Journal of Discrete Mathematical Sciences & Cryptography
ISSN 0972-0529 (Print), ISSN 2169-0065 (Online)
DOI : 10.47974/JDMSC-2297

Duality for distributive hyperlattice negatively ordered hypermonoids

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Abstract

This research presents a method for linking bounded distributive hyperlattice negatively ordered commutative monoids with a Priestley space negatively ordered commutative semihypergroups. More precisely, it constructs a categorical duality between the dual of bounded distributive hyperlattice negatively ordered commutative hypermonoids category and the Priestley spaces enriched with negatively ordered commutative semihypergroup structures category.

Subject Classification (2020): *06B05, 06B15.*

Keywords: *Ordred semigroup, Hyperlattice, Semihypergroup, Hypermonoid, Homomorphism of hyperlattice, Priestley space, Homomorphism of Priestley space, Priestley duality.*

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1. Introduction

The foundational work in hyperstructure theory began with Frédéric Martys 1934 paper [31]. In this seminal contribution, Marty extended traditional group theory by introducing a broader algebraic framework.

Marty sought to extend the algebraic structure of a group to a more general framework, which eventually led to the development of hypergroups.

With Marty's early death, the mathematical community suffered a significant loss since his published contributions to hypergroup theory were limited to just two basic works. Through careful study, these first publications established the basic traits of what we now understand as hypergroups sometimes known as multigroups. The relevance of this conceptual framework has expanded significantly and finds practical applications in several mathematical fields, including modern algebraic structures, combinatorial theory, and topological spaces.

In mathematics, we can think of an operation as a rule that takes one or more elements from a set and produces exactly one output element from another set. When we extend this idea to allow for multiple possible outputs (represented as a set of values rather than just one), we get what is called a hyperoperation. This generalized concept maintains the basic idea of transforming inputs but with potentially many results instead of a single one.

Algebraic hyperstructures are mathematical systems defined by a set equipped with one or more hyperoperations. This broad category includes various specialized structures such as hypergroups, hyperrings, and hyperlattices, all falling under the umbrella of hyperalgebraic theory.

The study of hyperstructures has developed through various mathematical approaches and found applications in different areas, including computer science and formal logic. Within logical systems, multialgebras have served as semantic foundations, some applications can also be found in [13], [14], [16] and [22]. A notable recent development involves Avron and colleagues' work on matrix semantics using multialgebras, referred to as non-deterministic matrices (Nmatrices). These structures have proven particularly valuable for analyzing certain paraconsistent logical systems, especially those categorized as Logics of Formal Inconsistency **LFIs** (see [9]), extended discussion of this topic is presented in [21]. Modern mathematicians have established this as an independent algebraic discipline, actively employed in cross-disciplinary studies by researchers worldwide, [1], [2], [29], [30], [36], [37], [38], Kehayopulu [25], [26], [27], [28], and books e.g. [15], [17], [42]

Recent advances in hyperalgebra theory have produced many fresh ideas and conceptual breakthroughs. Reviewing current literature holistically, however, presents significant difficulties since researchers have used different approaches to investigate essentially similar basic concepts.

M. Benado first proposed the idea of a hyperlattice, basically a lattice construction including hyperoperations; he called this a "multistructure" [10]. Morgado [32] later made contributions to this field, although most modern researchers favor the terms either hyperlattices or multilattices. See also [21] for more thorough reviews of these systems.

The conceptual framework of hyperlattices was independently developed through two significant contributions. Initial formalization appeared in the work of Koguep, Nkuimi, and Lele [29], while subsequent advancements by Konstantinidou and Mittas [30] established the theoretical foundations for ideals and filters within these algebraic systems

The structural analysis of prime ideals and filters in hyperlattice systems was pioneered in Ameri et al.'s foundational work [2]. Building upon this, Rasouli and Davvaz established the formal axiomatization of hyperlattices and demonstrated their fundamental connections to conventional lattice structures. Their subsequent contributions [37] and [38] included the construction of topological spaces derived from prime ideal collections in distributive hyperlattices.

Recent work by Ameri et al. [2] investigates prime ideal and filter theory within hyperlattice structures. The foundational definition and key properties of hyperlattices were established in earlier work by Rasouli and Davvaz, including their relationship to classical lattices. Subsequent research [37], [38], developed topological frameworks for prime ideal spaces in distributive hyperlattices.

The foundational work of Stone [40] established a crucial isomorphism between arbitrary Boolean algebras and their collections of prime ideals excluding specific elements [40]. This seminal result marked the beginning of substantial developments in representation theory for distributive lattice structures. Birkhoff's theorem [11] establishes that for every finite distributive lattice Ω, there exists an isomorphism between Ω and the ideal lattice formed by its join-irreducible elements under their induced partial ordering. Building on this foundation, Priestley [34], [35], developed a duality framework for distributive bounded lattices. Such representation theorems provide powerful tools for analyzing lattice-theoretic structures, offering both profound theoretical insights and practical applications for studying these algebraic systems.

Different types of topologies, such as distributive lattices [39] and Boolean algebras [40], can be used to study relationship semantics for propositional logic. You can look at these meanings algebraically with the help of canonical extensions, which were studied by Gehrke and Harding [18], Gehrke and Jónsson [19], and Gehrk [20], and Jónsson [23] and Jónsson and Tarski [24]. Topological methods, including those applied to distributive lattices [39] and Boolean algebras [40], play a crucial role in investigating logical semantics for propositional systems. These connections can be algebraically characterized through canonical extensions, as explored in [18], [19], [20], and further developed in [23], [24].

The theoretical foundation of this work builds upon Stone duality [20], which provides deep connections between syntactic formulations and semantic interpretations in both mathematical logic and theoretical computer science. These connections prove particularly valuable when examining formal specification languages and their corresponding computational models, where the interplay between algebraic descriptions and topological representations offers both profound theoretical understanding and practical computational advantages.

Present an extension of representation theory from classical bounded distributive lattices to a significantly more comprehensive categorical setting. Our work generalizes and unifies several important previous results [3], [4], [5], [6], [7], [8], [34], [35], [43] and [44] by establishing a fundamental representation theorem for *bounded distributive hyperlattices endowed with negatively ordered commutative hypermonoid structures*.

The organization of this article follows a natural progression: Section 2 introduces the necessary foundational concepts and notational conventions Section 3 presents our main theoretical breakthrough, establishing a categorical duality between:

The category having as objects all Priestley negatively ordered commutative semihypergroups and the category consisting of bounded distributive hyperlattices equipped with commutative hypermonoid structures that also exhibit negative ordering properties. The final section provides concrete examples illustrating these theoretical developments

2. Preliminary Notions

This section introduces foundational concepts in hyperlattice theory essential for our investigation. For additional references, see [1], [2], [8], [29], [30], [36], [37] and [38].

Definition 2.1: An ordered semigroup structure $\mathfrak{S} = (S, \star, \trianglelefteq)$ is characterized by:

- An underlying semigroup $(S, \star)$ where the operation $\star$ is associative;
- A partial order relation $\trianglelefteq$ making $(S, \trianglelefteq)$ a poset;
- Compatibility: The operation $\star$ is both right and left compatible with the order $\trianglelefteq$.

Such a structure is called *positively ordered* if $w_1 \trianglelefteq w_1 \star u_2$ and $u_2 \trianglelefteq w_1 \star u_2$ for all $w_1, u_2 \in S$, and *negatively ordered* if the reverse inequalities hold.

