) = [\Psi_4(w_\kappa \mu_\kappa) + \Psi_4(\mu_\kappa \lambda_\kappa) + \Psi_4(\mu_\kappa \mu_{\kappa+1}) + \Psi_4(\mu_\kappa w_{\kappa-1}) + \Psi_4(\mu_\kappa \lambda_{\kappa-1}) + \Psi_4(\mu_{\kappa-1}\mu_\kappa)] mod[6\ell\delta],$$

$$\Psi_4^*(\mu_\kappa) = \begin{cases} 0 & \text{if} \quad \kappa = 1; \\ \delta(17\ell + 3 - 2\kappa) mod[6\ell\delta] = \delta(5\ell + 3 - 2\kappa)\delta & \text{if} \quad \kappa \in [2, \ell]; \\ \delta(5\ell + 1) & \text{if} \quad \kappa = \ell + 1. \end{cases}$$

Since no two vertex labels can be seen to be identical, then the produced mapping Ψ^* is injective. The proof is achieve.

Example 4: In Fig. 6, we depict the wheel snake graph $S_7(W_3)$ with an edge 4− graceful labeling.

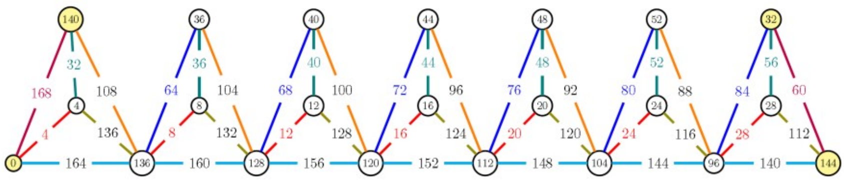

Figure 6
The wheel snake graph $S_7(W_3)$ with an E 4 −GL.

2.5 E δ −GL of a diagonal quadrilateral snake graph DQ_ℓ

Definition 2.9: A diagonal quadrilateral snake graph with ℓ blocks, indicated by DQ_ℓ, obtained from a quadrilateral snake by adding the edges $\lambda_\kappa \mu_{\kappa+1}$ and $\omega_\kappa \mu_\kappa, \kappa \in [1, \ell]$. Thus the vertex set is $V[DQ_\ell] = \{\mu_\kappa, \kappa \in [1, \ell + 1]\} \cup \{\lambda_\kappa, w_\kappa, \kappa \in [1, \ell]\}$ and the edge set is $E[DQ_\ell] = \{\mu_\kappa \lambda_\kappa, \lambda_\kappa w_\kappa, w_\kappa \mu_{\kappa+1},$

$\mu_\kappa\mu_{\kappa+1}, \lambda_\kappa\mu_{\kappa+1}, \omega_\kappa\mu_\kappa\kappa \in [1,\ell]\}$. The graph $DQ_\ell, \ell \geq 2$ has $\alpha = 3\ell + 1$ and $\beta = 6\ell$ edges, see Fig. 7

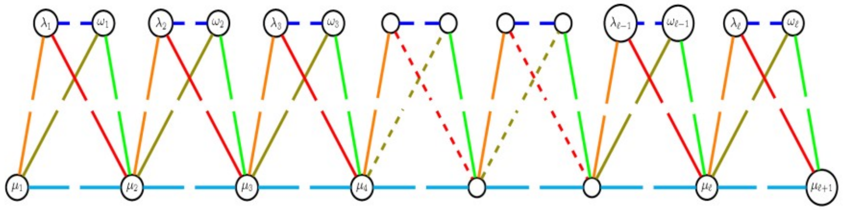

Figure 7
The diagonal quadrilateral snake graph DQ_ℓ

Theorem 2.10: *For an integer $\ell \geq 2$, the diagonal quadrilateral snake graph DQ_ℓ is an edge $\delta -$ graceful graph for $\delta \geq 1$.*

Proof: Construct the labeling transformation $\Psi_5: E[DQ_\ell] \to \{\delta, 2\delta, 3\delta, \cdots, 6\ell\delta\}$ by

$$\Psi_5(\lambda_\kappa\mu_\kappa) = \delta\kappa \quad \text{for} \quad \kappa \in [1,\ell],$$

$$\Psi_5(\omega_\kappa\mu_{\kappa+1}) = \delta(\ell + \kappa) \quad \text{for} \quad \kappa \in [1,\ell],$$

$$\Psi_5(\lambda_\kappa\omega_\kappa) = \begin{cases} \delta(2\ell + \kappa) & \text{if} \quad \kappa \in [1, \ell - 1]; \\ 6\ell\,\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_5(\lambda_\kappa\mu_{\kappa+1}) = \delta(4\ell - \kappa) \quad \text{for} \quad \kappa \in [1,\ell],$$

$$\Psi_5(\omega_\kappa\mu_\kappa) = \delta(6\ell - \kappa) \quad \text{for} \quad \kappa \in [1,\ell],$$

$$\Psi_5(\mu_\kappa\mu_{\kappa+1}) = \delta(5\ell - \kappa) \quad \text{for} \quad \kappa \in [1,\ell].$$

Then the produced vertex labels Ψ_5^* are given in the following manner:

$$\Psi_5^*(\lambda_\kappa) = \Psi_5(\mu_\kappa\lambda_\kappa) = \kappa\delta, \quad \kappa \in [1, \ell - 1],$$

$$\Psi_5^*(\lambda_\ell) = [\Psi_5(\lambda_\ell\mu_\ell) + \Psi_5(\lambda_\ell\mu_{\ell+1}) + \Psi_5(\lambda_\ell\omega_\ell)]\, mod\, [6\ell\delta] = 4\ell\delta,$$

$$\Psi_5^*(\omega_\kappa) = (9\ell + \kappa)\delta\, mod\, [6\ell\delta] = (3\ell + \kappa)\delta, \kappa \in [1, \ell - 1],$$

$$\Psi_5^*(w_\ell) = 13\ell\,\delta\, mod\, [6\ell\,\delta] = \ell\delta,$$

$$\Psi_5^*(\mu_1) = \Psi(\mu_1\mu_2) = 5\ell - 1, \quad \Psi^*(\mu_{\ell+1}) = \Psi(\lambda_\ell\mu_{\ell+1}) = 3\ell\delta,$$

$$\Psi_5^*(\mu_\kappa) = (21\ell - 2\kappa + 1)\delta\, mod\, [6\ell\delta] = (3\ell - 2\kappa + 1)\delta, \kappa \in [2, \ell].$$

The proof has been completed because there are no identical vertex labels.

Example 5: In Fig. 8, we present DQ_7 with an edge 3− graceful labeling.

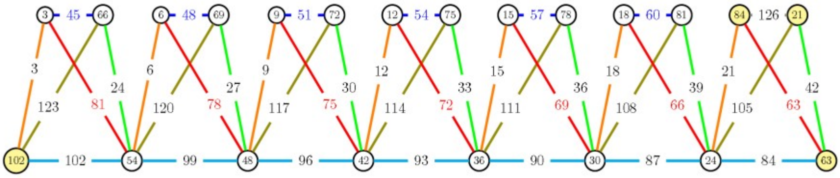

Figure 8

The diagonal quadrilateral snake graph DQ_7 with E3 −GL.

2.6: *Eδ −GL of the alternate wheel snake graph* $A[S_\ell(W_3)]$

Theorem 2.11: *For an integer* $\ell \geq 2$, *the alternate wheel snake graph* $A[S_\ell(W_3)]$ *is an edge* $\delta-$ *graceful graph for* $\delta \geq 1$.

Proof: The alternate wheel snake graph $A[S_\ell(W_3)]$ has $\alpha = 4\ell$ vertices and $\beta = 7\ell - 1$ edges, Fig. 9

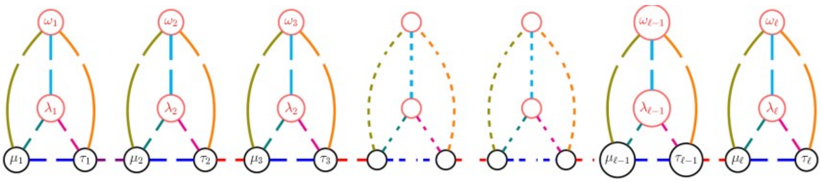

Figure 9

The graph $A[S_\ell(W_3)]$ with ordinary labeling .

(1) When $\ell \equiv 0 \; mod \; 2$, the labeling $\Psi_6 : E[A[S_\ell(W_3)]] \to \{\delta, 2\delta, 3\delta, \cdots, (7\ell - 1)\delta\}$ that we present is.

