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Bi-ideals and quasi-ideals on ternary semigroups of transformations

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Abstract

Let $(T_\Omega, [\ ])$ be a ternary semigroup of transformations on a nonempty set Ω, under a ternary operation $[\ ]$ defined by $[\alpha\beta\gamma] = \alpha\beta\gamma$ for all $\alpha, \beta, \gamma \in T_\Omega$. The present work investigates and identifies those ternary subsemigroups of T_Ω for which the collection of bi-ideals and the collection of quasi-ideals turn out to be identical.

Subject Classification: *20N10, 20M20.*

Keywords: *Transformation, Ternary semigroup, bi-ideal, Quasi-ideal.*

1. Introduction

A ternary semigroup was given by Lehmer [6]. Similarly to ideal theory on semigroups, ideal theory on ternary semigroups can lead to interesting algebraic insights and has applications in areas such as algebraic logic, information theory, and other branches of mathematics that explore generalized algebraic structures such as po-ternary semihypergroups (e.g., [8]). In this work we study a ternary semigroup of transformation for which the collection of bi-ideals and the collection of quasi-ideals turn out to be identical. Notions and definitions not provided in this work can be consulted in [1, 2, 7].

From now on, Let Θ be a ternary semigroup.

Remark 1.1: If Q is a quasi-ideal of Θ, then it is also a bi-ideal.

Relations between regularity and ideals in ternary semigroups is similar to the relation in semigroup but require complex interactions more than binary operations.

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Definition 1.2: ([7]) Let $a \in \Theta$. Then a is regular if $a = [axaya]$ for some $x, y \in \Theta$. We say that Θ is regular if a is regular for all $a \in \Theta$.

Remark 1.3: $a \in \Theta$ is regular if and only if $a = [axa]$ for some $x \in \Theta$.

Remark 1.4: If Θ is a regular ternary semigroup, then the collection of all bi-ideals and the collection of all quasi-ideals coincide.

From now on, the collection of all ternary semigroups whose bi-ideals are quasi-ideals is denoted by BQ.

Lemma 1.5: *Every regular ternary semigroup belongs to* BQ.

Let T_Ω be the set of all transformations on a nonempty set Ω. Define a ternary operation $[]$ on T_Ω by

$$[\alpha\beta\gamma] = \alpha\beta\gamma$$

for all $\alpha, \beta, \gamma \in T_\Omega$. Then $(T_\Omega, [])$ becomes a ternary semigroup. If $\alpha \in T_\Omega$, then the range of α is written by $ran\ \alpha$. From now on, we define

$$\begin{aligned}
T_1(\Omega) &= \{\alpha \in T_\Omega | \text{ran } \alpha \text{ is finite}\};\\
T_2(\Omega) &= \{\alpha \in T_\Omega | \Omega \backslash \text{ran } \alpha \text{ is infinite}\};\\
T_3(\Omega) &= \{\alpha \in T_\Omega | \alpha \text{ is one} - \text{to} - \text{one and } \Omega \backslash \text{ran } \alpha \text{ is infinite}\};\\
T_4(\Omega) &= \{\alpha \in T_\Omega | \Omega \backslash \text{ran } \alpha \text{ is finite}\};\\
T_5(\Omega) &= \{\alpha \in T_\Omega | \alpha \text{ is one} - \text{to} - \text{one and } \Omega \backslash \text{ran } \alpha \text{ is finite}\}.
\end{aligned}$$

Then they become ternary semigroups under a ternary operation $[]$. Throughout this paper, T_Ω is a ternary semigroup defined above unless we state otherwise.

2. Main Theorem

Lemma 2.1: *Let* Θ *be a ternary semigroup. Then* $\Theta \in BQ$ *if and only if*

$$B(\{s,t,w\}) = Q(\{s,t,w\})$$

for all $s, t, w \in \Theta$.

Proof: Assume that $\Theta \in BQ$. Let $s, t, w \in \Theta$. We have that $B(\{s,t,w\}) \subseteq Q(\{s,t,w\})$ since $Q(\{s,t,w\})$ is a bi-ideal of Θ containing $\{s,t,w\}$. Similarly, $Q(\{s,t,w\}) \subseteq B(\{s,t,w\})$ since $B(\{s,t,w\})$ is a quasi-ideal of Θ containing $\{s,t,w\}$. Thus, $B(\{s,t,w\}) = Q(\{s,t,w\})$.