For any poset $(P, \leq)$, a subset $U \subseteq P$ is:

- *Increasing* if $\xi_1 \in U$ and $\xi_1 \leq \xi_2$ implies $\xi_2 \in U$.
- *Decreasing* if $\xi_2 \in U$ and $\xi_1 \leq \xi_2$ implies $\xi_1 \in U$.

Given a nonvoid set Ω, and define $\mathcal{P}^+(\Omega)$ represents the set of all nonempty subsets of Ω [31].

Definition 2.2: A hyperstructure is an algebra $(\Upsilon, \star)$ that consists of

- A nonvoid carrier set Υ;
- A hyperoperation $\star : \Upsilon \times \Upsilon \to \mathcal{P}^+(\Upsilon)$.

For subsets $U, W \subseteq \Upsilon$, define $U \star W = \bigcup_{w_1 \in U, u_2 \in W} (w_1 \star u_2)$. The structure forms a *semihypergroup* when the hyperoperation is associative.

A semihypergroup Υ with hyperoperation $\star$ is *commutative* if the symmetric property:

$$\xi_1 \star \xi_2 = \xi_2 \star \xi_1 \text{holds for all } \xi_1, \xi_2 \in \Upsilon.$$

An element e of Υ is qualified,

- Weak identity if $w \in e \star w \cap w \star e$ for all $w \in \Upsilon$.
- Strong identity if $\{w\} = e \star w = w \star e$ for all $w \in \Upsilon$.

Definition 2.3: A partially ordered semihypergroup $(\Upsilon,\star,\preceq)$ satisfies:

- $(\Upsilon,\star)$ is a semihypergroup;
- $(\Upsilon,\preceq)$ is a poset;
- Order compatibility: $w_1 \preceq u_2$ implies $w_1 \star q \preceq^+ u_2 \star q$ and $q \star w_1 \preceq^+ q \star u_2$ for all $q \in \Upsilon$.

where $\preceq^+$ denotes the natural extension of $\preceq$ to non-empty subsets.

Definition 2.4: A hyperlattice $\mathcal{U} = (U, \curlywedge, \Cup)$ consists of:

- A meet operation $\curlywedge : U \times U \to U$;
- A join hyperoperation $\Cup: U \times U \to \mathcal{P}^+(U)$.

satisfying:

1. Idempotence: $w_1 \curlywedge w_1 = w_1$ and $w_1 \in w_1 \Cup w_1$;
2. Commutativity for both operations;
3. Absorption: $w_1 \in (w_1 \curlywedge (w_1 \Cup u_2)) \cap (w_1 \Cup (w_1 \curlywedge u_2))$;
4. Associativity for both operations;
5. Distributivity: $w_1 \curlywedge (u_2 \Cup q) \subseteq (w_1 \curlywedge u_2) \Cup (w_1 \curlywedge q)$.

The induced partial order satisfies $w_1 \preceq u_2 \Leftrightarrow u_2 \in w_1 \Cup u_2 \Leftrightarrow w_1 \curlywedge u_2 = w_1$. A hyperlattice is *bounded* if it contains universal bounds 0 and 1.

Definition 2.5: [33] The Nakano hyperoperation on a lattice $(\Upsilon,\wedge,\vee)$ is defined as:

$$w_1 \Cup u_2 = \{z \in \Upsilon | z \vee w_1 = z \vee u_2 = w_1 \vee u_2\}.$$

Definition 2.6: For a hyperlattice Υ:

- An *ideal* $I \subseteq \Upsilon$ satisfies:
  1. Downward closure: $w_1 \in I$ and $u_2 \preceq w_1$ implies $u_2 \in I$;
  2. Join absorption: $w_1, u_2 \in I$ implies $w_1 \Cup u_2 \subseteq I$.
- A *filter* $F \subseteq \Upsilon$ satisfies:
  1. Upward closure: $w_1 \in F$ and $w_1 \preceq u_2$ implies $u_2 \in F$;
  2. Meet closure: $w_1, u_2 \in F$ implies $w_1 \curlywedge u_2 \in F$.

2.1 *Hyperlattice negatively ordered commutative hypermonoids*

This subsection introduces key concepts of ordered commutative hypermonoids within hyperlattice frameworks, which will be fundamental for our subsequent analysis.

Definition 2.7: Let $\mp = (\Upsilon, \curlywedge, \Cup, \circledast)$ be an algebraic structure where:

- $(\Upsilon, \curlywedge, \Cup)$ forms a hyperlattice;
- $(\Upsilon, \circledast, \varepsilon)$ constitutes a commutative hypermonoid.

We call Υ a *bounded hyperlattice-ordered commutative hypermonoid* if it satisfies the following distributive laws:

- $w_1 \circledast (u_2 \curlywedge z) \preceq (w_1 \circledast u_2) \curlywedge (w_1 \circledast z)$;
- $w_1 \circledast (u_2 \Cup z) \subseteq (w_1 \circledast u_2) \Cup (w_1 \circledast z)$ for all $w_1, u_2, z \in \Upsilon$.

These conditions imply the order-preserving property: $w_1 \preceq u_2 \Rightarrow w_1 \circledast z \preceq u_2 \circledast z$. The structure is called *distributive* when it additionally satisfies $w_1 \curlywedge (u_2 \Cup z) \subseteq (w_1 \curlywedge u_2) \Cup (w_1 \curlywedge z)$.

A bounded structure contains universal bounds $\bot$ and $\top$ such that $\bot \preceq w_1 \preceq \top$ for all $w_1 \in \Upsilon$. In positively (negatively) ordered cases, the identity element ε coincides with $\bot$ (respectively $\top$).

Example 2.8: Consider the divisor lattice $\mathcal{D}_{18} = (D(18), gcd, lcm, 1{,}18)$. Its Hasse diagram is given by Figure 1:

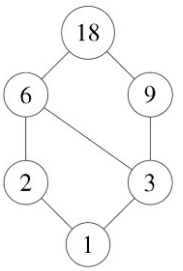

Figure 1
Hasse diagram of the divisor lattice $\mathcal{D}_{18}$

Define hyperoperations:

- $w_1 \Cup u_2 = \{k \in D(18) | \mathrm{lcm}(k, w_1) = \mathrm{lcm}(k, u_2) = \mathrm{lcm}(w_1, u_2)\}$.
- $w_1 \circledast u_2 = \begin{cases} \downarrow (\gcd(w_1, u_2)) & \text{if } w_1 = 18 \text{ or } u_2 = 18; \\ \{1\} & \text{otherwise.} \end{cases}$

Then $(\mathcal{D}_{18}, \gcd, \Cup, \circledast, 1, 18)$ forms a bounded distributive hyperlattice with negative ordering.

Theorem 2.9: *For each bounded distributive lattice $\mathcal{L} = (L, \curlywedge, \vee, \bot, \top)$, three distinct hypermonoid structures can be constructed:*

1. *Positive hypermonoid* with operations:

$$w_1 \Cup u_2 = \{z \in L | w_1 \vee u_2 = w_1 \vee z = u_2 \vee z\}.$$
$$w_1 \circledast u_2 = \begin{cases} \uparrow (w_1 \vee u_2) & \text{if } w_1 = \bot \text{ or } u_2 = \bot; \\ \{\top\} & \text{otherwise.} \end{cases}$$

2. Negative hypermonoid I with operations:

$$w_1 \Cup u_2 = \{z \in L | w_1 \vee u_2 = w_1 \vee z = u_2 \vee z\}$$
$$w_1 \circledast u_2 = \begin{cases} \downarrow (w_1 \curlywedge u_2) & \text{if } w_1 = \top \text{ or } u_2 = \top; \\ \{\bot\} & \text{otherwise.} \end{cases}$$

3. Negative hypermonoid II with operations:

$$w_1 \Cup u_2 = \{z \in L | z \geq w_1 \vee u_2\}.$$
$$w_1 \circledast u_2 = \begin{cases} \{t \in L | t < w_1 \curlywedge u_2\} & \text{if } w_1 = \top \text{ or } u_2 = \top; \\ \{\bot\} & \text{otherwise.} \end{cases}$$

Proof: The verification follows directly from Definition 3.1 by routine algebraic manipulation.