$\Psi_6(\lambda_\kappa \omega_\kappa) = \delta\kappa$ for $\kappa \in [1, \ell]$,

$\Psi_6(\mu_\kappa \omega_\kappa) = \delta(2\ell + \kappa)$ for $\kappa \in [1, \ell]$,

$\Psi_6(\tau_\kappa \omega_\kappa) = \delta(5\ell - 1 - \kappa)$ for $\kappa \in [1, \ell]$,

$\Psi_6(\mu_\kappa \tau_\kappa) = \delta(6\ell - 1 - \kappa)$ for $\kappa \in [1, \ell]$,

$$\Psi_6(\mu_\kappa \lambda_\kappa) = \begin{cases} (\ell + \kappa)\delta & \text{if} \quad \kappa \in [1, \ell - 1]; \\ \delta(6\ell - 1) & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_6(\tau_\kappa \lambda_\kappa) = \begin{cases} \delta(7\ell - 1 - \kappa) & \text{if} \quad \kappa \in [1, \ell - 1]; \\ \delta \; 2\ell & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_6(\tau_\kappa \mu_{\kappa+1}) = \begin{cases} \delta(3\ell + \kappa) & \text{if} \quad \kappa \in \left[1, \frac{\ell}{2} - 1\right]; \\ \delta(7\ell - 1) & \text{if} \quad \kappa = \frac{\ell}{2}; \\ \delta(3\ell + \kappa - 1) & \text{if} \quad \kappa \in \left[\frac{\ell}{2} + 1, \ell\right] \end{cases}$$

Regarding the labeling pattern above, the produced vertex labels Ψ_6^* are :

$\Psi_6^*(\omega_\kappa) = \Psi_6(\mu_\kappa \lambda_\kappa) \; mod \; [(7\ell - 1)\delta] = \kappa\delta, \quad \kappa \in [1, \ell]$,

$\Psi_6^*(\lambda_\kappa) = \Psi_6(\lambda_\kappa \mu_\kappa)\ mod\ [(7\ell - 1)\delta] = (\ell + \kappa)\delta,\ \kappa \in [1, \ell],$

$$\Psi_6^*(\mu_\kappa) = \begin{cases} \delta(2\ell + 1) & \text{if} \quad \kappa = 1; \\ (12\ell + 2\kappa - 2)\delta\ mod\ [(7\ell - 1)\delta] = (5\ell + 2\kappa - 1)\delta & \text{if} \quad \kappa \in \left[2, \frac{\ell}{2}\right]; \\ \frac{19\ell}{2} mod[(7\ell - 1)\delta] = (\frac{5\ell}{2} + 1)\delta & \text{if} \quad \kappa = \frac{\ell}{2} + 1; \\ (12\ell + 2\kappa - 3)\delta\ mod\ [(7\ell - 1)\delta] = (5\ell + 2\kappa - 2)\delta & \text{if} \quad \kappa \in \left[\frac{\ell}{2} + 2, \ell - 1\right]; \\ \delta(4\ell - 2) & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_6^*(\tau_\kappa) = \begin{cases} (21\ell - 2\kappa - 3)\delta\ mod\ [(7\ell - 1)\delta] = (7\ell - 2\kappa - 1)\delta & \text{if} \quad \kappa \in \left[1, \frac{\ell}{2} - 1\right]; \\ (\frac{5\ell}{2} - 1)\delta & \text{if} \quad \kappa = \frac{\ell}{2}; \\ (21\ell - 2\kappa - 4)\delta\ mod\ [(7\ell - 1)\delta] = (7\ell - 2\kappa - 2)\delta & \text{if} \quad \kappa \in \left[\frac{\ell}{2} + 1, \ell - 1\right]; \\ \delta(4\ell - 1) & \text{if} \quad \kappa = \ell, \end{cases}$$

(2) When $\ell \equiv 1\ mod\ 2$, the labeling transformation Ψ_6 proposed as follows:

$\Psi_6(\lambda_\kappa \tau_\kappa) = \delta\kappa$ for $\kappa \in [1, \ell]$,

$\Psi_6(\tau_\kappa \omega_\kappa) = \delta(\ell + \kappa)$ for $\kappa \in [1, \ell]$,

$\Psi_6(\tau_\kappa \mu_\kappa) = \delta(2\ell + \kappa)$ for $\kappa \in [1, \ell]$,

$$\Psi_6(\tau_\kappa \mu_{\kappa+1}) = \begin{cases} (7\ell - 1)\delta & \text{if} \quad \kappa = 1; \\ \delta(4\ell - \kappa) & \text{if} \quad \kappa \in [2, \ell], \end{cases}$$

$\Psi_6(\omega_\kappa \mu_\kappa) = \delta(5\ell - \kappa - 1)$ for $\kappa \in [1, \ell]$,

$\Psi_6(\omega_\kappa \lambda_\kappa) = \delta(6\ell - \kappa - 1)$ for $\kappa \in [1, \ell]$,

$\Psi_6(\lambda_\kappa \mu_\kappa) = \delta(7\ell - \kappa - 1)$ for $\kappa \in [1, \ell]$.

Applying the labeling pattern mentioned above, the generated vertex labels Ψ_6^* are;

$\Psi_6^*(\omega_\kappa) = \Psi_6(\mu_\kappa \omega_\kappa)\ mod\ [(7\ell - 1)\delta] = (5\ell - \kappa - 1)\delta, \quad \kappa \in [1, \ell],$

$\Psi_6^*(\lambda_\kappa) = \Psi_6(\lambda_\kappa \omega_\kappa)\ mod\ [(7\ell - 1)\delta] = (6\ell - \kappa - 1)\delta, \quad \kappa \in [1, \ell],$

$$\Psi_6^*(\mu_\kappa) = \begin{cases} (14\ell - \kappa - 2)\ mod\ [(7\ell - 1)\delta] = (7\ell - \kappa - 1)\delta & \text{if} \quad \kappa \in [1,2]; \\ (18\ell - 2\kappa - 1)\delta\ mod\ [(7\ell - 1)\delta] = (4\ell - 2\kappa + 1)\delta & \text{if} \quad \kappa \in [3, \ell], \end{cases}$$

$$\Psi_6^*(\tau_\kappa) = \begin{cases} (10\ell + 2)\delta\ mod\ [(7\ell - 1)\delta] = (3\ell + 3)\delta, & \text{if} \quad \kappa = 1; \\ (7\ell + 2\kappa)\delta\ mod\ [(7\ell - 1)\delta] = (2\kappa + 1)\delta & \text{if} \quad \kappa \in [2, \ell - 1], \\ 6\ell\ \delta & \text{if} \quad \kappa = \ell. \end{cases}$$

The proof is complete since the resulting mapping Ψ^* is injective due to the absence of identical vertex labels.

Example 6: The alternate wheel snake graph $A[S_7(W_3)]$ with an edge 4− graceful labeling and the alternate wheel snake $A[S_8(W_3)]$ with an edge 2− graceful labelling are depicted in Fig. 10, and Fig. 11, respectively.

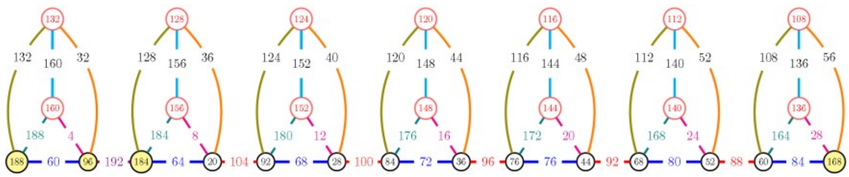

Figure 10
$\boldsymbol{A[S_7(K_4)]}$ **with an E4 −GL.**

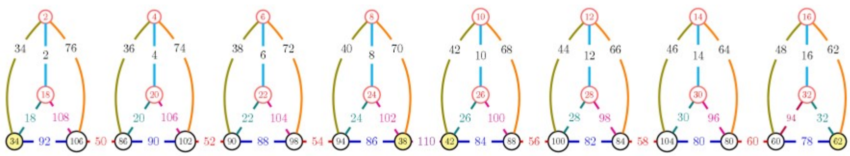

Figure 11
$\boldsymbol{A[S_8(K_4)]}$ **with an E2 −GL.**

2.7. *Eδ −GL of the wheel snake graph* $S_\ell(W_4)$

Definition 2.12: Wheel snake graphs, denoted by $S_\ell(W_4)$, are snake graphs with ℓ blocks where each block is W_4. This graph includes $V[S_\ell(W_4)] = \{\mu_\kappa, \kappa \in [1, \ell+1]\} \cup \{\lambda_\kappa, w_\kappa, x_\kappa, \kappa \in [1, \ell]\}$ and $E[S_\ell(W_4)] = \{\mu_\kappa\lambda_\kappa, \mu_\kappa w_\kappa, w_\kappa\mu_{\kappa+1}, \lambda_\kappa w_\kappa, \lambda_\kappa\mu_{\kappa+1}, \lambda_\kappa x_\kappa, \mu_{\kappa+1}x_\kappa, \mu_\kappa x_\kappa, \kappa \in [1, \ell]\}$. The wheel snake graph $S_\ell(W_4)$ has $\alpha = 4\ell + 1$ and $\beta = 8\ell$, see Fig. 12.