Conversely, we obtain from Remark 1.1 that every quasi-ideal is a bi-ideal. Let B be a bi-ideal of Θ. For any $b \in [B\Theta\Theta] \cap [\Theta B\Theta] \cap [\Theta\Theta B]$, we have

$$b \in [s\Theta\Theta] \cap [\Theta t\Theta] \cap [\Theta\Theta w]$$

for some $s, t, w \in B$. Thus,

$$\begin{aligned}
b &\in [Q(\{s,t,w\})\Theta\Theta] \cap [\Theta Q(\{s,t,w\})\Theta] \cap [\Theta\Theta Q(\{s,t,w\})]\\
&\subseteq Q(\{s,t,w\})\\
&= B(\{s,t,w\})\\
&\subseteq B.
\end{aligned}$$

Therefore, $[B\Theta\Theta] \cap [\Theta B\Theta] \cap [\Theta\Theta B] \subseteq B$. Let $b' \in [B\Theta\Theta] \cap [\Theta\Theta B\Theta\Theta] \cap [\Theta\Theta B]$. Then

$$b' \in [s'\Theta\Theta] \cap [\Theta\Theta t'\Theta\Theta] \cap [\Theta\Theta w']$$

for some $s', t', w' \in \Theta$. It follows that

$$\begin{aligned} b' &\in [Q(\{s', t', w'\})\Theta\Theta] \cap [\Theta\Theta Q(\{s', t', w'\})\Theta\Theta] \cap [\Theta\Theta Q(\{s', t', w'\})] \\ &\subseteq Q(\{s', t', w'\}) \\ &= B(\{s', t', w'\}) \\ &\subseteq B. \end{aligned}$$

Therefore, $[B\Theta\Theta] \cap [\Theta\Theta B\Theta\Theta] \cap [\Theta\Theta B] \subseteq B$. We infer that B is a quasi-ideal of Θ. Hence, $\Theta \in BQ$.

Theorem 2.2: $T_1(\Omega)$ *is a regular ternary semigroup.*

Proof: Let $\alpha \in T_1(\Omega)$. For any $y \in \mathrm{ran}\,\alpha$, we let x_y be a fixed element in $y\alpha^{-1}$. Then define $\beta: \Omega \to \Omega$ by

$$y\beta = \begin{cases} x_y & \text{if } y \in \mathrm{ran}\,\alpha \\ y^* & \text{if } y \notin \mathrm{ran}\,\alpha, \end{cases}$$

where y^* is a fixed element of Ω. We have that $\beta \in T_1(\Omega)$ and $\alpha = [\alpha\beta\alpha\beta\alpha]$. Thus, $T_1(\Omega)$ is regular.

Corollary 2.3: $T_1(\Omega) \in BQ$.

Theorem 2.4: *If* Ω *is infinite set, then* $T_2(\Omega) \in BQ$.

Proof: Assume that Ω is infinite. Let B be a bi-ideal of $T_2(\Omega)$. For any $\alpha \in [BT_2(\Omega)T_2(\Omega)] \cap [T_2(\Omega)BT_2(\Omega)] \cap [T_2(\Omega)T_2(\Omega)B]$, we have that

$$\alpha = [\beta_1\gamma_1\gamma_1'] = [\gamma_2\beta_2\gamma_2'] = [\gamma_3\gamma_3'\beta_3]$$

for some $\beta_1, \beta_2, \beta_3 \in B$ and $\gamma_1, \gamma_2, \gamma_3, \gamma_1', \gamma_2', \gamma_3' \in T_2(\Omega)$. Since T_Ω is regular, there exist $\eta_1, \eta_2 \in T_\Omega$ such that $\beta_1 = [\beta_1\eta_1\beta_1]$ and $[\gamma_3\gamma_3'\beta_3] = [[\gamma_3\gamma_3'\beta_3]\eta_2 [\gamma_3\gamma_3'\beta_3]]$. It follows that

$$\begin{aligned} \alpha &= [\beta_1\gamma_1\gamma_1'] \\ &= [[\beta_1\eta_1\beta_1]\gamma_1\gamma_1'] \\ &= [\beta_1\eta_1[\gamma_3\gamma_3'\beta_3]] \\ &= [\beta_1\eta_1[[\gamma_3\gamma_3'\beta_3]\eta_2[\gamma_3\gamma_3'\beta_3]]] \\ &= [\beta_1[\eta_1\gamma_3\gamma_3']\beta_3[\eta_2\gamma_3\gamma_3']\beta_3]] \\ &\in BT_2(\Omega)BT_2(\Omega)B \\ &\subseteq B. \end{aligned}$$