Corollary 2.10: *The Boolean algebra* $\mathbb{B}_2 = \{0,1\}$ *forms a bounded distributive negatively ordered hypermonoid with operations defined by:*

$\Cup$	0	1
0	{0}	{1}
1	{1}	{0,1}

$\circledast$	0	1
0	{0}	{0}
1	{0}	{0,1}

Given a bounded distributive hyperlattice $\mathcal{H} = (H, \wedge, \sqcup, \bot, \top)$, its *spectral dual space* $\mathfrak{S}(\mathcal{H})$ is constructed as follows:

Definition 2.11: The **carrier set** $\mathfrak{P}$ consists of all non-trivial hyperlattice homomorphisms:

$$\mathfrak{P} := \{\phi : H \to \mathbf{2} | \phi \text{ preserves operations } \phi(\bot) = 0, \phi(\top) = 1\}$$

where $\mathbf{2} = (\{0,1\}, \wedge, \vee, 0, 1)$ is the 2-element lattice.

Definition 2.12: A point-by-point order $\preccurlyeq$ is defined in the following manner:

$$\phi \preccurlyeq \psi \Leftrightarrow \forall q \in H, \phi(q) \leq \psi(q).$$

Definition 2.13: The **spectral topology** $\mathcal{T}$ is the initial topology induced by the evaluation maps:

$$\hat{h} : \mathfrak{P} \to \mathbf{2}, \quad \phi \mapsto \phi(h) \quad (h \in H).$$

Theorem 2.14: *The triple* $\mathfrak{S}(\mathcal{H}) = (\mathfrak{P}, \mathcal{T}, \preccurlyeq)$ *forms a Priestley space.*

Lemma 2.15: (Dual Structure Construction). *Let* $\mathcal{P} = (\mathfrak{K}, \mathcal{T}, \preceq)$ *be a Priestley space. Then there are canonical hyperoperations* $\curlyvee$ *and* $\boxtimes$ *endowing the collection*

$$\mathcal{L}(\mathcal{P}) = \{Y \subseteq \mathfrak{K} \mid Y \text{ is } \mathcal{T} - \text{clopen and } \preceq - \text{increasing}\}$$

with a structure of bounded distributive hyperlattice with negatively ordered commutative hypermonoid operations $(\mathcal{L}(\mathcal{P}), \cap, \curlyvee, \boxtimes, \emptyset, \mathfrak{K})$, where:

(i) The hyperjoin $\curlyvee$ is given by:

$$\Upsilon_1 \curlyvee \Upsilon_2 = \{\Upsilon \in \mathcal{L}(\mathcal{P}) | \Upsilon_1 \cup \Upsilon_2 \subseteq \Upsilon \text{ and } \Upsilon_1 \cup \Upsilon = \Upsilon_2 \cup \Upsilon\};$$

(ii) The hyperproduct $\boxtimes$ is defined as:

$$\Upsilon_1 \boxtimes \Upsilon_2 = \begin{cases} \downarrow (\Upsilon_1 \cap \Upsilon_2) & \text{if } \Upsilon_1 = \Re \text{ or } \Upsilon_2 = \Re; \\ \{\min(\mathcal{L}(\mathcal{P}))\} & \text{otherwise.} \end{cases}$$

for all clopen increasing sets $\Upsilon_1, \Upsilon_2 \in \mathcal{L}(\mathcal{P})$.

2.2 *Hyperideals and hyperfilters in hyperlattice ordered commutative hypermonoid*

This section introduces the concepts of (prime) hyperideals and (prime) hyperfilters in hyperlattice negatively ordered commutative hypermonoid see [22], and [41]

Definition 2.16: Let $\mathcal{H} = (\Upsilon, \wedge, \sqcup, \circledast)$ be a commutative hypermonoid with hyperlattice ordering. A subset $\Im \subseteq \Upsilon$ is said to be a hyperideal when it satisfies these axioms:

1. **Order Compatibility**: For $\xi_1 \in \Im$ and $\xi_2 \preceq \xi_1$, we have $\xi_2 \in \Im$;
2. **Join Closure**: Given $\xi_1, \xi_2 \in \Im$, their hyperjoin satisfies $\xi_1 \sqcup \xi_2 \subset \Im$;
3. **Product Absorption**: If either ξ_1 or ξ_2 lies in $\Im$, then their hyperproduct $\xi_1 \circledast \xi_2$ is contained in $\Im$ for all $\xi_1, \xi_2 \in \Upsilon$;
4. **Prime Property**: A proper hyperideal $\Im$ is prime when $\xi_1 \wedge \xi_2 \in \Im$ necessitates that at least one of ξ_1 or ξ_2 belongs to $\Im$;
5. **Intersection Property**: For arbitrary elements $\xi_1, \xi_2 \in \Upsilon$, non-empty intersection $(\xi_1 \circledast \xi_2) \cap \Im \neq \emptyset$ implies $\xi_1 \in \Im$ or $\xi_2 \in \Im$.

Definition 2.17: Consider an algebraic structure $\mathcal{A} = (U, \wedge, \sqcup, \star)$ where $(U, \wedge, \sqcup)$ forms a hyperlattice and $(U, \star)$ constitutes a commutative hypermonoid with compatible ordering. A non-void subset $\mathfrak{F} \subseteq U$ qualifies as a **hyperfilter** when it meets these criteria:

1. **Upward Closure**: For any elements $u_1 \in \mathfrak{F}$ and $u_2 \in U$, if $u_1 \preccurlyeq u_2$, then u_2 must belong to $\mathfrak{F}$;
2. **Meet Stability**: For two elements $u_1, u_2 \in \mathfrak{F}$, their meet $u_1 \wedge u_2$ is necessarily contained in $\mathfrak{F}$;
3. **Product Condition**: Given $u_1 \in \mathfrak{F}$ and $u_2 \in U$, if their hyperproduct $u_1 \star u_2$ intersects $\mathfrak{F}$ non-emptily, then u_2 must be in $\mathfrak{F}$;
4. **Prime Property** (for proper hyperfilters): $\mathfrak{F}$ is prime if whenever $u_1 \sqcup u_2$ meets $\mathfrak{F}$, then either $u_1 \in \mathfrak{F}$ or $u_2 \in \mathfrak{F}$;
5. **Complete Absorption**: If the hyperproduct $u_1 \star u_2$ has non-empty intersection with $\mathfrak{F}$, then both u_1 and u_2 must belong to $\mathfrak{F}$.