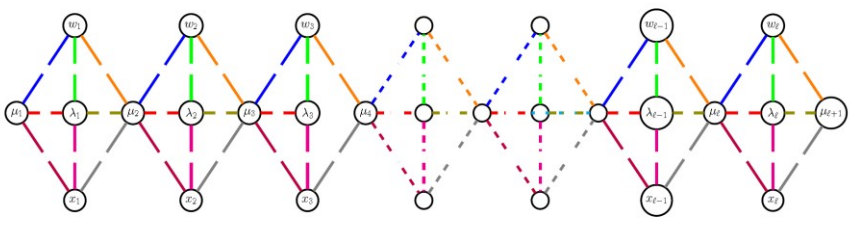

Figure 12
The wheel snake graph $\boldsymbol{S_\ell(W_4)}$ **with ordinary labeling**

Theorem 2.13: For an integer $\ell \geq 2$, the wheel snake graph $S_\ell(W_4)$ is an edge $\delta-$ graceful graph for $\delta \geq 1$.

Proof: The transformation $\Psi_7: E[A[S_\ell(W_4)]] \rightarrow \{\delta, 2\delta, 3\delta, \cdots, 8\ell\delta\}$ that we introduce is.

$\Psi_7(\lambda_\kappa \mu_\kappa) = \delta\kappa$ for $\kappa \in [1, \ell]$,
$\Psi_7(\lambda_\kappa \mu_{\kappa+1}) = \delta(\ell + \kappa)$ for $\kappa \in [1, \ell]$,
$\Psi_7(\lambda_\kappa x_\kappa) = \delta(2\ell + \kappa)$ for $\kappa \in [1, \ell]$,
$\Psi_7(w_\kappa \mu_{\kappa+1}) = \delta(3\ell + \kappa)$ for $\kappa \in [1, \ell]$,

$$\Psi_7(x_\kappa \mu_{\kappa+1}) = \begin{cases} \delta(2\ell + \kappa) & \text{if} \quad \kappa \in [1, \ell - 1]; \\ 8\ell\,\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

$\Psi_7(\mu_\kappa x_\kappa) = \delta(6\ell - \kappa)$ for $\kappa \in [1, \ell]$,
$\Psi_7(w_\kappa \lambda_\kappa) = \delta(7\ell - \kappa)$ for $\kappa \in [1, \ell]$,
$\Psi_7(w_\kappa \mu_\kappa) = \delta(8\ell - \kappa)$ for $\kappa \in [1, \ell]$.

The generated vertex labels Ψ_7^* are provided as follows, taking into consideration the labelling pattern mentioned above:

$$\Psi_7^*(w_\kappa) = [\Psi_7(w_\kappa \mu_\kappa) + \Psi_7(w_\kappa \lambda_\kappa) + \Psi_7(w_\kappa \mu_{\kappa+1})]\ mod\ [8\ell\delta] = (2\ell - \kappa)\delta, \kappa \in [1, \ell],$$

$$\Psi_7^*(x_\kappa) = [\Psi_7(x_\kappa \mu_\kappa) + \Psi_7(x_\kappa \lambda_\kappa) + \Psi_7(x_\kappa \mu_{\kappa+1})]\ mod\ [8\ell\delta] = (4\ell + \kappa)\delta, \kappa \in [1, \ell - 1],$$

$$\Psi_7^*(x_\ell) = [\Psi_7(x_\ell \mu_\ell) + \Psi_7(x_\ell \lambda_\ell) + \Psi_7(x_\ell \mu_{\ell+1})]\ mod\ [8\ell\delta] = 0,$$

$$\Psi_7^*(\lambda_\kappa) = [\Psi_7(\lambda_\kappa \mu_\kappa) + \Psi_7(\lambda_\kappa w_\kappa) + \Psi_7(\lambda_\kappa \mu_{\kappa+1}) + \Psi_7(\lambda_\kappa x_\kappa)]\ mod\ [8\ell\delta] = 2(\ell + \kappa)\delta, \kappa \in [1, \ell],$$

$$\Psi_7^*(\mu_\kappa) = [\Psi_7(\mu_\kappa \lambda_\kappa) + \Psi_7(\mu_\kappa w_\kappa) + \Psi_7(\mu_\kappa x_\kappa) + \Psi_7(\mu_\kappa w_{\kappa-1}) + \Psi_7(\mu_\kappa \lambda_{\kappa-1}) + \Psi_7(x_{\kappa-1} \mu_\kappa)]\ mod\ [8\ell\,\delta],$$

$$\Psi_7^*(\mu_\kappa) = \begin{cases} (6\ell - 1)\delta & \text{if} \quad \kappa = 1; \\ (22\ell - 3 + 2\kappa)\delta\ mod\ [8n\delta] = (6\ell - 3 + 2\kappa)\delta & \text{if} \quad \kappa \in [2, \ell]; \\ 6\ell\,\delta & \text{if} \quad \kappa = \ell + 1. \end{cases}$$

Since there are no identical vertex labels, the resulting mapping Ψ_7^* is injective, making the proof complete.

Example 7: In Fig. 13, we present S7(W4) with an edge 3− graceful labeling which is consistent with the labelling considering previously.

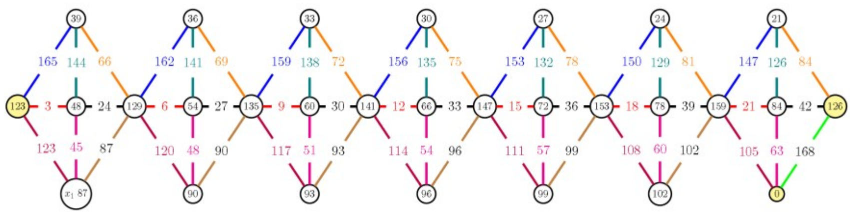

Figure 13
The wheel snake graph $S_7(W_4)$ with an E3 −GL.

2.8. *$E\delta$ −GL of a diagonal pentagonal snake graph* DPS_ℓ

Definition 2.14: A pentagonal snake graph, symbolled by PS_ℓ, is formed from a path $P_\ell = \{\mu_1, \mu_2, \cdots, \mu_{\ell+1}\}$ by replacing every edge by a cycle $C_5 = \{\mu_\kappa, \lambda_\kappa, \omega_\kappa, \tau_\kappa, \mu_{\kappa+1}, \kappa \in [1, \ell]\}$. A diagonal pentagonal snake graph, denoted by DPS_ℓ, [Fig.:15]Fig. Fig. 14, is obtained from PS_ℓ by adding the edges $\{\lambda_\kappa\mu_{\kappa+1}, \tau_\kappa\mu_\kappa, \kappa \in [1, \ell]\}$, this graph includes $V[DPS_\ell] = \{\mu_\kappa, \kappa \in [1, \ell+1]\} \cup \{\lambda_\kappa, \omega_\kappa, \tau_\kappa, \kappa \in [1, \ell]\}$ and $E[DPS_\ell] = \{\mu_\kappa\lambda_\kappa, \mu_\kappa\mu_{\kappa+1}, , \mu_\kappa\tau_\kappa, \lambda_\kappa\omega_{\kappa+1}, \lambda_\kappa\mu_{\kappa+1}, \omega_\kappa\tau_\kappa, \tau_\kappa\mu_{\kappa+1}\kappa \in [1, \ell]\}$. So, the graph DPS_ℓ has $\alpha = 4\ell + 1$ and $\beta = 7\ell$.

