

## **Classification of $n$ -dimensional generalized Heisenberg Lie algebras of rank $(n - 4)$**

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### **Abstract**

In this paper, we classify all  $n$ -dimensional generalized Heisenberg Lie algebras of rank  $(n - 4)$ , and to this end, we demonstrate that for any  $n$ -dimensional generalized Heisenberg Lie algebra, if the rank is  $(n - 4)$ , then such an algebra does not exist when  $n \geq 11$ . We also show that for a  $n$ -dimensional 2-step nilpotent Lie algebra  $L$  with the derived subalgebra of dimension  $r$ , we have  $r \leq n + \left\lceil \frac{1 - \sqrt{8n + 1}}{2} \right\rceil$ .

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### **1. Introduction**

Classification of algebras is one of the important and old topics in the theory of finite dimensional Lie algebras. The first work in the field related to the classification of Lie algebras on the field of complex numbers was done in 1896 by Cartan [1]. Then, Gantmacher classified them on the field of real numbers in 1939 [2]. The classification of solvable Lie algebras has been done for dimension  $n \leq 5$  in the field of real numbers and partially over the field of complex numbers in [3, 4]. The first research on nilpotent Lie algebras for six dimensions was done by Umlauf [5] in his thesis in the late 19th century, and then continued by

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Morozov [6]. Since then, many attempts have been made to complete the classification of nilpotent Lie algebras in various fields in the literature [7]. In dimension 7, the first classification was done according to Morozov's method [8]. However, due to the incompleteness of the results and the complexity of these calculations, the issue of calculating complete lists was a discussed issue in the literature. Seeley [8] and Gong [18] presented a classification of nilpotent Lie algebras of dimension 7 in complex ( $\mathbb{C}$ ) and real ( $\mathbb{R}$ ) spaces. In dimension 8, there are only partial results [9].

A Lie algebra  $A$  is called 2-step nilpotent if  $A$  is not Abelian and  $[A, A]$  is placed in the center of  $A$ . They are useful in studying some geometrical problems such as commutative Riemannian manifolds, weakly symmetric Riemannian manifolds, etc. [10]. For the 2-step nilpotent case, authors in [11] present a classification of 8-dimensional 2-step nilpotent Lie algebras with a 2-dimensional center. A complete classification of this type of Lie algebras with dimension 8 and in the complex numbers is given in [19]. The 9-dimensional 2-step nilpotent 3-ary Filippov algebras is studied in [12]. Furthermore, a classification of 2-step nilpotent Lie algebras with 9-dimension with 2-dimensional center is presented in [10]. In addition, rigid 2-step nilpotent in a variety of nilpotent Lie algebras has been carried out in [13]. We define a Lie algebra  $L$  to be a generalized Heisenberg algebra of rank  $k$  if, if  $L^2 = Z(L)$  and  $\dim L^2 = k$ . For more information about Heisenberg groups, the readers refer to [14, 15, 16]. There are the following result about classification of  $n$ -dimensional generalized Heisenberg Lie (GHL) algebra for  $n = 6, 7$  and 8 in the literature.

**Theorem 1:** ([17]). *If  $L$  is a 6-dimensional GHL of rank 2, then  $L$  is isomorphic to  $L_{6,22}(\varepsilon)$ .*

**Theorem 2:** ([18]). *If  $L$  is a 7-dimensional GHL algebra of rank 3, then  $L$  is isomorphic to 37A, 37B, 37C or 37D.*

**Theorem 3:** ([19]). *If  $L$  is an 8-dimensional GHL algebra of rank 4, then  $L$  is isomorphic to  $N_1^{8,4}$ ,  $N_2^{8,4}$  or  $N_3^{8,4}$ .*

We also have the following theorem about the GHL algebra of rank 1.