Therefore, $[BT_2(\Omega)T_2(\Omega)] \cap [T_2(\Omega)BT_2(\Omega)] \cap [T_2(\Omega)T_2(\Omega)B] \subseteq B$. We can prove similarly that

$$[BT_2(\Omega)T_2(\Omega)] \cap [T_2(\Omega)T_2(\Omega)BT_2(\Omega)T_2(\Omega)] \cap [T_2(\Omega)T_2(\Omega)B] \subseteq B.$$

Hence, B is a quasi-ideal of $T_2(\Omega)$.

It was proved in [4] that Θ is a right simple ternary semigroup if and only if $[a\Theta\Theta] = \Theta$ for all $a \in T$. We know that if S is a right simple semigroup, then a ternary semigroup, induced by S with a ternary operation defined by

$$[abc] = (ab)c$$

for all $a, b, c \in S$, becomes a right simple ternary semigroup. In 1992, Higgins [3] proved that if Ω is a countably infinite set, then $T_3(\Omega)$ is a right simple semigroup under the composition. Then we have the following lemma:

Lemma 2.5: *If Ω is a countably infinite set, then $T_3(\Omega)$ is a right simple ternary semigroup.*

In 1969, Kapp [5] prove that if S is right simple, then the set of bi-ideals of S coincide the set of quasi-ideals of S. We apply this concept to prove the following theorem.

Theorem 2.6: $T_3(\Omega) \in BQ$ if and only if Ω is countably infinite.

Proof: Assume that Ω is countably infinite. By Lemma 2.5, $T_3(\Omega)$ is a right simple ternary semigroup. Let B be a bi-ideal of $T_3(\Omega)$. Then

$$\begin{aligned}[BT_3(\Omega)T_3(\Omega)] \cap [T_3(\Omega)BT_3(\Omega)] \cap [T_3(\Omega)T_3(\Omega)B] &\subseteq [T_3(\Omega)T_3(\Omega)B] \\ &= [BT_3(\Omega)T_3(\Omega)T_3(\Omega)B] \\ &= [BT_3(\Omega)BT_3(\Omega)T_3(\Omega)T_3(\Omega)B] \\ &= [BT_3(\Omega)BT_3(\Omega)B] \\ &\subseteq B.\end{aligned}$$

We can prove on the same way that

$$[BT_3(\Omega)T_3(\Omega)] \cap [T_3(\Omega)T_3(\Omega)BT_3(\Omega)T_3(\Omega)] \cap [T_3(\Omega)T_3(\Omega)B] \subseteq B.$$

Therefore, B is a quasi-ideal of $T_3(\Omega)$.

Conversely, assume that Ω is uncountable. Then there exists an infinite subset Σ of Ω such that

$$|\Omega\backslash\Sigma| = |\Omega| = |\Sigma|.$$

Thus, there is a bijection $\alpha: \Omega \to \Omega\backslash\Sigma$. Let Υ be a countably infinite subset of Ω. Since Ω is uncountable, we obtain $|\Omega\backslash\Upsilon| = |\Omega|$. Then there exists a bijection $\beta: \Omega \to \Omega\backslash\Upsilon$. It is clear that $\alpha, \beta \in T_3(\Omega)$ and $[\alpha^{-1}\beta\alpha]: \Omega\backslash\Sigma \to \Omega\backslash\Sigma$ is injective. Since Υ is infinite, there exists $\Psi_1 \subseteq \Upsilon$ such that

$$|\Upsilon\backslash\Psi_1| = |\Psi_1| = |\Upsilon|.$$

Then there is a bijection $\phi_1: \Upsilon \to \Upsilon\backslash\Psi_1$. Let $\varepsilon: \Omega \to \Omega$ be an identity transformation. Define a transformation $\beta^*: \Omega \to \Omega$ by

$$x\beta^* = \begin{cases} x\varepsilon & \text{if } x \in \Omega\backslash\Upsilon \\ x\phi_1 & \text{if } x \in \Upsilon \end{cases}.$$