Theorem 2.18: *Let $\mathcal{H} = (U, \wedge, \sqcup, \odot)$ be a commutative hypermonoid with hyperlattice structure. There exists a complement duality between:*

1. *Prime hyperideals $\mathfrak{I} \subseteq U$ and their set complements $U \backslash \mathfrak{I}$ which form prime hyperfilters;*
2. *Prime hyperfilters $\mathfrak{Q} \subseteq U$ and their set complements $U \backslash \mathfrak{Q}$ which form prime hyperideals.*

Proof: We establish both directions of the correspondence, based on the duality Principle:

Part (i): Uet $\mathfrak{J}$ be a prime hyperideal. To show $U\backslash\mathfrak{J}$ is a prime hyperfilter:

(α) Whenever $u_1 \in U\backslash\mathfrak{J}$ and u_2 dominates u_1 (i.e., $u_2 \succeq u_1$), if $u_2 \in \mathfrak{J}$ then u_1 belongs necesarely to $\mathfrak{J}$ by the hyperideal property, contradicting $u_1 \in U\backslash\mathfrak{J}$. Thus $u_2 \in U\backslash\mathfrak{J}$.

(β) For $u_1, u_2 \in U\backslash\mathfrak{J}$, if $u_1 \wedge u_2$ must lie in $\mathfrak{J}$, then primality provided both u_1 and u_2 are elements of $\mathfrak{J}$. Contradiction. Hence $u_1 \wedge u_2 \in U\backslash\mathfrak{J}$.

(γ) If the hyperproduct $u_1 \odot u_2$ meets $U\backslash\mathfrak{J}$ and $u_1 \in U\backslash\mathfrak{J}$, but $u_2 \in \mathfrak{J}$, then $u_1 \odot u_2 \subseteq \mathfrak{J}$ by the hyperideal property, contradicting the non-empty intersection.

Primality follows since if $u_1 \sqcup u_2$ intersects $U\backslash\mathfrak{J}$ but neither u_1 nor u_2 does, then $u_1 \sqcup u_2 \subseteq \mathfrak{J}$, making the intersection impossible. The product condition follows similarly.

Part (ii): For a prime hyperfilter $\mathfrak{Q}$, its complement $U\backslash\mathfrak{Q}$ inherits:

- Downward closure from $\mathfrak{Q}$'s upward closure,
- Join absorption from meet stability,
- Primality from the filter conditions,

through direct logical complementation of the definitions.

Proposition 2.19: [8] If $\Phi \neq \emptyset$ of a hyperlattice ordered commutative hypermonoid $(\Upsilon,\wedge,\sqcup,\circledast)$, then the minimal hyperlattice filter encompassing Φ is expressed as:

$$\langle\Phi\rangle = \{\mathfrak{a} \in \Upsilon | \zeta_1 \wedge \ldots \wedge \zeta_n \preccurlyeq \mathfrak{a}, \text{for certain elements } \zeta_1, .., \zeta_n \in \Phi\}.$$

If $\Phi = \{\zeta\}$, we write $\langle\Phi\rangle = \uparrow \zeta = \{w \in \Upsilon | \zeta \preccurlyeq w\}$.

Proposition 2.20: If δ is a hyperlattice filter of a distributive hyperlattice negatively ordered commutative hypermonoid $(\Upsilon,\wedge,\sqcup,\circledast)$, then the smallest hyperlattice ordered commutative hypermonoid hyperfilter encompassing δ admits the representation,

$$[\delta) = \{w \in \Upsilon | \zeta_1 \circledast \ldots \circledast \zeta_n \preceq w, \text{for certain elements } \zeta_1, .., \zeta_n \in \delta\}.$$

Proof: Take $a \in \delta$. The reflexivity of $\preceq$ gives $a \preceq a$, and we obtain a belongs to $[\delta)$. Hence, the interval $[\delta)$ is non-empty. To prove that the set $[\delta)$ is a hyperfilter, consider $q, o \in \Upsilon$ verifying $q \preceq o$, then $\xi_1 \circledast \ldots \circledast \xi_n \preceq q \preceq o$, so $o \in [\delta)$. On the other hand, for every $q, o \in [\delta)$, there exists $\xi_1, \xi_2, \ldots, \xi_n, o_1, o_2, \ldots, o_m$ such that $\xi_1 \circledast \xi_2 \circledast \ldots \circledast \xi_n \preceq q$ and $o_1 \circledast o_2 \circledast \ldots \circledast o_m \preceq o$. Then $(\xi_1 \circledast \xi_2 \circledast \ldots \circledast \xi_n) \wedge (o_1 \circledast o_2 \circledast \ldots \circledast o_m) \preceq q \wedge o$. Since Υ is negatively ordered so, $q \wedge o \in [\delta)$. Now let $q \in [\delta)$ and $(q \circledast \mathrm{o}) \cap [\delta) \neq \phi$, then there exis $w \in (q \circledast \mathrm{o}) \cap [\delta)$. Since $w \in [\delta)$, by the definition of $[\delta)$, there exist $\zeta_1, .., \zeta_n \in [\delta)$ such that $\zeta_1 \circledast \ldots \circledast \zeta_n \preceq w$. But $w \in q \circledast o$. As $q \in [\delta)$,

closure of $[\delta)$ under hyperproducts implies that o must also belong to $[\delta)$, otherwise, $(q \circledast o)$ could not yield an element of $[\delta)$. Hence $o \in [\delta)$.

Next, consider any element $e \in \delta$. By reflexivity of the order relation, we have $e \preceq e$, which immediately implies $e \in [\delta)$. This establishes the inclusion $\delta \subseteq [\delta)$. Now, let $\mathcal{F}$ be an arbitrary hyperfilter containing $\delta \subseteq \mathcal{F}$. Subsequently, for any $q \in [\delta)$,there exists a finite collection of elements $\{\xi_i\}_{i=1}^{n} \subseteq \delta$ such that the hyperproduct $\circledast_{i=1}^{n} \xi_i \preceq q$, then $q \in \mathcal{F}$. Therefore $[\delta) \subseteq \mathcal{F}$

Theorem 2.21: *For a distributive hyperlattice negatively ordered commutative hypermonoid* $(\Upsilon,\wedge,\sqcup,\circledast)$. *Consider a hyperfilter* $\mathcal{F}$ *and a hyperideal* Γ *of* Υ. *If* $\mathcal{F}$ *and* Γ *are disjoint, then there is a prime hyperfilter* $\mathcal{P}$ *containing* $\mathcal{F}$ *and disjoint with* Γ.

Proof: Similarly to [8, Theorem 2.9].

Corollary 2.22: *Consider a distributive hyperlattice negatively ordered commutative hypermonoid* $(\Upsilon,\wedge,\sqcup,\circledast)$. *If* Γ *is a hyperideal and* $a \notin \Gamma$, *then there exists a prime hyperfilter* $\mathcal{P}$ *containing* a *and disjoint with* Γ.

Proof: Let Γ be a hyperideal and take $\mathcal{F} = \{x \in \Upsilon \backslash \Gamma | a \circledast \ldots \circledast a \preceq x\}$, it follows $\mathcal{F}$ and Γ are disjoint. It is easy to show that $\mathcal{F}$ is a hyperfilter. According to Theorem 2.21, a prime hyperfilter $\mathcal{P}$ exists that contains $\mathcal{F}$ and is disjoint from Γ.

2.3 *Hyperlattice ordered commutative hypermonoid homomorphisms*

We now examine structure-preserving mappings between hyperlattice-ordered commutative hypermonoids. These homomorphisms play a crucial role in our subsequent analysis, and we establish their fundamental properties below.

Definition 2.23: Given two algebraic structures $(\Upsilon, \{\wedge, \Cup, \circledast\})$ and $(\Upsilon', \{\curlywedge, \Subset, \circledcirc\})$ combining hypermonoid and hyperlattice axioms, a function $j: \Upsilon \to \Upsilon'$ is a **morphism** when:

- Meets are preserved exactly: $j(r \wedge s) = j(r) \curlywedge j(s)$,
- Joins are mapped inclusively: $j(r \Cup s) \subseteq j(r) \Subset j(s)$,
- Multiplication is order-respecting: $j(r \circledast s) \preceq j(r) \circledcirc j(s)$.