**Theorem 4:** ([20]). *Every Heisenberg Lie algebra has dimension  $2m + 1$ , and it is isomorphic to*

$$H(m) = \langle x, x_1, \dots, x_{2m} : [x_{2i-1}, x_{2i}] = x, i = 1, \dots, m \rangle.$$

*for some natural number  $m$ .*

In the next section, we describe a classification of all  $n$ -dimensional GHL algebras of rank  $(n - 4)$ , and show that there is no  $n$ -dimensional GHL algebra of rank  $(n - 4)$  for  $n \geq 11$ . Moreover, we show that if  $L$  be a  $n$ -dimensional GHL algebra of rank  $r$ , then  $r \leq n + \lfloor \frac{1-\sqrt{8n+1}}{2} \rfloor$ .

## 2. Main results

In this section, we categorize all  $n$ -dimensional generalized Heisenberg Lie algebras that have a rank of  $(n - 4)$ . To this end, we first classify 9-dimensional GHL algebras of rank 5. Let  $L$  be a 9-dimensional nilpotent Lie algebra with basis  $\{e_1, \dots, e_9\}$  and  $L^2 = Z(L) = \langle e_5, e_6, e_7, e_8, e_9 \rangle$ . Since,  $L/\langle e_9 \rangle$  is a 8-dimensional GHL algebra of rank 4, by using Theorem 3,  $A/\langle e_9 \rangle$  is isomorphic to  $N_1^{8,4}$ ,  $N_2^{8,4}$  or  $N_3^{8,4}$ . So, we can have the following lemmas.

**Lemma 1:** *Let  $L/\langle e_9 \rangle \cong N_1^{8,4}$ , then  $L$  is isomorphic to  $N_1^{9,5}$  or  $N_2^{9,5}$ .*

**Proof:** Let  $L/\langle e_9 \rangle \cong N_1^{8,4}$ . In this situation, the multiplication table for  $L$  can be expressed as

$$\begin{aligned} [e_1, e_2] &= e_5 + \alpha e_9, & [e_2, e_3] &= e_6 + \beta e_9, & [e_3, e_4] &= e_7 + \gamma e_9, \\ [e_1, e_4] &= e_8 + \lambda e_9, & [e_1, e_3] &= \alpha_1 e_9, & [e_2, e_4] &= \alpha_2 e_9. \end{aligned}$$

By applying an appropriate change of basis, it is possible to assume that  $\alpha = \beta = \gamma = \lambda = 0$ .

Since  $\dim(L/\langle e_5, e_6, e_7, e_8 \rangle)^2 = 1$ , we have  $L/\langle e_5, e_6, e_7, e_8 \rangle \cong H(1) \oplus A(1)$  or  $H(2)$ .

If  $L/\langle e_5, e_6, e_7, e_8 \rangle \cong H(1) \oplus A(1)$ , according to the structure of Heisenberg Lie algebras, the only one of the coefficients  $\alpha_1$  and  $\alpha_2$  is one and the others are zero. We can assume that  $\alpha_1 = 1$  and  $\alpha_2 = 0$ .

Thus,

$$L = \langle e_1, \dots, e_9 : [e_1, e_2] = e_5, [e_2, e_3] = e_6, [e_3, e_4] = e_7, [e_1, e_4] = e_8, [e_1, e_3] = e_9 \rangle.$$

This algebra is denoted by  $N_1^{9,5}$ .

If  $L/\langle e_5, e_6, e_7, e_8 \rangle \cong H(2)$ , Based on the characteristics of Heisenberg Lie algebras, we find that

$$\begin{aligned} L &\cong \langle e_1, \dots, e_9 : [e_1, e_2] = e_5, [e_2, e_3] = e_6, [e_3, e_4] = e_7, \\ &\quad [e_1, e_4] = e_8, [e_1, e_3] = [e_2, e_4] = e_9 \rangle. \end{aligned}$$

This algebra is denoted by  $N_2^{9,5}$ .