We have $\beta^* \in T_3(\Omega)$. Since Σ is infinite, there exists $\Phi \subseteq \Sigma$ such that

$$|\Sigma\backslash\Phi| = |\Phi| = |\Sigma|.$$

Then there is a bijection $\phi_2: \Sigma \to \Phi$. Define a transformation $\gamma: \Omega \to \Omega$ by

$$x\gamma = \begin{cases} x\alpha^{-1}\beta\alpha & \text{if } \in \Omega\backslash\Sigma \\ x\phi_2 & \text{if } x \in \Sigma. \end{cases}$$

Since $(\Omega\backslash\Sigma) \cap \Phi = \emptyset$, it follows that γ is injective. Since $ran\ \gamma \subseteq (\Omega\backslash\Sigma) \cup \Phi$, we obtain $\Sigma\backslash\Phi \subseteq \Omega\backslash\text{ran}\ \gamma$. Thus,

$$|\Omega| = |\Sigma\backslash\Phi| \leq |\Omega\backslash\text{ran}\ \gamma|.$$

This implies that $\Omega\backslash\text{ran}\ \gamma$ is infinite and hence $\gamma \in T_3(\Omega)$. Since $\Omega\backslash ran\ \gamma$ is infinite, there exists $\Psi_2 \subseteq \Omega\backslash\text{ran}\ \gamma$ such that

$$|\Omega\backslash\text{ran}\ \gamma| = |\Psi_2| = |(\Omega\backslash\text{ran}\ \gamma)\backslash\Psi_2|.$$

Then there exists a bijection $\phi_3 : \Omega\backslash\text{ran}\ \gamma \to (\Omega\backslash\text{ran}\ \gamma)\backslash\Psi_2$. Define a transformation $\gamma^* : \Omega \to \Omega$ by

$$x\gamma^* = \begin{cases} x\varepsilon & \text{if } x \in \text{ran}\ \gamma \\ x\phi_3 & \text{if } x \in \Omega\backslash\text{ran}\ \gamma. \end{cases}$$

It is obvious that $\gamma^* \in T_3(\Omega)$. Moreover, for any $x \in \Omega$, we have that

$$\begin{aligned} x\alpha\gamma\gamma^* &= x\alpha\gamma\varepsilon \\ &= x\alpha\alpha^{-1}\beta\alpha\varepsilon \\ &= x\beta\alpha\varepsilon \\ &= x\beta\varepsilon\alpha \\ &= x\beta\beta^*\alpha. \end{aligned}$$

Thus, $[\alpha\gamma\gamma^*] = [\beta\beta^*\alpha]$. Define a transformation $\lambda : \Omega \to \Omega$ by

$$x\lambda = \begin{cases} x\varepsilon & \text{if } x \in \Omega\backslash\Sigma \\ x\phi_2 & \text{if } x \in \Sigma. \end{cases}$$

Since $\Phi \cap (\Omega\backslash\Sigma) = \emptyset$, it follows that λ is injective. Since $ran\ \lambda = (\Omega\backslash\Sigma) \cup \Phi$, we have that $\Sigma\backslash\Phi \subseteq \Omega\backslash\text{ran}\ \lambda$. Thus,

$$|\Omega| = |\Sigma\backslash\Phi| \leq |\Omega\backslash\text{ran}\ \lambda|.$$

This implies that $\Omega\backslash\text{ran}\ \lambda$ is infinite and hence, $\lambda \in T_3(\Omega)$. It can be observed that for any $x \in \Omega$,

$$x\beta\alpha\lambda = x\beta\alpha\varepsilon = x\beta\varepsilon\alpha = x\beta\beta^*\alpha.$$

Therefore, $[\alpha\gamma\gamma^*] = [\beta\beta^*\alpha] = [\beta\alpha\lambda]$. This implies that $\beta\beta^*\alpha \in Q(\alpha)$. Now, we consider the following cases:

Case 1: If $[\beta\beta^*\alpha] = \alpha$, then for any $x \in \Omega$,

$$x = x\alpha\varepsilon\alpha^{-1} = x\beta\beta^*\alpha\varepsilon\alpha^{-1} = x\beta\varepsilon\alpha\varepsilon\alpha^{-1} = x\beta\varepsilon\varepsilon = x\beta.$$

Thus, β is an identity transformation. This contradicts to $\beta \in T_3(\Omega)$.