A bijective morphism, we call it a **structure isomorphism** between these algebraic systems.

Proposition 2.24: Let $(\Upsilon,\wedge,\sqcup,\circledast)$ be a hyperlattice-ordered commutative hypermonoid and $(\mathbb{U},\wedge,\sqcup,\diamond,0,1)$ the bounded structure from Lemma 2.10 with negative order. Every prime hyperfilter $\mathcal{F} \subset \Upsilon$ induces a surjective structure-preserving map $\hbar : \Upsilon \twoheadrightarrow \{0,1\}$ characterized by $\mathcal{F} = \hbar^{-1}(1)$.

Proof: Put $\hbar(\Theta) = \begin{cases} 1, & \text{if } \Theta \subseteq \mathcal{F}, \\ 0, & \text{otherwise.} \end{cases}$

For the conditions $\hbar(r \wedge s) = \hbar(r) \wedge \hbar(s)$ and $\hbar(r \sqcup s) \subseteq \hbar(r) \sqcup \hbar(s)$

similarly to [8, Proposition 2.12]. For the third homomorphism axiom, we have $\hbar(r \circledast s) = 1 \Rightarrow r \circledast s \subseteq \mathcal{F} \Rightarrow (r \in \mathcal{F}$ and $s \in \mathcal{F}) \Rightarrow (\hbar(r) = 1$ and $\hbar(s) = 1) \Rightarrow \hbar(r) \diamond \hbar(s) = \{0,1\}$. Hence, $\hbar(r \circledast s) \subseteq \hbar(r) \diamond \hbar(s)$.

Corollary 2.25: *For a distributive hyperlattice negatively ordered commutative hypermonoid Y. If $r, s \in Y$ are such that $r \not\preceq s$, there exists a prime hyperfilter $\mathcal{F}$ containing s but not containing r.*

Proof: Taking $\downarrow$ s in Corollary 2.22, which is a hyperideal of a distributive hyperlattice negatively ordered commutative hypermonoid and $r \notin \downarrow s$.

3. Priestley-style topological representation theorem

We establish a representation theorem extending Priestley's classical result to the category of bounded distributive hyperlattices endowed with negative orderings and commutative hypermonoid operations.

We begin by recalling essential topological-algebraic structures that underpin our investigation.

Definition 3.1: (Order-Embedded Topological System). A order-embedded topological system is a composite structure $\mathcal{S} = (Y, \mathcal{F}, \trianglelefteq)$ where:

1. $(Y, \mathcal{F})$ represents a topological environment and establishes a partial ordering
2. A subcollection $A \subseteq Y$ is called *bi-stable* when it simultaneously satisfies closure and openness conditions relative to $\mathcal{F}$
3. The system exhibits *complete order isolation* if for any $p \not\trianglelefteq q$ there exist:
   - A $\trianglelefteq$-increasing bi-stable set $A_p \ni p$
   - A $\trianglelefteq$-decreasing bi-stable set $A_q \ni q$
   - With $A_p \cap A_q = \emptyset$
4. A *Priestley system* is a compact order-embedded topological environment with complete order isolation [34, 35]

Definition 3.2: Let $\mathcal{A} = (\mathfrak{A}, \wedge, \sqcup, \bot, \top)$ be a bounded distributive hyperlattice. The spectral dual of $\mathcal{A}$, denoted $\Subset(\mathcal{A})$, consists of:

- The carrier set $\mathfrak{P}$ of all nontrivial hyperlattice morphisms $\phi: \mathcal{A} \to \mathbf{2}$, where $\mathbf{2} = (\{\bot, T)\}, \wedge, \sqcup, \bot, \top))$ is the two-elements Boolean algebra (structure).
- A point-by-point order $\preccurlyeq$ on $\mathfrak{P}$, where $h \preccurlyeq l$ iff $h(d) \leq l(d)$ for all $d \in \mathfrak{A}$.
- The initial topology $\mathcal{T}$ inherited from the product space $\mathbf{2}^{\mathcal{A}}$ equipped with the discrete topology.

Theorem 3.3: (Duality Correspondence*). Let* $\mathcal{Z} = (Z,\wedge,\sqcup,\bot,\top)$ *be a bounded distributive hyperlattice. Then the spectral representation space* $T(Z) = (X,\tau,\leq)$, *consisting of:*

- X *represents the collection of all proper hyperlattice morphisms from* Z *to* $\mathbf{2}$.
- $\leq$ *the pointwise order on* X.

forms a Priestley space in the sense of [8].

Definition 3.4: A Priestley-ordered commutative semi hypergroup is characterized as a structure $\mathfrak{X} = (\mathcal{X},\tau,\leq,*)$, in which,

- The structure $(\mathcal{X},\tau,\leq)$ is a space of Priestley.
- The algebraic structure $(\mathcal{X},\leq,*)$ constitutes a commutative semi hypergroup with negative ordering.

Definition 3.5: A *topological-algebraic structure* $\mathfrak{X} = (\mathcal{X},\tau,\leq,*)$ is classified as a Priestley-ordered commutative semi hypergroup when:

- The triple $(\mathcal{X},\tau,\leq)$ forms a space of Priestley,
- The algebraic structure $(\mathcal{X},\leq,*)$ constitutes a commutative semi hypergroup with negative ordering.

Definition 3.6: Given a bounded distributive hyperlattice hyperlattice endowed with a negatively ordered commutative hypermonoid structure $\mathcal{E} = (E,\ \wedge,\ \sqcup,\ \circledcirc,\bot,\top)$. Its dual representation space $\mathfrak{D}(\mathcal{E})$ is the quadruple $(\mathcal{X},\mathcal{T},\preccurlyeq,*)$ where:

(i) $\mathcal{X}$ denotes the collection of all structure-preserving morphisms $\phi: \mathcal{E} \to \mathbf{2}$, where $\mathbf{2} = (\{\bot,\top\},\wedge,\sqcup,\diamond,\bot,\top)$ is the two-element Boolean algebra (structure)

(ii) The partial order $\preccurlyeq$ on $\mathcal{X}$ is characterized by the following definition:

$$\phi \preccurlyeq \psi \Leftrightarrow \forall q \in E, \phi(q) \leq \psi(q).$$

(iii) The initial topology $\mathcal{T}$ is induced by the product topology on $\mathbf{2}^E$.

(iv) The hyperoperation $*: \mathcal{X} \times \mathcal{X} \to \mathcal{P}^*(\mathcal{X})$ is defined in a pointwise manner:

$$(\phi * \psi)(q) = \begin{cases} \downarrow [\phi(q) \wedge \psi(q)] & \text{if } \phi\ (q) \sqcup \psi(q) = \top, \\ \{\bot\} & \text{otherwise.} \end{cases}$$

Proposition 3.7: Given a bounded distributive hyperlattice $\mathcal{H} = (L,\wedge,\vee,\odot,0,1)$ equipped with a negatively ordered commutative hypermonoid structure, its dual space $\mathcal{P}(\mathcal{H}) = (Y,\mathcal{O},\preccurlyeq,\diamond)$ forms a Priestley space with a compatible negatively ordered commutative hypergroupoid structure.