**Lemma 2:** *Let  $L/\langle e_9 \rangle \cong N_2^{8,4}$ , then  $L$  is isomorphic to  $N_1^{9,5}$ .*

**Proof:** Let  $L/\langle e_9 \rangle \cong N_2^{8,4}$ . In this situation, the multiplication table for  $L$  can be expressed as

$$\begin{aligned} [e_1, e_2] &= e_5 + \alpha e_9, & [e_1, e_3] &= e_6 + \beta e_9, & [e_2, e_3] &= e_7 + \gamma e_9, \\ [e_1, e_4] &= e_8 + \lambda e_9, & [e_2, e_4] &= \alpha_1 e_9, & [e_3, e_4] &= \alpha_2 e_9. \end{aligned}$$

By applying an appropriate change of basis, it is possible to assume that  $\alpha = \beta = \gamma = \lambda = 0$ .

Since  $\dim(L/\langle e_5, e_6, e_7, e_8 \rangle)^2 = 1$ , we have  $L/\langle e_5, e_6, e_7, e_8 \rangle \cong H(1) \oplus A(1)$ . Based on the characteristics of Heisenberg Lie algebras, the only one of the coefficients  $\alpha_1$  and  $\alpha_2$  is one, and the others are zero. We can assume that  $\alpha_1 = 1$  and  $\alpha_2 = 0$ . Thus,

$L = \langle e_1, \dots, e_9 : [e_1, e_2] = e_5, [e_2, e_3] = e_6, [e_3, e_4] = e_7, [e_1, e_4] = e_8, [e_1, e_3] = e_9 \rangle$ .  
This algebra is isomorphic to  $N_1^{9,5}$ .

**Lemma 3:** Let  $L/\langle e_9 \rangle \cong N_3^{8,4}$ , then  $L$  is isomorphic to  $N_1^{9,5}$  or  $N_2^{9,5}$ .

**Proof:** Let  $L/\langle e_9 \rangle \cong N_2^{8,4}$ . In this situation, the multiplication table for  $L$  can be expressed as

$$\begin{aligned} [e_1, e_2] &= e_5 + \alpha e_9, & [e_1, e_3] &= e_6 + \beta e_9, & [e_2, e_3] &= e_7 + \gamma e_9, \\ [e_1, e_4] &= e_8 + \lambda e_9, & [e_2, e_4] &= e_6 + \theta e_9, & [e_3, e_4] &= \delta e_9. \end{aligned}$$

By applying an appropriate change of basis, it is possible to assume that  $\alpha = \beta = \gamma = \lambda = 0$ . Thus, the multiplication in  $L$  can be written as follows:

$$\begin{aligned} [e_1, e_2] &= e_5, & [e_1, e_3] &= e_6, & [e_2, e_3] &= e_7, \\ [e_1, e_4] &= e_8, & [e_2, e_4] &= e_6 + \theta e_9, & [e_3, e_4] &= \delta e_9. \end{aligned}$$

Since  $\dim(L/\langle e_5, e_6, e_7, e_8 \rangle)^2 = 1$ , we have  $L/\langle e_5, e_6, e_7, e_8 \rangle \cong H(1) \oplus A(1)$ . According to the structure of Heisenberg Lie algebras, the only one of the coefficients  $\theta$  and  $\delta$  is one the others are zero. We have two possibilities:

(a) If  $\theta = 1, \delta = 0$ , we have the following algebra:

$$\begin{aligned} L \cong \langle e_1, \dots, e_9 : [e_1, e_2] &= e_5, [e_1, e_3] = e_6, [e_2, e_3] = e_7, \\ [e_1, e_4] &= e_8, [e_2, e_4] = e_6 + e_9 \rangle. \end{aligned}$$

By replacing  $e_9$  by  $e_6 + e_9$ , this algebra is isomorphic to

$$\begin{aligned} L &= \langle e_1, \dots, e_9 : [e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_2, e_3] = e_7, \\ [e_1, e_4] &= e_8, [e_2, e_4] = e_9 \rangle. \end{aligned}$$

This algebra is isomorphic to  $N_1^{9,5}$ .

(b) If  $\delta = 1, \theta = 0$ , we have the following algebra:

$$\begin{aligned} L \cong \langle e_1, \dots, e_9 : [e_1, e_2] &= e_5, [e_1, e_3] = [e_2, e_4] = e_6, \\ [e_2, e_3] &= e_7, [e_1, e_4] = e_8, [e_3, e_4] = e_9 \rangle. \end{aligned}$$

It can be easily seen that this algebra is isomorphic to  $N_2^{9,5}$ , and this completes the proof.