Case 2: If $[\beta\beta^*\alpha] = [\alpha^3]$, then

$$\beta = [\beta\varepsilon\alpha\alpha^{-1}\varepsilon] = [\beta\beta^*\alpha\alpha^{-1}\varepsilon] = [\alpha^3\alpha^{-1}\varepsilon] = [\alpha\alpha\varepsilon].$$

Thus, $\text{ran}\ \beta = \text{ran}\ \alpha\alpha\varepsilon$. Since $\text{ran}\ \alpha\alpha\varepsilon \subseteq \text{ran}\ \alpha$, we obtain that $\Omega\backslash\text{ran}\ \alpha \subseteq \Omega\backslash\text{ran}\ \alpha\alpha\varepsilon$. This implies that $|\Sigma| = |\Omega\backslash\text{ran}\ \alpha| \leq |\Omega\backslash\text{ran}\ \alpha\alpha\varepsilon|$. Therefore, $\Omega\backslash\text{ran}\ \alpha\alpha\varepsilon$ is uncountable. This contradicts to $|\Omega\backslash\text{ran}\ \alpha\alpha\varepsilon| = |\Omega\backslash\text{ran}\ \beta| = |\Upsilon|$.

Case 3: If $[\beta\beta^*\alpha] = [\alpha\varphi_1\alpha\varphi_2\alpha]$ for some $\varphi_1, \varphi_2 \in T_3(\Omega)$, then

$$\begin{aligned}\beta &= \beta\varepsilon\alpha\alpha^{-1}\varepsilon \\ &= \beta\beta^*\alpha\alpha^{-1}\varepsilon \\ &= \alpha\varphi_1\alpha\varphi_2\alpha\alpha^{-1}\varepsilon \\ &= \alpha\varphi_1\alpha\varphi_2\varepsilon.\end{aligned}$$

Thus,

$$\begin{aligned}\Upsilon &= \Omega\backslash\operatorname{ran}\beta \\ &= \Omega\backslash\operatorname{ran}\alpha\varphi_1\alpha\varphi_2\varepsilon \\ &= \Omega\backslash\operatorname{ran}\alpha\varphi_1\alpha\varepsilon\varphi_2 \\ &\supseteq \Omega\varphi_2\backslash\operatorname{ran}\alpha\varphi_1\alpha\varepsilon\varphi_2 \\ &= (\Omega\backslash\operatorname{ran}\alpha\varphi_1\alpha)\varphi_2 \\ &\supseteq (\Omega\backslash\operatorname{ran}\alpha)\varphi_2 \\ &= \Sigma\varphi_2.\end{aligned}$$

This implies that $|\Sigma\varphi_2| \leq |\Upsilon|$. Since Σ is uncountable and φ_2 is injective, we have that $\Sigma\varphi_2$ is uncountable and so Υ is uncountable. This contradicts to the determination of Υ.

By the above three cases, we conclude that $\beta\beta^*\alpha \notin B(\alpha)$. Therefore, $T_3(\Omega) \notin BQ$.

Theorem 2.7: *$T_4(\Omega) \in BQ$ if and only if Ω is finite.*

Proof: Assume that Ω is finite. Then $T_4(\Omega)$ is regular. This implies that $T_4(\Omega) \in BQ$. Conversely, Assume that Ω is infinite. Let

$A = \{b_n | n \in \mathbb{N}\}$ where $b_i \neq b_j$ if $i \neq j$.