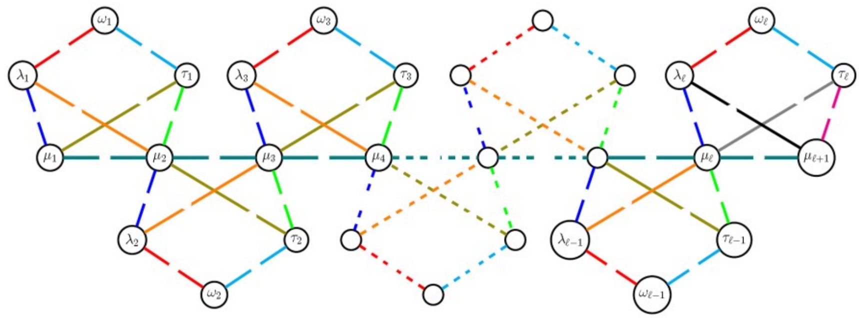

Figure 14
The diagonal pentagonal snake graph DPS_ℓ with ordinary labeling

Theorem 2.15: *For an integer $\ell \geq 2$, the diagonal pentagonal snake graph DPS_ℓ is an edge $\delta-$ graceful graph for $\delta \geq 1$.*

Proof:

Case 1: $\ell \equiv 0 \bmod 2$. Define the labeling $\Psi_8: E(DPS_\ell) \to \{\delta, 2\delta, 3\delta, \cdots, 7\ell\delta\}$ by

$\Psi_8(\mu_\kappa\mu_{\kappa+1}) = \delta\kappa$ for $\kappa \in [1, \ell]$,

$\Psi_8(\lambda_\kappa\mu_{\kappa+1}) = \delta(\ell + \kappa)$ for $\kappa \in [1, \ell]$,

$\Psi_8(\mu_\kappa\tau_\kappa) = \delta(2\ell + \kappa)$ for $\kappa \in [1, \ell]$,

$\Psi_8(\lambda_\kappa\mu_\kappa) = \delta(7\ell - \kappa)$ for $\kappa \in [1, \ell]$,

$$\Psi_8(\lambda_\kappa\omega_\kappa) = \begin{cases} \delta(3\ell + \kappa) & \text{if } \kappa \in [1, \ell - 1]; \\ 5\ell\,\delta & \text{if } \kappa = \ell, \end{cases}$$

$$\Psi_8(\tau_\kappa\omega_\kappa) = \begin{cases} \delta(4\ell + \kappa) & \text{if } \kappa \in [1, \ell - 1]; \\ 4\ell\,\delta & \text{if } \kappa = \ell, \end{cases}$$

$$\Psi_8(\tau_\kappa\mu_{\kappa+1}) = \begin{cases} \delta(6\ell - \kappa) & \text{if } \kappa \in [1, \ell - 1]; \\ 7\ell\,\delta & \text{if } \kappa = \ell, \end{cases}$$

The produced vertex labels Ψ_8^*, given the above labeling pattern are;

$\Psi_8^*(\omega_\kappa) = [\Psi_8(\lambda_\kappa\omega_\kappa) + \Psi_8(\tau_\kappa\omega_\kappa)] \bmod [7\ell\delta] = 2\kappa\delta, \kappa \in [1, \ell]$,

$\Psi_8^*(\mu_\kappa) = [\Psi_8(\mu_\kappa\tau_\kappa) + \Psi_8(\mu_\kappa\mu_{\kappa-1})] \bmod [7\ell\delta] = (2\ell + 2\kappa - 1)\delta, \kappa \in [1, \ell]$,

$\Psi_8^*(\mu_{\ell+1}) = (10\ell)\delta \bmod [7\ell\delta] = 3\ell\delta$,

$\Psi_8^*(\lambda_\kappa) = [\Psi_8(\lambda_\kappa\mu_\kappa) + \Psi_8(\lambda_\kappa\omega_\kappa) + \Psi_8(\lambda_\kappa\mu_{\kappa+1})]\ mod\ [7\ell\delta]$,

$\Psi_8^*(\lambda_\kappa) = \begin{cases} (11\ell+\kappa)\delta\ mod\ [7\ell\delta] = (4\ell+\kappa)\delta, & \text{if} \quad \kappa \in [1, \ell-1]; \\ 13\ell\delta\ mod\ [7\ell\delta] = 6\ell\ \delta & \text{if} \quad \kappa = \ell, \end{cases}$

$\Psi_8^*(\tau_\kappa) = [\Psi_8(\tau_\kappa\omega_\kappa) + \Psi_8(\mu_\kappa\tau_\kappa) + \Psi_8(\tau_\kappa\mu_{\kappa+1}]\ mod\ [7\ell\delta]$,

$\Psi_8^*(\tau_\kappa) = \begin{cases} (12\ell+\kappa)\delta\ mod\ [7\ell\delta] = (5\ell+\kappa)\delta & \text{if} \quad \kappa \in [1, \ell-1]; \\ 14\ell\delta\ mod\ [7\ell\delta] & \text{if} \quad \kappa = \ell, \end{cases}$

Case 2: $\ell \equiv 1\ mod\ 2$. The vertex labeling function Ψ_8 can be defined similarly to the previous case, but with the replacements listed below:

$\Psi_8(\lambda_\kappa\mu_{\kappa+1}) = \begin{cases} \delta(\ell+\kappa) & \text{if} \quad \kappa \in [1, \ell-1]; \\ 3\ell\ \delta & \text{if} \quad \kappa = \ell, \end{cases}$

$\Psi_8(\mu_\kappa\tau_\kappa) = \begin{cases} \delta(2\ell+\kappa) & \text{if} \quad \kappa \in [1, \ell-1]; \\ 2\ell\ \delta & \text{if} \quad \kappa = \ell, \end{cases}$

$\Psi_8(\lambda_\kappa\omega_\kappa) = \delta(3\ell+\kappa)$ for $\kappa \in [1, \ell]$,

$\Psi_8(\tau_\kappa\omega_\kappa) = \delta(4\ell+\kappa)$ for $\kappa \in [1, \ell]$.

The produced vertex labels Ψ_8^* in light of the labeling structure above are;

$\Psi_8^*(\mu_{\ell+1}) = 11\ell\delta\ mod\ [7\ell\delta] = 4\ell\delta$,

$\Psi_8^*(\mu_\ell) = (3\ell-1)\delta\ mod\ [7\ell\delta] = (3\ell-1)\delta$.

Because there are no identical vertex labels, the injective property of Ψ_8^* is proven. Thus, DPS_ℓ allows for edge $\delta-$ graceful labeling.

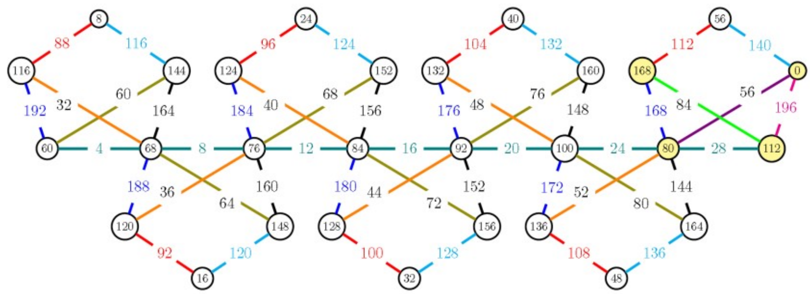

Figure 15
DPS_7 with an E4 −GL.

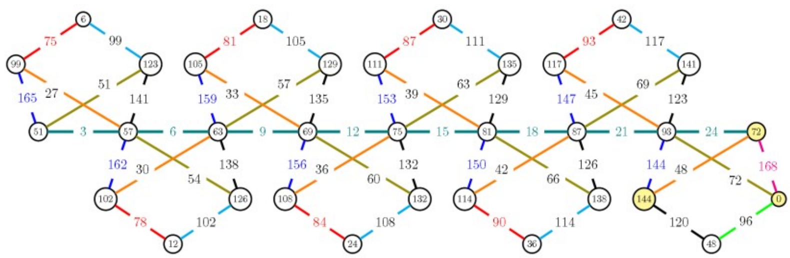

Figure 16
DPS_8 with an E3 −GL.

Example 8: The diagonal pentagonal snake graph DPS_7 with an edge 4− graceful labeling and the diagonal pentagonal snake graph DPS_8 with an edge 3− graceful labeling according to the labeling earlier are depicted in Fig. 15, and Fig. 16, respectively.

3. Eδ −GL of chain graphs

3.1 *Eδ −GL of the linear quadrilateral chain LQ_ℓ*

Definition 3.1: A linear quadrilateral chain on the plane, shown as LQ_ℓ, is constructed by linking ℓ quadrilaterals linearly by participating of a common edge. Every two neighbouring inner faces share precisely one edge, and that edge is parallel to the other. The graph LQ_ℓ with ℓ blocks includes $V[LQ_\ell] = \{\mu_\kappa, \lambda_\kappa, \kappa \in [1,\ell]\}$ and $E[LQ_\ell] = \{\mu_\kappa\mu_{\kappa+1},, \lambda_\kappa\lambda_{\kappa+1}, \kappa \in [1,\ell-1]\} \cup \{\mu_\kappa\lambda_\kappa, \kappa \in [1,\ell]\}$. So, LQ_ℓ has $\alpha = 2\ell$ and $\beta = 3\ell - 2$. A linear quadrilateral chain LQ_ℓ is isomorphic to the ladder graph $L_\ell = P_2 \times P_\ell$

Theorem 3.2: *For an integer $\ell \geq 3$, the linear quadrilateral chain LQ_ℓ is an edge $\delta-$ graceful graph.*

Proof: The labeling function $\Psi_9: E[LQ_\ell] \to \{\delta, 2\delta, 3\delta, \cdots, (3\ell-2)\delta\}$ that we suggest is.