Proof: Let $\mathcal{H}^* = (Y,\mathcal{O},\preccurlyeq)$ denote the Priestley space associated with $\mathcal{H}$ as established in prior work. We verify that the hyperoperation $\diamond$ endows $(Y,\preccurlyeq)$ with the required algebraic structure:

1. **Commutativity**: For any $\phi_1, \phi_2 \in Y$, the equality $\phi_1 \diamond \phi_2 = \phi_2 \diamond \phi_1$ follows immediately from the commutativity of $\odot$.
2. **Associativity**: The relation $(\phi_1 \diamond \phi_2) \diamond \phi_3 = \phi_1 \diamond (\phi_2 \diamond \phi_3)$ holds through direct computation.
3. **Order Compatibility**: For $\phi_1 \preccurlyeq \phi_2$, we observe:
$$\begin{aligned}(\phi_1 \diamond \phi_3)(a) &= \inf\{\phi_1(b) \wedge \phi_3(c) | b \odot c \leq a\}, \\ &\leq \inf\{\phi_2(b) \wedge \phi_3(c) | b \odot c \leq a\}, \\ &= (\phi_2 \diamond \phi_3)(a).\end{aligned}$$
establishing $\phi_1 \diamond \phi_3 \preccurlyeq \phi_2 \diamond \phi_3$.
4. **Negative Ordering**: The inequalities $\phi_1 \diamond \phi_2 \preccurlyeq \phi_1$ and $\phi_1 \diamond \phi_2 \preccurlyeq \phi_2$ follow from:
$$(\phi_1 \diamond \phi_2)(a) \preccurlyeq (\phi_1 \wedge \phi_2)(a) \preccurlyeq \phi_i(a) \quad (i = 1,2).$$

Thus, $\mathcal{P}(\mathcal{H})$ satisfies all axioms of a negatively ordered commutative hypergroupoid in the category of Priestley spaces.

In the following, we give a definition of *Priestley space negatively ordered commutative semi hypergroup homomorphism*.

Definition 3.8: Consider be two a Priestley spaces negatively ordered commutative semi hypergroups. $\mathcal{Q} = (\mathfrak{Q}, \mathcal{T}, \sqsubseteq, *)$, and $\mathcal{Q}' = (\mathfrak{Q}', \mathcal{T}', \precsim, \diamond)$. A map $\phi: \mathfrak{Q} \to \mathfrak{Q}'$ is called:

(HP1) **Monotonic** if for each $\xi_1, \xi_2 \in \mathfrak{Q}$, $\xi_1 \sqsubseteq \xi_2 \Rightarrow \phi(\xi_1) \precsim \phi(\xi_2)$;

(HP2) **Structure-preserving** if it satisfies:
- Monotonicity (HP1);
- Topological continuity;
- Submultiplicativity: $\phi(u_1 * u_2) \precsim \phi(u_1) \diamond \phi(u_2)$ for all $u_1, u_2 \in \mathfrak{Q}$.

A bijective structure-preserving map ϕ whose inverse also preserves structure is called a **structural isomorphism** between these spaces.

In what follows, we will give necessary conditions that the dual of a *Priestley space negatively ordered commutative semi hypergroup* be a *bounded distributive hyperlattice negatively ordered commutative hypermonoid*.

Lemma 3.9: (Dual Structure Construction). *Let $\mathcal{P} = (\mathfrak{X}, \mathcal{T}, \sqsubseteq)$ be a space of Priestley.*

There are canonical hyperoperations $\curlyvee$ with $\boxtimes$ endowing the collection
$$\mathcal{L}(\mathcal{P}) = \{Y \subseteq \mathfrak{X} \mid Y \text{ is } \mathcal{T} - \text{clopen and } \sqsubseteq - \text{increasing}\},$$
with a structure of a bounded distributive hyperlattice with negatively ordered commutative hypermonoid operations $(\mathcal{L}(\mathcal{P}), \cap, \curlyvee, \boxtimes, \emptyset, \mathfrak{X})$, where:

1. *The hyperjoin $\curlyvee$ is given by:*
$$U_1 \curlyvee U_2 = \{Q \in \mathcal{L}(\mathcal{P}) | U_1 \cup U_2 \subseteq Q \text{ and } U_1 \cup Q = U_2 \cup Q\}.$$

2. *The hyperproduct* $\boxtimes$ *is defined as:*

$$U_1 \boxtimes U_2 = \begin{cases} \downarrow (U_1 \cap U_2) & \text{if } U_1 = \mathfrak{X} \text{ or } U_2 = \mathfrak{X}, \\ \{\min(\mathcal{L}(\mathcal{P}))\} & \text{otherwise.} \end{cases}$$

for all clopen increasing sets $U_1, U_2 \in \mathcal{L}(\mathcal{P})$.

Proof: A straightforward verification shows that the algebra $(\mathcal{L}(\mathcal{P}),\cap,\cup,\emptyset,\mathfrak{X})$ constitutes a bounded distributive lattice, as referenced in [34, 35]. According to Definition 2.7, the algebra $(\mathcal{L}(\mathcal{P}),\cap,\curlyvee,\boxtimes,\emptyset,\mathfrak{X})$ is a bounded distributive hyperlattice that is negatively ordered and a commutative hypermonoid.

Lemma 3.10: (Representation Theorem). *For any bounded distributive hyperlattice* $\mathcal{E} = (E, \curlywedge, \curlyvee, \circledast, \bot, \top)$ *equipped with negatively ordered commutative hypermonoid structure, the canonical embedding*

$$\Phi_{\mathcal{E}}: \mathcal{E} \hookrightarrow \mathcal{L}(\mathfrak{T}(\mathcal{E}))$$

given by

$$\Phi_{\mathcal{E}}(e) = \{\phi \in \mathrm{Spec}(\mathcal{E}) | \phi(e) = \top\}$$

establishes an isomorphism in the category of hyperlattice-ordered commutative hypermonoids.

Proof: Since E hyperlattice then $\Phi_{\mathcal{E}}$ is an isomorphism of hyperlattices see [8]. $\mathcal{L}(\mathfrak{T}(\mathcal{E}))$ is hyperlattice negatively ordered commutative hypermonoid by Lemma 3.9. For the condition $\Phi_E(t \circledast s) \subseteq \Phi_E(t) \circledcirc \Phi_E(s)$, for all $t, s \in E$ and $\phi \in Spec(\mathcal{E})$; $\phi \in \Phi_E(t \circledast s) \Rightarrow \phi(t \circledast s) = 1 \Rightarrow \phi(t) \diamond \phi(s) = \{0,1\} \Rightarrow (\phi(t) = 1$ and $\phi(s) = 1) \Rightarrow \phi \in \Phi_E(t) \cap \Phi_E(s) \Rightarrow \phi \in \Phi_E(t) \circledcirc \Phi_E(s) = \downarrow (\Phi_E(t) \cap \Phi_E(s))$. So $\Phi_E(t \circledast s) \preceq \Phi_E(t) \circledcirc \Phi_E(s)$. Therefore, $\Phi_{\mathcal{E}}$ is an isomorphism of E onto $\mathcal{L}(\mathfrak{T}(\mathcal{E}))$.

Lemma 3.11: (Dual Mapping Preservation (dual-homomorphism)). *Given a structure-preserving map* $\phi: \mathcal{H}_1 \to \mathcal{H}_2$ *between hyperlattice negatively ordered commutative hypermonoid homomorphism, the induced mapping*

$$\mathcal{D}(\phi): \mathcal{D}(\mathcal{H}_2) \to \mathcal{D}(\mathcal{H}_1), \psi \mapsto \psi \circ \phi$$

constitutes a morphism of ordered topological semi hypergroups.