Define $\alpha: \Omega \to \Omega$ by

$$x\alpha = \begin{cases} b_{\frac{n}{2}} & \text{if } x = b_n \text{ for some even } n \\ b_1 & \text{if } x = b_n \text{ for some odd } n \\ x & \text{if otherwise.} \end{cases}$$

Then $\operatorname{ran}\alpha = \Omega$ which implies that $\Omega\backslash\operatorname{ran}\alpha = \emptyset$. Thus, $\alpha \in T_4(\Omega)$. Define $\beta: \Omega \to \Omega$ by

$$x\beta = \begin{cases} b_{n-2} & \text{if } x = b_n \text{ for some } n \in \mathbb{N}\backslash\{1,2\} \\ b_1 & \text{if } x = b_2 \\ x & \text{if otherwise.} \end{cases}$$

Then $\operatorname{ran}\beta = \Omega$ and $\beta \in T_4(\Omega)$. Define $\gamma: \Omega \to \Omega$ by

$$x\gamma = \begin{cases} b_{n-1} & \text{if } x = b_n \text{ for some } n \in \mathbb{N}\backslash\{1\} \\ x & \text{if otherwise.} \end{cases}$$

Then $\operatorname{ran}\gamma = \Omega$ and $\gamma \in T_4(\Omega)$. Let $x \in \Omega$. If $x \in \Omega\backslash A$, then

$$x\beta\alpha\varepsilon = x\beta\varepsilon\alpha = x = x\alpha\varepsilon\gamma.$$

If $x = b_2$, then

$$x\beta\alpha\varepsilon = x\beta\varepsilon\alpha = b_2\beta\varepsilon\alpha = b_1 = b_2\alpha\varepsilon\gamma = x\alpha\varepsilon\gamma.$$

If $x \in \{b_{2n-1} | n \geq 1\}$, then

$$x\beta\alpha\varepsilon = x\beta\varepsilon\alpha = b_{2n-1}\beta\varepsilon\alpha = b_1 = b_{2n-1}\alpha\varepsilon\gamma = x\alpha\varepsilon\gamma.$$

If $x \in \{b_{2n} | n \geq 2\}$, then

$$x\beta\alpha\varepsilon = x\beta\varepsilon\alpha = b_{2n}\beta\varepsilon\alpha = b_{n-1} = b_{2n}\alpha\varepsilon\gamma = x\alpha\varepsilon\gamma.$$

Therefore, $\beta\alpha\varepsilon = \beta\varepsilon\alpha = \alpha\varepsilon\gamma \in Q(\alpha)$. Since $b_6\alpha = b_3$ and $b_6\beta\varepsilon\alpha = b_2$, it follows that $\alpha \neq \beta\varepsilon\alpha$. Since $b_3\alpha\alpha\alpha = b_1$, it follows that $\beta\varepsilon\alpha \neq \alpha^3$. Suppose that $\alpha\varepsilon\gamma \in B(\alpha)$. Then $\alpha\varepsilon\gamma = \alpha\lambda_1\alpha\lambda_2\alpha$ for some $\lambda_1, \lambda_2 \in T_4(\Omega)$. Let $y \in \Omega$. Then there exists $x \in \Omega$ such that $y = y\varepsilon = x\alpha$. That is

$$y\gamma = x\alpha\varepsilon\gamma = x\alpha\lambda_1\alpha\lambda_2\alpha = y\varepsilon\lambda_1\alpha\lambda_2\alpha.$$

Thus, $\gamma = [\varepsilon\lambda_1\alpha\lambda_2\alpha]$. If $x \in \Omega\backslash A$, then

$$x = x\gamma = x\varepsilon\lambda_1\alpha\lambda_2\alpha.$$

By definition of α, we have that $x\lambda_1\alpha\lambda_2 = x$. If $x = b_n$ for some $n \geq 3$, then

$$b_{n-1} = b_n\gamma = b_n\varepsilon\lambda_1\alpha\lambda_2\alpha.$$

By definition of α, we have $x\lambda_1\alpha\lambda_2 = b_{2(n-1)}$. Thus,

$$\text{ran}\ \lambda_1\alpha\lambda_2 = \Omega\backslash A \cup \{b_{2(n-1)} | n \geq 3\} \cup \{b_1\lambda_1\alpha\lambda_2, b_2\lambda_1\alpha\lambda_2\}.$$

Therefore,

$$\begin{aligned}\Omega\backslash\text{ran}\ \lambda_1\alpha\lambda_2 &= A \cap (\Omega\backslash\{b_{2(n-1)} | n \geq 3\}) \cap (\Omega\backslash\{b_1\lambda_1\alpha\lambda_2, b_2\lambda_1\alpha\lambda_2\}) \\ &= (\{b_1, b_2\} \cup \{b_{2n+1} | n \geq 1\})\backslash\{b_1\lambda_1\alpha\lambda_2, b_2\lambda_1\alpha\lambda_2\}.\end{aligned}$$

We can observe that $\Omega\backslash\text{ran}\ \lambda_1\alpha\lambda_2$ is infinite which contradicts to $\lambda_1\alpha\lambda_2 \in T_4(\Omega)$. Therefore, $\alpha\varepsilon\gamma \notin B(\alpha)$. Hence, $T_4(\Omega) \notin BQ$.