Now, we can have.

**Theorem 5:** Let  $L$  be an 9-dimensional GHL algebra of rank 5, then  $L$  is isomorphic to  $N_1^{9,5}$  or  $N_2^{9,5}$ .

**Proof:** From lemmas 1, 2 and 3, we can immediate consequence this theorem.

Now, we classify 10-dimensional GHL algebras of rank 6.

**Theorem 6:** If  $L$  be a 10-dimensional GHL algebra of rank 6, then  $L$  is isomorphic to  $N_1^{10,6}$ .

**Proof:** For a 10-dimensional nilpotent Lie algebra  $L$  with basis  $\{e_1, \dots, e_{10}\}$  such that  $L^2 = Z(L) = \langle e_5, e_6, e_7, e_8, e_9, e_{10} \rangle$ . Since  $L/\langle e_{10} \rangle$  is an 9-dimensional

GHL algebra of rank 5, by using Theorem 5,  $A/\langle e_{10} \rangle$  is isomorphic to  $N_1^{9,5}$  or  $N_2^{9,5}$ . We discuss following two cases.

**Case 1:** Let  $L/\langle e_{10} \rangle \cong N_1^{9,5}$ . In this situation, the multiplication table for  $L$  can be expressed as

$$\begin{aligned} [e_1, e_2] &= e_5 + \alpha e_{10}, & [e_1, e_3] &= e_6 + \beta e_{10}, & [e_2, e_3] &= e_7 + \gamma e_{10}, \\ [e_1, e_4] &= e_8 + \lambda e_{10}, & [e_2, e_4] &= e_9 + \delta e_{10}, & [e_3, e_4] &= \theta e_{10}. \end{aligned}$$

It can be assumed that  $\alpha = \beta = \gamma = \lambda = \delta = 0$ .

Since  $\dim(L/\langle e_5, e_6, e_7, e_8, e_9 \rangle)^2 = 1$ , we have  $L/\langle e_5, e_6, e_7, e_8, e_9 \rangle \cong H(1) \oplus A(2)$ . Based on the characteristics of Heisenberg Lie algebras,

$$\begin{aligned} L \cong \langle e_1, \dots, e_{10} : [e_1, e_2] &= e_5, [e_1, e_3] = e_6, [e_2, e_3] = e_7, [e_1, e_4] = e_8, \\ &= e_9, [e_3, e_4] = e_{10} \rangle. \end{aligned}$$

This algebra is denoted by  $N_1^{10,6}$ .

**Case 2:** Let  $L/\langle e_{10} \rangle \cong N_2^{9,5}$ . In this situation, the multiplication table in  $L$  can be written as

$$\begin{aligned} [e_1, e_2] &= e_5 + \alpha e_{10}, & [e_1, e_3] &= e_6 + \beta e_{10}, & [e_2, e_3] &= e_7 + \gamma e_{10}, \\ [e_1, e_4] &= e_8 + \lambda e_{10}, & [e_2, e_4] &= e_6 + \delta e_{10}, & [e_3, e_4] &= e_9 + \theta e_{10}. \end{aligned}$$

It can be assumed that  $\alpha = \beta = \gamma = \lambda = \theta = 0$ . Thus, the multiplication in  $L$  can be written as follows:

$$\begin{aligned} [e_1, e_2] &= e_5, & [e_1, e_3] &= e_6, & [e_2, e_3] &= e_7, \\ [e_1, e_4] &= e_8, & [e_2, e_4] &= e_6 + \delta e_{10}, & [e_3, e_4] &= e_9. \end{aligned}$$

Since  $\dim(L/\langle e_5, e_6, e_7, e_8, e_9 \rangle)^2 = 1$ , we have  $L/\langle e_5, e_6, e_7, e_8, e_9 \rangle \cong H(1) \oplus A(2)$ . According to the structure of Heisenberg Lie algebras,  $\delta = 1$ . By replacing  $e_{10}$  by  $e_6 + e_{10}$ , this algebra is isomorphic to

$$\begin{aligned} L \cong \langle e_1, \dots, e_{10} : [e_1, e_2] &= e_5, [e_1, e_3] = e_6, [e_2, e_3] = e_7, [e_1, e_4] = e_8, \\ &= e_9, [e_2, e_4] = e_{10} \rangle. \end{aligned}$$

This algebra is isomorphic to  $N_1^{10,6}$ , and this completes the proof.