Proof: We establish the required properties:

1. **Order and Continuity**: The mapping $\mathcal{D}(\phi)$ preserves both the specialization order and topological structure, as shown in prior work [4], [5], [6], [7], [8].
2. **Operation Compatibility**: For any pair $\rho_1, \rho_2 \in \mathcal{D}(\mathcal{H}_2)$ and element $h \in H_1$:

   1. When $\rho_1(\phi(h)) = 1$, the composition satisfies:

$$(\mathcal{D}(\phi)(\rho_1) \diamond \mathcal{D}(\phi)(\rho_2))(h) = \rho_2(\phi(h))$$

   The relation $\rho_1 \star \rho_2 \preceq \rho_2$ ensures:

$$\mathcal{D}(\phi)(\rho_1 \star \rho_2)(h) \leq (\mathcal{D}(\phi)(\rho_1) \diamond \mathcal{D}(\phi)(\rho_2))(h)$$

2. The symmetric case $\rho_2(\phi(h)) = 1$ follows similarly.
3. When $\rho_1(\phi(h)) = \rho_2(\phi(h)) = 0$, we obtain:

$$\mathcal{D}(\phi)(\rho_1) \diamond \mathcal{D}(\phi)(\rho_2) = \mathcal{D}(\phi)(\rho_1 \star \rho_2) = 0$$

Thus $\mathcal{D}(\phi)$ preserves all necessary algebraic and topological structure.

Lemma 3.12: (Inverse Image Homomorphism). *Consider two Priestley spaces* $\mathcal{P}_1 = (X, \mathcal{T}, \leq, \star)$ *and* $\mathcal{P}_2 = (Y, \mathcal{T}', \preceq, \diamond)$ *endowed with negatively ordered commutative semi hypergroup structures. For any structure-preserving map* $\omega: \mathcal{P}_1 \to \mathcal{P}_2$, *the inverse image mapping*

$$\mathcal{L}(\omega): \mathcal{L}(\mathcal{P}_2) \to \mathcal{L}(\mathcal{P}_1), V \mapsto \omega^{-1}(V)$$

constitutes a surjective homomorphism between the corresponding algebraic structures.

Proof: We verify the homomorphism conditions:

1. **Well-definedness**: The preimage $\omega^{-1}(V)$ of any increasing $\mathcal{T}'$-clopen set $V \subseteq Y$ remains increasing and $\mathcal{T}$-clopen in X, as established in [4], [5], [6], [7], [8].
2. **Operation Preservation**:
   • For meets: $\mathcal{L}(\omega)(V \cap W) = \mathcal{L}(\omega)(V) \cap \mathcal{L}(\omega)(W)$
   • For hyperjoins: $\mathcal{L}(\omega)(V \curlyvee W) \subseteq \mathcal{L}(\omega)(V) \curlyvee \mathcal{L}(\omega)(W)$
3. **Hyperproduct Compatibility**:
   1. When $V = Y$:

$$\mathcal{L}(\omega)(V \diamond W) = \mathcal{L}(\omega)(\downarrow W) = \downarrow \mathcal{L}(\omega)(W) = X \star \mathcal{L}(\omega)(W)$$

   2. The symmetric case $W = Y$ follows similarly.
   3. For $V, W \neq Y$:

$$\mathcal{L}(\omega)(V \diamond W) = \emptyset \preceq \mathcal{L}(\omega)(V) \star \mathcal{L}(\omega)(W)$$

   Thus $\mathcal{L}(\omega)$ preserves all required operations.

Theorem 3.13: *For any homomorphism* $f: \mathcal{H}_1 \to \mathcal{H}_2$ *between negatively ordered commutative hypermonoids, there exists a canonical correspondence making the following square commutative:*

$$\varphi_{H_2} \circ f = \tilde{f} \circ \varphi_{H_1}$$

where:

- $\varphi_{H_i}: \mathcal{H}_i \to \mathcal{L}(\mathfrak{Spec}(\mathcal{H}_i))$ *are the spectral embeddings*
- $\tilde{f} = \mathcal{L}(T(f))$ *corresponds to the induced dual morphism*
- *The map* $T(f): \mathfrak{Spec}(\mathcal{H}_2) \to \mathfrak{Spec}(\mathcal{H}_1)$ *is given by the composition* $T(f)(\psi) = \psi \circ f$.

Proof: The equality follows by direct computation for any $x \in \mathcal{H}_1$:

$$\begin{aligned}(\varphi_{H_2} \circ f)(x) &= \{\psi \in \mathfrak{Spec}(\mathcal{H}_2) | \psi(f(x)) = 1\} \\ &= \{\psi \circ f \in \mathfrak{Spec}(\mathcal{H}_1) | \psi(f(x)) = 1\} \\ &= (\tilde{f} \circ \varphi_{H_1})(x)\end{aligned}$$

This proof adapts the arguments from [12, Lemma 3.11] to our generalized framework.

Lemma 3.14: (Dual Space Isomorphism). *For any Priestley space* $\mathcal{P} = (X, \tau, \preceq, \star)$ *endowed with a negatively ordered commutative semi hypergroup structure, the evaluation map*

$$\Xi_{\mathcal{P}}: \mathcal{P} \to \mathfrak{D}(\mathcal{L}(\mathcal{P}))$$

defined by

$$\Xi_{\mathcal{P}}(\rho)(\mathcal{Y}) = \begin{cases} \top & \text{if } \rho \in \mathcal{Y}; \\ \bot & \text{otherwise.} \end{cases}$$

for all clopen increasing sets $Y \in \mathcal{L}(\mathcal{P})$, *establishes an isomorphism in the category of such structures.*

Proof: We verify the necessary conditions:

1. **Well-definedness**: By Lemma 2.15, $\mathcal{L}(\mathcal{P})$ forms a bounded distributive hyperlattice, and by Lemma 3.9, its dual $\mathfrak{D}(\mathcal{L}(\mathcal{P}))$ is again a Priestley space with semi hypergroup structure.
2. **Morphism Properties**:
   - Order-preservation and continuity follow from standard results: [4], [5], [6], [7] [8], [34], and [35].
   - Bijectivity is established through Stone-type duality arguments see [37], and [41].
3. **Operation Compatibility**: For $\rho, \sigma \in X$ and $\mathcal{Y} \in \mathcal{L}(\mathcal{P})$:
   - When $\rho \star \sigma \notin \mathcal{Y}$, the inequality holds trivially,
   - When $\rho \star \sigma \in \mathcal{Y}$, the increasing property ensures $\{\rho, \sigma\} \subseteq \mathcal{Y}$, making $\Xi_{\mathcal{P}}(\rho)(\mathcal{Y}) \diamond \Xi_{\mathcal{P}}(\sigma)(\mathcal{Y}) = \{\bot, \top\}$ dominate $\Xi_{\mathcal{P}}(\rho \star \sigma)(\mathcal{Y})$.

   Thus $\Xi_{\mathcal{P}}$ preserves all required structure.

Theorem 3.15: *Let* $\gamma: P_1 \to P_2$ be *a Priestley space negatively ordered commutative semi hypergroup isomorphism. Then* $T(L(\gamma)) \circ \pi_{P_1} = \pi_{P_2} \circ \gamma$.

Proof: Similarly to [12, Lemma 3.12].

Our following theorem establishes a duality correspondence in the Priestley style for algebraic structures combining:

- Bounded distributive hyperlattice properties,
- Commutative hypermonoid operations,
- Negative order relations.

Theorem 3.16: (Duality for Hyperlattice Structures). *There exists a dual equivalence between:*

(a) *The category of bounded distributive hyperlattices with negatively ordered commutative hypermonoid structures.*

(b) *The category of Priestley spaces carrying compatible negatively ordered commutative semihypergroup operations.*

Proof: The conclusions in Lemmas 3.10 and 3.14, as well as Theorems 3.13 and 3.15, immediately dictate this.