Theorem 2.8: $T_5(\Omega) \in BQ$ *if and only if* Ω *is finite.*

Proof: Assume that Ω is finite. Then $T_5(\Omega)$ is regular. Thus, $T_5(\Omega) \in BQ$. Conversely, Assume that Ω is infinite. Let

$$A = \{b_n | n \in \mathbb{N}\} \text{ where } b_i \neq b_j \text{ if } i \neq j.$$

Define $\alpha: \Omega \to \Omega$ by

$$x\alpha = \begin{cases} b_{n+3} & \text{if } x = b_n \text{ for some } n \in \mathbb{N} \\ x & \text{otherwise.} \end{cases}$$

Then α is injective and $\Omega\backslash\text{ran}\ \alpha = \{b_1, b_2, b_3\}$. Define $\beta: \Omega \to \Omega$ by

$$x\beta = \begin{cases} b_{n+1} & \text{if } x = b_n \text{ for some } n \in \mathbb{N}\backslash\{1\} \\ x & \text{otherwise.} \end{cases}$$

Then β is injective and $\Omega\backslash\text{ran}\ \beta = \{b_2\}$. Define $\gamma: \Omega \to \Omega$ by

$$x\gamma = \begin{cases} b_{n+1} & \text{if } x = b_n \text{ for some } n \in \mathbb{N}\backslash\{1,2,3,4\} \\ x & \text{otherwise.} \end{cases}$$

Then γ is injective and $\Omega\backslash\text{ran}\ \gamma = \{b_5\}$. Thus, $\alpha, \beta, \gamma \in T_5(\Omega)$. By definitions of α, β and γ, we have that

$$x\beta\varepsilon\alpha = x\beta\alpha\varepsilon = \begin{cases} x & \text{if } x \in \Omega\backslash A \\ b_4 & \text{if } x = b_1 \\ b_{n+4} & \text{if } x \in A\backslash\{b_1\}. \end{cases}$$

Thus, $\beta\varepsilon\alpha = \beta\alpha\varepsilon = \alpha\gamma\varepsilon \in Q(\alpha)$. Suppose that $\beta\varepsilon\alpha \in B(\alpha)$. Since $b_2\beta\varepsilon\alpha = b_6, b_2\alpha = b_5$ and $b_2\alpha^3 = b_{11}$, we obtain that

$$\beta\varepsilon\alpha \neq \alpha \text{ and } \beta\varepsilon\alpha \neq \alpha^3.$$

Thus, $\beta\varepsilon\alpha = \alpha\lambda_1\alpha\lambda_2\alpha$ for some $\lambda_1, \lambda_2 \in T_5(\Omega)$. Since α is injective, it follows that

$$\beta = \alpha\lambda_1\alpha\lambda_2\varepsilon = \alpha\varepsilon\lambda_1\alpha\lambda_2.$$

Since $b_1, b_2, b_3 \notin ran\ \alpha$ and $\lambda_1\alpha\lambda_2$ is injective, we have that

$$b_1\lambda_1\alpha\lambda_2, b_2\lambda_1\alpha\lambda_2, b_3\lambda_1\alpha\lambda_2 \notin \text{ran } \alpha\varepsilon\lambda_1\alpha\lambda_2.$$

Thus, $|\Omega\backslash\text{ran } \alpha\varepsilon\lambda_1\alpha\lambda_2| \geq 3$. . This contradicts to $1 = |\Omega\backslash\text{ran } \beta| = |\Omega\backslash\text{ran } \alpha\varepsilon\lambda_1\alpha\lambda_2|$. Therefore, $\beta\varepsilon\alpha \notin B(\alpha)$. Hence, $T_5(\Omega) \notin BQ$.

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