$$\Psi_9(\lambda_\kappa\lambda_{\kappa+1}) = \begin{cases} \frac{\delta}{2}(\ell+\kappa) & \text{if} \quad \kappa \in [1,\ell-2], \ \ \kappa \equiv 1 \, mod \, 2; \\ \frac{\delta}{2}(5\ell-\kappa-3) & \text{if} \quad \kappa \in [2,\ell-1], \ \ \kappa \equiv 0 \, mod \, 2; \end{cases}$$

$$\Psi_9(\mu_\kappa\mu_{\kappa+1}) = \begin{cases} \delta(3\ell-2) & \text{if} \quad \kappa = 1; \\ \frac{\delta}{2}(\kappa+1) & \text{if} \quad \kappa \in [3,\ell-2], \ \ \kappa \equiv 1 \, mod \, 2; \\ \frac{\delta}{2}(2\ell+\kappa-2) & \text{if} \quad \kappa \in [2,\ell-1], \ \ \kappa \equiv 1 \, mod \, 2; \end{cases}$$

$$\Psi_9(\lambda_\kappa\mu_\kappa) = \begin{cases} \delta\kappa & \text{if} \quad \kappa = 1; \\ \frac{\delta}{2}(4\ell-\kappa-1) & \text{if} \quad \kappa \in [3,\ell], \ \ \kappa \equiv 1 \, mod \, 2; \\ \frac{\delta}{2}(6\ell-\kappa-4) & \text{if} \quad \kappa \in [3,\ell-1], \ \ \kappa \equiv 0 \, mod \, 2, \end{cases}$$

Following the labeling pattern previously indicated, the produced map Ψ_9^* on the vertices are;

$$\Psi_9^*(\lambda_\kappa) = [\Psi_9(\lambda_\kappa\lambda_{\kappa+1}) + \Psi_9(\lambda_\kappa\mu_\kappa) + \Psi_9(\lambda_\kappa\lambda_{\kappa-1})]mod[(3\ell-2)\delta],$$

$$\Psi_9^*(\lambda_\kappa) = \begin{cases} \left(\frac{\ell+3}{2}\right)\delta \; mod\; [(3\ell-2)\delta] = \frac{\delta}{2}(\ell+3) & \text{if} \quad \kappa = 1; \\ \left(6\ell - \frac{\kappa}{2} - 4\right)\delta \; mod\; [(3\ell-2)\delta] = & \text{if} \quad \kappa \in [2, \ell-1], \\ \quad \frac{\delta}{2}(6\ell-\kappa-4) & \qquad \kappa \equiv 0 \; mod\; 2; \\ \left(5\ell - \frac{\kappa}{2} - \frac{3}{2}\right)\delta \; mod\; [(3\ell-2)\delta] = & \text{if} \quad \kappa \in [3, \ell-2], \\ \quad \frac{\delta}{2}(4\ell-\kappa+1) & \qquad \kappa \equiv 1 \; mod\; 2; \\ \left(\frac{7\ell-3}{2}\right)\delta \; mod\; [(3\ell-2)\delta] = \frac{\delta}{2}(\ell+1) & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_9^*(\mu_\kappa) = [\Psi_9(\mu_\kappa\mu_{\kappa+1}) + \Psi_9(\mu_\kappa\lambda_\kappa) + \Psi_9(\mu_\kappa\mu_{\kappa-1})] \; mod\; [(3\ell-2)\delta],$$

$$\Psi_9^*(\mu_\kappa) = \begin{cases} (3\ell-1)\delta \; mod\; [(3\ell-2)\delta] = \delta & \text{if} \quad \kappa = 1; \\ (7\ell-5)\delta \; mod\; [(3\ell-2)\delta] = (\ell-1)\delta & \text{if} \quad \kappa = 2; \\ \left(\frac{\kappa+1}{2}\right)\delta \; mod\; [(3\ell-2)\delta] = \frac{\delta}{2}[\kappa+1], & \text{if} \quad \kappa \in [3, \ell-2], \\ & \qquad \kappa \equiv 1 \; mod\; 2; \\ \left(4\ell + \frac{\kappa}{2} - 3\right)\delta \; mod\; [(3\ell-2)\delta] = & \text{if} \quad \kappa \in [4, \ell-1], \\ \frac{\delta}{2}[2\ell+\kappa-2] & \qquad \kappa \equiv 0 \; mod\; 2; \\ (3\ell-2)\delta \; mod\; [(3\ell-2)\delta] = 0 & \text{if} \quad \kappa = \ell, \end{cases}$$

It is confirmed that Ψ_9^* has an injective property. Thus, an edge $\delta-$graceful labeling is admitted by LQ_ℓ.

Example 9: In Fig. 17, we present linear quadrilateral chain $LQ_{11} \cong L_{11}$ with an edge 5− graceful labeling.

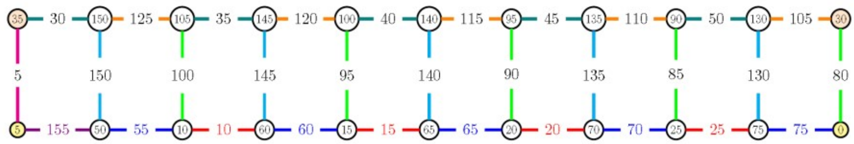

Figure 17
The linear quadrilateral chain LQ_{11} with an E5 − GL.

3.2 *E$\delta-$ GL of a diagonal linear hexagonal chain*

Definition 3.3: A diagonal linear hexagonal chain, indicated by DLH_ℓ as depicted in Fig. 18, is created by linearly combining ℓ hexagons by sharing a common edge. Any two adjacent inside faces have precisely one edge in common and these common edges are parallel to each other. The graph DLH_ℓ with ℓ blocks consists of $V[DLH_\ell] = \{\mu_\kappa, \lambda_\kappa, \kappa \in [1, \ell+1]\} \cup \{w_\kappa, \tau_\kappa, \kappa \in [1, \ell]\}$ and $E[DLH_\ell] = \{\mu_\kappa w_\kappa, w_\kappa\mu_{\kappa+1},, \lambda_\kappa\tau_\kappa, \tau_\kappa\lambda_{\kappa+1}, \mu_\kappa\lambda_\kappa, \kappa \in [1, \ell]\} \cup \{\mu_\kappa\lambda_\kappa, \kappa \in [1, \ell+1]\}$. So, $DLH_\ell, \ell \geq 3$ has $\alpha = 4\ell + 2$ and $\beta = 6\ell + 1$.

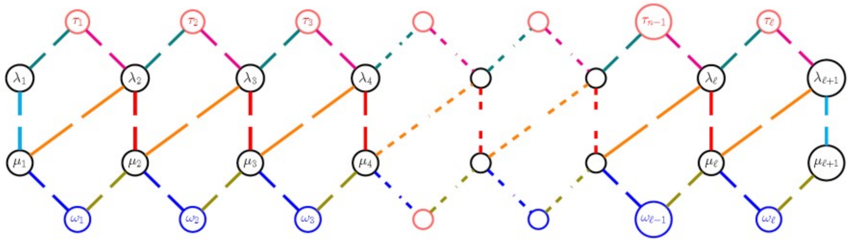

Figure 18
The diagonal linear hexagonal chain DLH_ℓ with ordinary labeling

Theorem 3.4: *For an integer $\ell \geq 2$, the diagonal linear hexagonal chain DLH_ℓ is an edge $\delta-$ graceful graph for $\delta \geq 1$.*

Proof:

Case 1: $\ell \equiv 0 \bmod 2$. The labeling $\Psi_{10}: E[DLH_\ell] \to \{\delta, 2\delta, \cdots, (6\ell+1)\delta\}$ that we propose is.

$$\Psi_{10}(\tau_\kappa \lambda_{\kappa+1}) = \delta\kappa \text{ for } \kappa \in [1,\ell],$$

$$\Psi_{10}(\tau_\kappa \lambda_\kappa) = \delta(\ell+\kappa) \text{ for } \kappa \in [1,\ell],$$

$$\Psi_{10}(\mu_\kappa \omega_\kappa) = \begin{cases} \delta(2\ell+\kappa) & \text{if } \kappa \in [1,\ell-1]; \\ \delta(5\ell+1) & \text{if } \kappa = \ell, \end{cases}$$

$$\Psi_{10}(\omega_\kappa \mu_{\kappa+1}) = \delta(3\ell+\kappa+1) \text{ for } \kappa \in [1,\ell],$$

$$\Psi_{10}(\mu_\kappa \lambda_{\kappa+1}) = \begin{cases} \delta(4\ell+1+\kappa) & \text{if } \kappa \in [1,\ell-1]; \\ \delta(6\ell+1) & \text{if } \kappa = \ell, \end{cases}$$

$$\Psi_{10}(\mu_\kappa \lambda_\kappa) = \begin{cases} \delta(3\ell+1) & \text{if } \kappa = 1; \\ \delta(6\ell-\kappa+2) & \text{if } \kappa \in [2,\ell]; \\ \delta(3\ell) & \text{if } \kappa = \ell+1. \end{cases}$$