In the following theorem, we show that there is no  $n$ -dimensional GHL algebras of rank  $(n - 4)$  when  $n \geq 11$ .

**Theorem 7:** *There is no  $n$ -dimensional GHL algebra of rank  $(n - 4)$  for  $n \geq 11$ .*

**Proof:** First, we prove that this theorem is correct for  $n = 11$ .

If  $L$  be an 11-dimensional nilpotent Lie algebra with basis  $\{e_1, \dots, e_{11}\}$  such that  $L^2 = Z(L) = \langle e_5, e_6, e_7, e_8, e_9, e_{10}, e_{11} \rangle$ . Since  $L/\langle e_{11} \rangle$  is a 10-dimensional GHL algebra of rank 6, by using Theorem 6,  $L/\langle e_{11} \rangle$  is isomorphic to  $N_1^{10,6}$ . In this situation, the multiplication table in  $L$  can be written as

$$\begin{aligned} [e_1, e_2] &= e_5 + \alpha e_{11}, & [e_1, e_3] &= e_6 + \beta e_{11}, & [e_2, e_3] &= e_7 + \gamma e_{11}, \\ [e_1, e_4] &= e_8 + \lambda e_{11}, & [e_2, e_4] &= e_9 + \delta e_{11}, & [e_3, e_4] &= e_{10} + \theta e_{11}. \end{aligned}$$

It can be assumed that  $\alpha = \beta = \gamma = \lambda = \delta = \theta = 0$ . Thus, the multiplication in  $L$  can be written as follows:

$$\begin{aligned} [e_1, e_2] &= e_5, & [e_1, e_3] &= e_6, & [e_2, e_3] &= e_7, \\ [e_1, e_4] &= e_8, & [e_2, e_4] &= e_9, & [e_3, e_4] &= e_{10}. \end{aligned}$$

In this algebra  $L^2 = \langle e_5, e_6, e_7, e_8, e_9, e_{10} \rangle$ , thus  $L^2 \neq Z(L)$ . Therefore, there is no 11-dimensional GHL algebra of rank 7. Now, if  $L$  be a 12-dimensional GHL algebra of rank 8, then  $L/\langle e_{12} \rangle$  is a 11-dimensional GHL algebra of rank 7. Similarly, this is a contradiction. The correctness of theorem is proven by continuing this process.

Therefore, we have:

**Theorem 8:** *Let  $L$  be a  $n$ -dimensional GHL algebra of rank  $(n - 4)$ , then  $L$  is isomorphic to  $H(2)$ ,  $L_{6,22}(\varepsilon)$ ,  $37A$ ,  $37B$ ,  $37C$ ,  $37D$ ,  $N_1^{8,4}$ ,  $N_2^{8,4}$ ,  $N_3^{8,4}$ ,  $N_1^{9,5}$ ,  $N_2^{9,5}$  or  $N_1^{10,6}$ .*

**Proof:** From Theorems 1, 2, 3, 5, 6 and 7, these results are obtained easily.

The obtained  $n$ -Lie algebras in this article have been listed in Table 1. Note that the obtained algebras are non-isomorphic.

Let  $L$  be an  $n$ -dimensional GHL algebra of rank  $r$ . Since  $\dim L/Z(L) = n - r$ , by using Lemma 14 of [21] we have:

$$r \leq \frac{1}{2}(n - r)(n - r - 1). \quad (1)$$

Therefore, we can have the following theorem.