Theorem 3.17: *A contravariant equivalence exists between:*

(i) *The category $\mathcal{BDHL}$ consists of bounded distributive hyperlattice structures that are endowed with compatible negatively ordered commutative hypermonoid operations.*

(ii) *The category $\mathcal{PSH}$ consists of Priestley spaces equipped with negatively ordered commutative semihypergroup operations.*

Proof: The verification follows by synthesizing the essential components from:

(ii) The structural characterization in Lemma 3.10.

(iii) The representation theorem in Lemma 3.14.

(iv) The categorical framework established in Theorems 3.13 and 3.15.

4. Illustrative Cases

Example 4.1: Consider the bounded distributive lattice $\mathcal{E} = (\mathbb{E}, \wedge, \vee, \bot, \top)$ represented by Figure 2.

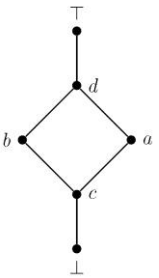

Figure 2
Hasse diagram representation of $\mathcal{E}$

We equip $\mathbb{E}$ with two hyperoperations defined as follows:

- The join hyperoperation: $\omega_1 \sqcup \omega_2 = \{r \in \mathbb{E} | r \vee \omega_1 = r \vee \omega_2 = \omega_1 \vee \omega_2\}$
- The meet hyperoperation:

$$\omega_1 \odot \omega_2 = \begin{cases} \downarrow (\omega_1 \wedge \omega_2) & \text{if } \omega_1 = \top \text{ or } \omega_2 = \top, \\ \{\bot\} & \text{otherwise.} \end{cases}$$

According to Theorem 2.9 (2), the structure $(\mathbb{E}, \wedge, \sqcup, \odot, \bot, \top)$ forms a bounded hyperlattice with negatively ordered commutative hypermonoid properties. The set $T(\mathbb{E})$ consists of all homomorphisms from $\mathbb{E}$ to the

two-elements lattice $\{\bot, \top\}$, yielding four distinct mappings: $\{\sigma_1, \sigma_2, \sigma_3, \sigma_4\} = X$, given by Table 1. This is what we call $T(\mathbb{E})$.

Table 1

Complete list of lattice homomorphisms

$\mathbb{E}$	$\bot$	c	a	b	d	$\top$
σ_1	0	0	0	0	0	1
σ_2	0	0	0	1	1	1
σ_3	0	0	1	0	1	1
σ_4	0	1	1	1	1	1

Its bidual is $L(T(\mathbb{E})) = \{\emptyset, \{\sigma_4\}, \{\sigma_3, \sigma_4\}, \{\sigma_2, \sigma_4\}, \{\sigma_2, \sigma_3, \sigma_4\}, X\}$. Then $(L(T(\mathbb{E})), \cap, \sqcup, \odot, \emptyset, X)$ is a bounded distributive hyperlattice negatively ordered commutative hypermonoid where,

$A \sqcup B = \{C \in L(T((\mathbb{E}))) | A \cup B = C \cup A = C \cup B\}$; and

$$A \odot B = \begin{cases} A \cap B \text{ if } A = X \text{ or } B = X, \\ \phi \text{ otherwise.} \end{cases}$$

for all $A, B \in L(T(\mathbb{E}))$.

Finally, $\phi_{\mathbb{E}}: \mathbb{E} \to L(T(\mathbb{E}))$ is given by Table 2.

Table 2

The homomorphism $\phi_{\mathbb{E}}$

$\mathbb{E}$	$\bot$	c	a	b	d	$\top$
$\phi_{\mathbb{E}}(a_i) \mid i = 1 \text{ to } 6$	$\emptyset$	$\{\sigma_4\}$	$\{\sigma_3, \sigma_4\}$	$\{\sigma_2, \sigma_4\}$	$\{\sigma_2, \sigma_3, \sigma_4\}$	X

is a bounded distributive hyperlattice negatively ordered commutative hypermonoid isomorphism.

Example 4.2: Let us examine a four-element Priestley space $\mathfrak{P} = (P, \tau, \preccurlyeq, \diamond)$ where $P = \{p, q, r, s\}, x \diamond y = \inf\{x, y\}$ and whose order relation is shown in Figure 3.

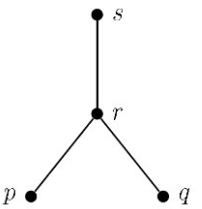

Figure 3

Poset structure of $\mathfrak{P}$

The collection of clopen increasing subsets forms the lattice $\mathcal{C}(P) = \{\emptyset, \{s\}, \{r, s\}, \{q, r, s\}, \{p, r, s\}, P\}$. We introduce two algebraic operations on $\mathcal{C}(P)$:

- The hyperjoin $\curlyvee$ is given by:
$$\Lambda_1 \curlyvee \Lambda_2 = \{\Lambda \in (\mathcal{C}(P) | \Lambda_1 \cup \Lambda_2 \subseteq \Lambda \text{ and } \Lambda_1 \cup \Lambda = \Lambda_2 \cup \Lambda\}.$$
- The hyperproduct $\boxtimes$ is defined as:
$$\Lambda_1 \boxtimes \Lambda_2 = \begin{cases} \downarrow (\Lambda_1 \cap \Lambda_2) & \text{if } \Lambda_1 = P \text{ or } \Lambda_2 = P, \\ \{\emptyset\} & \text{otherwise.} \end{cases}$$

By using Lemma 3.9, we derive that $(\mathcal{C}(P),\ \cap,\ \curlyvee,\ \boxtimes,\ \emptyset,\ P)$ constitutes a bounded hyperlattice possessing negatively ordered commutative hypermonoid characteristics. The complete set of structure-preserving maps $\mathcal{H}(\mathcal{C}(P)) = \{\gamma_1, \gamma_2, \gamma_3, \gamma_4\}$ is fully determined by Table 3.

Table 3

Structure-preserving mappings

$\mathcal{C}(P)$	$\emptyset$	$\{s\}$	$\{r,s\}$	$\{q,r,s\}$	$\{p,r,s\}$	P
γ_1	0	0	0	1	0	1
γ_2	0	0	0	0	1	1
γ_3	0	0	1	1	1	1
γ_4	0	1	1	1	1	1

The natural correspondence $\iota_P: P \to \mathcal{H}(\mathcal{C}(P))$ defined as:
$$\iota_P(q) = \gamma_1$$
$$\iota_P(p) = \gamma_2$$
$$\iota_P(r) = \gamma_3$$
$$\iota_P(s) = \gamma_4$$
yields an isomorphism within the category of Priestley spaces enhanced with negatively ordered commutative semihypergroup operations.

Conclusion

This paper establishes a Priestley-type duality between the category of bounded distributive hyperlattices equipped with a negatively ordered commutative hypermonoid structure and the category of Priestley spaces associated with negatively ordered commutative semihypergroups. To this end, we introduced and formalized key notions, including negatively ordered semihypergroups, hyperlattice-ordered monoids, hyperideals, and hyperfilters, which form the foundation of the duality framework.

Acknowledgement

We express our gratitude for the support received from the "General Directorate of Scientific Research and Technological Development (GDSRTD)", Algeria.

Dedication

We dedicate this work to the cherished memory of my dear colleague, brother, and student, Ali Oumhani. His presence, wisdom, and spirit continue to inspire me every day.

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Received December, 2024