Considering the labeling pattern stated previously, the produced map Ψ_{10}^* on the set of vertices are;

$$\Psi_{10}^*(\tau_\kappa) = [\Psi_{10}(\tau_\kappa\lambda_\kappa) + \Psi_{10}(\tau_\kappa\lambda_{\kappa+1})] \, mod \, [(6\ell+1)\delta] = (\ell+2\kappa)\delta, \kappa \in [1,\ell],$$

$$\Psi_{10}^*(\lambda_\kappa) = [\Psi_{10}(\lambda_\kappa\tau_\kappa) + \Psi_{10}(\lambda_\kappa\tau_{\kappa-1}) + \Psi_{10}(\lambda_\kappa\mu_{\kappa-1}) + \Psi_{10}(\lambda_\kappa\mu_\kappa)] \, mod \, [(6\ell+1)\delta],$$

$$\Psi_{10}^*(\lambda_\kappa) = \begin{cases} (4\ell+2)\delta & \text{if } \kappa = 1; \\ (5\ell+2\kappa)\delta mod[(6\ell+1)\delta] & \text{if } \kappa \in [2,\ell]; \\ 4\ell\,\delta & \text{if } \kappa = \ell+1. \end{cases}$$

$$\Psi_{10}^*(\mu_\kappa) = [\Psi_{10}(\mu_\kappa\lambda_\kappa) + \Psi_{10}(\mu_\kappa\lambda_{\kappa+1}) + \Psi_{10}(\mu_\kappa w_{\kappa-1}) + \Psi_{10}(\omega_\kappa\mu_\kappa)] \, mod \, [(6\ell+1)\delta],$$

$$\Psi_{10}^*(\mu_\kappa) = \begin{cases} (15\ell+3+2\kappa)\delta mod[(6\ell+1)\delta] = & \text{if } \kappa \in [1,\ell-1]; \\ \quad (3\ell+2\kappa+1)\delta & \\ (2\ell+1)\delta & \text{if } \kappa = \ell; \\ \ell\,\delta & \text{if } \kappa = \ell+1. \end{cases}$$

$$\Psi_{10}^*(\omega_\kappa) = \begin{cases} (5\ell+2\kappa+1)\delta \;\; mod \; [(6\ell+1)\delta] & \text{if } \kappa \in [1,\ell-1]; \\ (3\ell+1) & \text{if } \kappa = \ell. \end{cases}$$

Case 2: $\ell \equiv 1 \, mod \, 2$. The labeling Ψ_{10} can be defined with the following substitutions in place of the ones in the preceding case:

$\Psi_{10}(\mu_\kappa \omega_\kappa) = \delta(2\ell + \kappa)$ for $\kappa \in [1, \ell]$,

$$\Psi_{10}(\omega_\kappa \mu_{\kappa+1}) = \begin{cases} \delta(3\ell + 1 + \kappa) & \text{if} \quad \kappa \in [1, \ell - 1]; \\ \delta(5\ell + 1) & \text{if} \quad \kappa = \ell. \end{cases}$$

$\Psi_{10}(\mu_\ell \lambda_{\ell+1}) = \delta(3\ell + 1)$,

$$\Psi_{10}(\mu_\kappa \lambda_\kappa) = \begin{cases} \delta(4\ell + 1) & \text{if} \quad \kappa = 1; \\ \delta(6\ell + 1) & \text{if} \quad \kappa = \ell + 1. \end{cases}$$

then the derived vertex labels are;

$\Psi_{10}^*(\omega_\ell) = 2\ell\delta$,

$$\Psi_{10}^*(\lambda_\kappa) = \begin{cases} (5\ell + 2)\delta & \text{if} \quad \kappa = 1; \\ (4\ell + 1)\delta & \text{if} \quad \kappa = \ell + 1. \end{cases}$$

$$\Psi_{10}^*(\mu_\kappa) = \begin{cases} (4\ell + 3)\delta & \text{if} \quad \kappa = 1; \\ (3\ell + 2\kappa + 1)\delta & \text{if} \quad \kappa \in [2, \ell - 1]; \\ (3\ell + 1)\delta & \text{if} \quad \kappa = \ell; \\ (5\ell + 1)\delta & \text{if} \quad \kappa = \ell + 1. \end{cases}$$

Since there aren't any duplicated vertex labels, then DLH_ℓ permits an edge $\delta -$ graceful labeling.

Example 10: In Fig. 19, we set up the diagonal linear hexagonal chain DLH_7 with an edge 5− graceful labeling. Also, In Fig. 20, we set up DLH8 with an edge 4− graceful labeling.

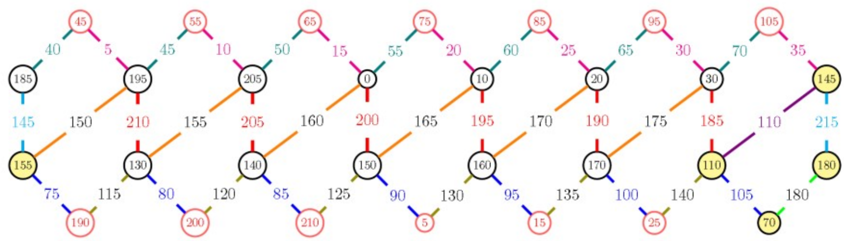

Figure 19
The diagonal linear hexagonal chain DLH_7 with an E5 − GL.

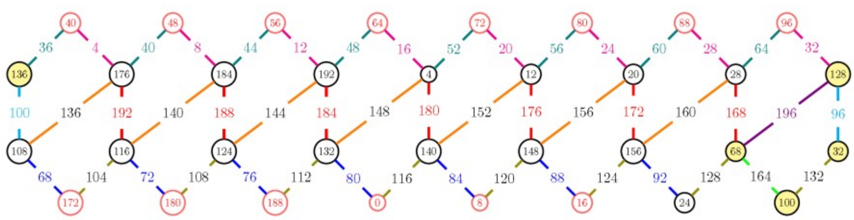

Figure 20
The diagonal linear hexagonal chain DLH_8 with an E4 − GL.

3.3 *$E\delta$ −GL of a linear chain* $[\ell K_3|(\ell-1)K_4]$

Definition 3.5: A linear chain , denoted by $[\ell K_3|(\ell-1)K_4]$ as depicted in Fig. 21, consists of ℓ copies of complete graph K_3 and $(\ell-1)$ copies of complete graph K_4 linearly through collaboration of an identical edge. There is only one edge that all neighbouring internal faces share, and that edge is parallel to the other. A linear $[\ell K_3|(\ell-1)K_4]$ chain consists of $V[\ell K_3|(\ell-1)K_4]=\{\mu_\kappa,\lambda_\kappa,\omega_\kappa,\kappa\in[1,\ell]\}$ and $E[\ell K_3|(\ell-1)K_4]=\{\mu_\kappa\lambda_\kappa,\omega_\kappa\mu_\kappa,,\lambda_\kappa\omega_\kappa,\kappa\in[1,\ell]\}\cup\{\omega_\kappa\lambda_{\kappa+1},\mu_\kappa\mu_{\kappa+1},\mu_\kappa\lambda_{\kappa+1},\omega_\kappa\mu_{\kappa+1}\kappa\in[1,\ell-1]\}$. So, the graph $[\ell K_3|(\ell-1)K_4],\ell\geq 3$ includes $\alpha=3\ell$ and $\beta=7\ell-4$.

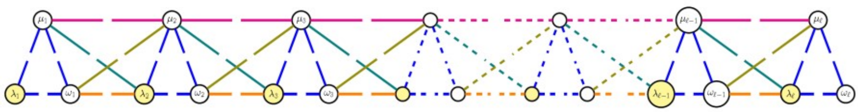

Figure 21
A linear chain $[\boldsymbol{\ell}\boldsymbol{K}_3|(\boldsymbol{\ell}-\mathbf{1})\boldsymbol{K}_4]$ with standard labeling.