**Theorem 9:** *For an  $n$ -dimensional 2-step nilpotent Lie algebra  $L$  with a derived subalgebra of dimension  $r$ , it follows that  $r \leq n + \lfloor \frac{1 - \sqrt{8n+1}}{2} \rfloor$ , and the equality is hold, if and only if  $\dim L = \frac{1}{2}t(t + 1)$  and  $\dim L^2 = \frac{1}{2}t(t - 1)$  for some  $t \geq 2$  and  $L = \langle e_1, \dots, e_t, y_{ij} : y_{ij} = [e_i, e_j], 1 \leq i < j \leq t \rangle$ .*

**Proof:** By using Lemma 4.5 of [22], there exist a generalized Heisenberg Lie algebra  $H$  and abelian Lie algebra  $A$  such that  $L = H \oplus A$ . As  $\dim H^2 = \dim L^2 = r$ , by using relation (1),  $r \leq \frac{1}{2}(h - r)(h - r - 1)$  when  $\dim H = h$ . Thus  $r \leq \frac{1}{2}(n - r)(n - r - 1)$ , and then  $r^2 - (2n + 1)r + n^2 - n \geq 0$ . By solving this inequality, we get  $rn + \lfloor \frac{1 - \sqrt{8n+1}}{2} \rfloor$ . The equality is hold, if and only if  $r = \frac{2n+1-\sqrt{8n+1}}{2}$ . Expression  $8n + 1$  is a perfect square if and only if  $n = \frac{1}{2}t(t + 1)$  for some integer  $t \geq 2$ . Thus, equality is hold, if and only if  $r = \frac{1}{2}t(t - 1)$ . Finally, the structure of this Lie algebras is  $L = \langle e_1, \dots, e_t, y_{ij} : y_{ij} = [e_i, e_j], 1 \leq i < j \leq t \rangle$ .

**Table 1**  
**List of all algebras obtained for  $n$ -dimensional GHL algebra of rank  $(n - 4)$ .**

| Name                 | Non-zero multiplication                                                                                         |
|----------------------|-----------------------------------------------------------------------------------------------------------------|
| $L_{6,22}(\epsilon)$ | $[e_1, e_2] = [e_3, e_4] = e_5, [e_1, e_3] = e_6, [e_2, e_4] = \epsilon e_6$                                    |
| $37A$                | $[e_1, e_2] = e_5, [e_2, e_3] = e_6, [e_2, e_4] = e_7$                                                          |
| $37B$                | $[e_1, e_2] = [e_3, e_4] = e_5, [e_2, e_3] = e_6, [e_3, e_4] = e_7$                                             |
| $37C$                | $[e_1, e_2] = [e_3, e_4] = e_5, [e_2, e_3] = e_6, [e_2, e_4] = e_7$                                             |
| $37D$                | $[e_1, e_2] = [e_3, e_4] = e_5, [e_1, e_3] = e_6, [e_2, e_4] = e_7$                                             |
| $N_1^{8,4}$          | $[e_1, e_2] = e_5, [e_2, e_3] = e_6, [e_3, e_4] = e_7, [e_1, e_4] = e_8$                                        |
| $N_2^{8,4}$          | $[e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_2, e_3] = e_7, [e_1, e_4] = e_8$                                        |
| $N_3^{8,4}$          | $[e_1, e_2] = e_5, [e_2, e_3] = e_6, [e_3, e_4] = e_7, [e_1, e_3] = [e_2, e_4] = e_8$                           |
| $N_1^{9,5}$          | $[e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_2, e_3] = e_7, [e_1, e_4] = e_8, [e_2, e_4] = e_9$                      |
| $N_2^{9,5}$          | $[e_1, e_2] = e_5, [e_1, e_3] = e_2, e_4] = e_6, [e_2, e_3] = e_7, [e_1, e_4] = e_8, [e, e_4] = e_9$            |
| $N_1^{10,6}$         | $[e_1, e_2] = e_5, [e_1, e_3] = e_6, [e_2, e_3] = e_7, [e_1, e_4] = e_8, [e_2, e_4] = e_9, [e_3, e_4] = e_{10}$ |

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