Theorem 3.6: *For an integer $\ell\geq 3$, the linear chain $[\ell K_3|(\ell-1)K_4]$ is an edge $\delta-$ graceful graph for $\delta\geq 1$.*

Proof:

Case 1: $\ell\equiv 1\, mod\, 2$. The labeling function $\Psi_{11}: E[\ell K_3|(\ell-1)K_4]\to\{\delta,2\delta,\cdots,(7\ell-4)\delta\}$ that we propose is.

$$\Psi_{11}(\mu_\kappa\omega_\kappa)=\begin{cases}\delta\kappa & \text{if} \quad \kappa\in[1,\ell-1];\\ \delta(5\ell-4) & \text{if} \quad \kappa=\ell,\end{cases}$$

$$\Psi_{11}(\mu_\kappa\lambda_\kappa)=\begin{cases}\delta(7\ell-4) & \text{if} \quad \kappa=1;\\ \delta(\ell+\kappa-2) & \text{if} \quad \kappa\in[2,\ell],\end{cases}$$

$$\Psi_{11}(\lambda_\kappa\omega_\kappa)=\delta(2\ell+2\kappa-3) \ \text{ for } \ \kappa\in[1,\ell],$$

$$\Psi_{11}(\omega_\kappa\lambda_{\kappa+1})=\delta(2\ell+2\kappa-2) \ \text{ for } \ \kappa\in[1,\ell-1],$$

$$\Psi_{11}(\mu_\kappa\mu_{\kappa+1})=\begin{cases}\delta(4\ell+\kappa-3) & \text{if} \quad \kappa\in[1,\ell-2];\\ \delta(5\ell-3) & \text{if} \quad \kappa=\ell-1,\end{cases}$$

$$\Psi_{11}(\mu_\kappa\lambda_{\kappa+1})=\delta(6\ell-\kappa-3) \ \text{ for } \ \kappa\in[1,\ell-1],$$

$$\Psi_{11}(\omega_\kappa\mu_{\kappa+1})=\delta(7\ell-\kappa-4) \ \text{ for } \ \kappa\in[1,\ell-1].$$

Considering the labeling pattern mentioned above, the produced map Ψ_{11}^* on the vertices are;

$$\Psi_{11}^*(\mu_\kappa)=[\Psi_{11}(\mu_\kappa\omega_{\kappa-1})+\Psi_{11}(\mu_\kappa\lambda_\kappa)+\Psi_{11}(\mu_\kappa\omega_\kappa)+\Psi_{11}(\mu_\kappa\lambda_{\kappa+1})+\Psi_{11}(\mu_\kappa\mu_{\kappa+1})+\Psi_{11}(\mu_\kappa\mu_{\kappa-1})]\, mod\, [(7\ell-4)\delta],$$

$$\Psi_{11}^*(\mu_\kappa)=\begin{cases}(3\ell-1)\delta & \text{if} \quad \kappa=1;\\ (\ell+2\kappa-3)\delta & \text{if} \quad \kappa\in[2,\ell-2];\\ (3\ell-4)\delta & \text{if} \quad \kappa=\ell-1;\\ (4\ell-4)\delta & \text{if} \quad \kappa=\ell,\end{cases}$$

$$\Psi_{11}^*(\lambda_\kappa)=[\Psi_{11}(\lambda_\kappa\omega_\kappa)+\Psi_{11}(\lambda_\kappa\mu_\kappa)+\Psi_{11}(\lambda_\kappa\omega_{\kappa-1})+\Psi_{11}(\lambda_\kappa\mu_{\kappa-1})]\, mod\, [(7\ell-4)\delta],$$

$$\Psi_{11}^{*}(\lambda_\kappa) = \begin{cases} (2\ell-1)\delta & \text{if} \quad \kappa = 1; \\ (4\ell+4\kappa-7)\delta \; mod \; [(7\ell-4)\delta] & \text{if} \quad \kappa \in [2,\ell], \end{cases}$$

$$\Psi_{11}^{*}(\omega_\kappa) = [\Psi_{11}(\omega_\kappa\lambda_\kappa) + \Psi_{11}(\omega_\kappa\mu_\kappa) + \Psi_{11}(\omega_\kappa\lambda_{\kappa+1}) + \Psi_{11}(\omega_\kappa\mu_{\kappa+1})] \; mod \; [(7\ell-4)\delta],$$

$$\Psi_{11}^{*}(\omega_\kappa) = \begin{cases} (4\ell+4\kappa-5)\delta \; mod \; [(7\ell-4)\delta] & \text{if} \quad \kappa \in [1,\ell-1]; \\ (2\ell-3)\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

Case 2: $\ell \equiv 0 \; mod \; 2$. After making the following substitution, The labeling Ψ_{11} can be defined in a manner analogues to the preceding case

$$\Psi_{11}(\mu_\kappa\omega_\kappa) = \begin{cases} \delta\kappa & \text{if} \quad \kappa \in [1,\ell-1]; \\ \delta(5\ell-3) & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_{11}(\lambda_\kappa\omega_\kappa) = \begin{cases} \delta(2\ell) & \text{if} \quad \kappa = 1; \\ \delta(2\ell+2\kappa-3) & \text{if} \quad \kappa \in [2,\ell], \end{cases}$$

$$\Psi_{11}(\omega_\kappa\lambda_{\kappa+1}) = \begin{cases} \delta(2\ell-1) & \text{if} \quad \kappa = 1; \\ \delta(2\ell+2\kappa-2) & \text{if} \quad \kappa \in [2,\ell-1], \end{cases}$$

$$\Psi_{11}(\mu_\kappa\mu_{\kappa+1}) = \delta(4\ell+\kappa-3) \text{ for } \kappa \in [1,\ell-1].$$

The produced map Ψ_{11}^{*} on the vertices are as follows:

$$\Psi_{11}^{*}(\mu_\kappa) = [\Psi_{11}(\mu_\kappa\omega_{\kappa-1}) + \Psi_{11}(\mu_\kappa\lambda_\kappa) + \Psi_{11}(\mu_\kappa\omega_\kappa) + \Psi_{11}(\mu_\kappa\lambda_{\kappa+1}) + \Psi_{11}(\mu_\kappa\mu_{\kappa+1}) + \Psi_{11}(\mu_\kappa\mu_{\kappa-1})] \; mod \; [(7\ell-4)\delta],$$

$$\Psi_{11}^{*}(\mu_\kappa) = \begin{cases} (3\ell-1)\delta & \text{if} \quad \kappa = 1; \\ (\ell+2\kappa-3)\delta & \text{if} \quad \kappa \in [2,\ell-1]; \\ (4\ell-4)\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

$$\Psi_{11}^{*}(\lambda_\kappa) = [\Psi_{11}(\lambda_\kappa\omega_\kappa) + \Psi_{11}(\lambda_\kappa\mu_\kappa) + \Psi_{11}(\lambda_\kappa\omega_{\kappa-1}) + \Psi_{11}(\lambda_\kappa\mu_{\kappa-1})] \; mod \; [(7\ell-4)\delta],$$

$$\Psi_{11}^{*}(\lambda_\kappa) = \begin{cases} (2\ell)\delta & \text{if} \quad \kappa = 1; \\ (4\ell)\delta & \text{if} \quad \kappa = 2; \\ (4\ell+4\kappa-7)\delta \; mod \; [(7\ell-4)\delta] & \text{if} \quad \kappa \in [3,\ell], \end{cases}$$

$$\Psi_{11}^{*}(\omega_\kappa) = [\Psi_{11}(\omega_\kappa\lambda_\kappa) + \Psi_{11}(\omega_\kappa\mu_\kappa) + \Psi_{11}(\omega_\kappa\lambda_{\kappa+1}) + \Psi_{11}(\omega_\kappa\mu_{\kappa+1})] \; mod \; [(7\ell-4)\delta],$$

$$\Psi_{11}^{*}(\omega_\kappa) = \begin{cases} (4\ell+4\kappa-5)\delta \; mod \; [(7\ell-4)\delta] & \text{if} \quad \kappa \in [1,\ell-1]; \\ (2\ell-2)\delta & \text{if} \quad \kappa = \ell, \end{cases}$$

Because there are no repeated vertex labels. Consequently, $[\ell K_3|(\ell-1)K_4]$ enables an edge $\delta-$ graceful labeling.

Example 11: A linear $[7K_3 \,|\, 6K_4]$ chain with its explicit edge 3− graceful labeling and a linear $[6K_3 \,|\, 5K_4]$ chain with its explicit edge 4− graceful labeling, according to the labeling earlier, are illustrated in Fig. 22 and is illustrated in Fig. 23, respectively.

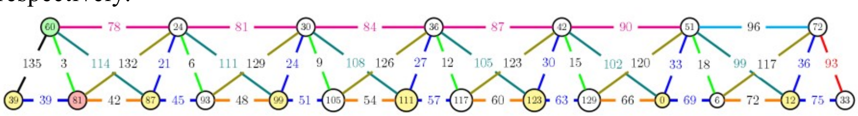

Figure 22
$[7K_3|6K_4]$ chain with an E3 − GL.

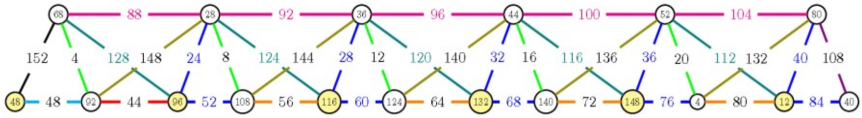

Figure 23
$[6K_3|5K_4]$ chain with an E4 − GL.

3.4 *$E\delta$ −GL of a linear chain* $[\ell K_3|\ell K_4]$

Theorem 3.7: *For an integer $\ell \geq 3$, the linear chain $[\ell K_3|\ell K_4]$ is an edge $\delta-$ graceful graph for $\delta \geq 1$.*

Proof: A linear chain $[\ell K_3|\ell K_4]$ consists of $V[\ell K_3|\ell K_4] = \{\omega_\kappa, \kappa \in [1,\ell]\} \cup \{\mu_\kappa, \lambda_\kappa, \kappa \in [1, \ell+1]\}$ and $E[\ell K_3|\ell K_4] = \{\omega_\kappa\mu_\kappa, \lambda_\kappa\omega_\kappa, \omega_\kappa\lambda_{\kappa+1}, \mu_\kappa\mu_{\kappa+1}, \mu_\kappa \lambda_{\kappa+1}, \omega_\kappa\mu_{\kappa+1}, \kappa \in [1,\ell]\} \cup \{\mu_\kappa\lambda_\kappa, \kappa \in [1,\ell+1]\}$. So, the graph $[\ell K_3|\ell K_4], \ell \geq 3$ includes $\alpha = 3\ell+2$ and $\beta = 7\ell+1$.

Case 1: $\ell \equiv 1 \; mod \; 2$. The labeling $\Psi_{12}: E[\ell K_3|\ell K_4] \to \{\delta, 2\delta, \cdots, (7\ell+1)\delta\}$ that we propose is.

$$\Psi_{12}(\mu_\kappa\omega_\kappa) = \delta\kappa \text{ for } \kappa \in [1,\ell],$$

$$\Psi_{12}(\mu_\kappa\lambda_\kappa) = \begin{cases} \delta(7\ell+1) & \text{if } \kappa = 1; \\ \delta(\ell+\kappa-1) & \text{if } \kappa \in [2,\ell-1], \end{cases}$$

$$\Psi_{12}(\lambda_\kappa\omega_\kappa) = \begin{cases} \delta(2\ell+2) & \text{if } \kappa = 1; \\ \delta(2\ell+2\kappa-1) & \text{if } \kappa \in [2,\ell], \end{cases}$$

$$\Psi_{12}(\omega_\kappa\lambda_{\kappa+1}) = \begin{cases} \delta(2\ell+1) & \text{if } \kappa = 1; \\ \delta(2\ell+2\kappa) & \text{if } \kappa \in [2,\ell], \end{cases}$$

$$\Psi_{12}(\mu_\kappa\mu_{\kappa+1}) = \delta(4\ell+\kappa) \text{ for } \kappa \in [1,\ell],$$

$$\Psi_{12}(\mu_\kappa\lambda_{\kappa+1}) = \delta(6\ell-\kappa+1) \text{ for } \kappa \in [1,\ell],$$

$$\Psi_{12}(\omega_\kappa\mu_{\kappa+1}) = \delta(7\ell-\kappa+1) \text{ for } \kappa \in [1,\ell].$$

Following the labeling pattern described earlier, the produced map Ψ^*_{12} on the vertices are;

$$\Psi^*_{12}(\mu_\kappa) = [\Psi_{12}(\mu_\kappa\omega_{\kappa-1}) + \Psi_{12}(\mu_\kappa\lambda_\kappa) + \Psi_{12}(\mu_\kappa\omega_\kappa) + \Psi_{12}(\mu_\kappa\lambda_{\kappa+1}) + \Psi_{12}(\mu_\kappa\mu_{\kappa+1}) + \Psi_{12}(\mu_\kappa\mu_{\kappa-1})] \; mod \; [(7\ell+1)\delta],$$

$$\Psi^*_{12}(\mu_\kappa) = \begin{cases} (3\ell+1)\delta & \text{if } \kappa = 1; \\ (\ell+2\kappa-2)\delta & \text{if } \kappa \in [2,\ell]; \\ 6\ell\,\delta & \text{if } \kappa = \ell+1, \end{cases}$$

$$\Psi^*_{12}(\lambda_\kappa) = [\Psi_{12}(\lambda_\kappa\omega_\kappa) + \Psi_{12}(\lambda_\kappa\mu_\kappa) + \Psi_{12}(\lambda_\kappa\omega_{\kappa-1}) + \Psi_{12}(\lambda_\kappa\mu_{\kappa-1})] \; mod \; [(7\ell+1)\delta],$$

$$\Psi^*_{12}(\lambda_\kappa) = \begin{cases} (2\ell+2)\delta & \text{if } \kappa = 1; \\ (4\ell+4)\delta & \text{if } \kappa = 2; \\ (4\ell+4\kappa-3)\delta \; mod \; [(7\ell+1)\delta] & \text{if } \kappa \in [3,\ell]; \\ 4\ell\,\delta & \text{if } \kappa = \ell+1, \end{cases}$$

$$\Psi^*_{12}(\omega_\kappa) = [\Psi_{12}(\omega_\kappa\lambda_\kappa) + \Psi_{12}(\omega_\kappa\mu_\kappa) + \Psi_{12}(\omega_\kappa\lambda_{\kappa+1}) + \Psi_{12}(\omega_\kappa\mu_{\kappa+1})] \; mod \; [(7\ell+1)\delta],$$

$\Psi_{12}^{*}(\omega_\kappa) = (4\ell + 4\kappa - 1)\delta \; mod \; [(7\ell + 1)\delta]$ for $\kappa \in [1, \ell]$.

Case 2: If $\ell \equiv 0 \; mod \; 2$. After making the following substitutions, the labeling function Ψ_{12} can be defined in a manner identical to the preceding case:

$\Psi_{12}(\lambda_\kappa \omega_\kappa) = \delta(2\ell + 2\kappa - 1)$ for $\kappa \in [1, \ell]$,

$\Psi_{12}(\omega_\kappa \lambda_{\kappa+1}) = \delta(2\ell + 2\kappa)$ for $\kappa \in [1, \ell]$.

The produced map Ψ_{12}^{*} on the vertices are;

$$\Psi_{12}^{*}(\lambda_\kappa) = [\Psi_{12}(\lambda_\kappa \omega_\kappa) + \Psi_{12}(\lambda_\kappa \mu_\kappa) + \Psi_{12}(\lambda_\kappa \omega_{\kappa-1}) + \Psi_{12}(\lambda_\kappa \mu_{\kappa-1})] \; mod \; [(7\ell + 1)\delta],$$

$$\Psi_{12}^{*}(\lambda_\kappa) = \begin{cases} (2\ell + 1)\delta & \text{if} \quad \kappa = 1; \\ (4\ell + 4\kappa - 3)\delta \; mod \; [(7\ell + 1)\delta] & \text{if} \quad \kappa \in [2, \ell]; \\ 4\ell \, \delta & \text{if} \quad \kappa = \ell + 1, \end{cases}$$

Upon considering case (1) and case(2), and due to the absence of duplicate vertex labels, it can be determine that the chain $[\ell K_3 | \ell K_4]$ has the characteristic of being an edge $\delta -$ graceful graph.

Example 12: A linear $[6K_3 \,|\, 6K_4]$ chain with its explicit edge 3− graceful labeling, according to the labeling function defined above, is illustrated in Fig. 24.

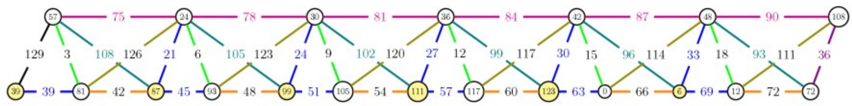

Figure 24
$[6K_3|6K_4]$ chain with an E3 − GL.

4. *Application of $E\delta - GL$*

An electrical network is represented by a graph $\Gamma = (V(\Gamma), E(\Gamma))$ where $|V(\Gamma)| = \alpha$ and $|E(\Gamma)| = \beta$. The set of vertices $V(\Gamma) = \lambda_\kappa$ indicates the nodes where the wires are joined, and the set of edges $E(\Gamma) = e_\kappa$ represents the wires where the electric current is travelling. We examine weighted connected simple graphs, which are weighted graphs that are connected but undirected and do not have multi-edges or self-loops. The labeling number that indicates the current flowing through this wire is the weight of each edge.

Let us consider a collection of charges $\{\delta, 2\delta, 3\delta, \cdots, \beta\delta\}$. The challenge is to develop various designs for electric networks that enable the use of a single charge from the set $\{\delta, 2\delta, 3\delta, \cdots, \beta\delta\}$ to each wire in the network so that the amount of accumulation charges at each nodes (connection points) must vary. We provide eight designs for electric networks associated with snake graphs and four designs for electric networks associated with chain graphs. Every single one of these designs matches our required features and specifications.

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