

A class of $\mathbb{Z}_r$ -simplex codes and its gray images

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Abstract

The various parameters of $\mathbb{Z}_r$ -Simplex code under the Homogeneous and Lee weight metric for some particular cases of r with rank k are considered. Some fundamental properties like weight distributions of $\mathbb{Z}_r$ -Simplex code are determined under the same metric. It is

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demonstrated that the Gray map generates several intriguing p -ary nonlinear codes of $\mathbb{Z}_r$ -Simplex codes.

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1. Introduction

Linear codes utilizing finite chain rings, algebraic structures often called finite principal ideal rings, are a prominent subject of study in the field of coding theory.

The exploration of codes over finite rings plays a crucial role in coding theory, as it has led to the development of various families of nonlinear codes through the utilization of Gray images derived from linear codes over finite rings [2, 8]. Hammons et al. [9] made a remarkable contribution to codes over $\mathbb{Z}_4$, which captivated many coding theorists who were conducting research on codes over $\mathbb{Z}_4$ [1] over the past few decades. The generalized Kerdock codes came to light as a result of Carlet's further refinement of this idea to $\mathbb{Z}_{2^k}$ with Homogeneous weight in [3]. Substantial minimum distances have been noticed in these nonlinear binary codes.

The study of weight distribution is pivotal because it quantifies the inherent capacity of a code to identify and rectify errors. Beyond these structural properties, weight enumerators allow for the precise calculation of error probability performance during the decoding process. Additionally, in [14], the author explored and discovered the weight distribution properties of various well-known linear codes. Recent research includes the examination of $[n, k, d]$ $\mathbb{Z}_r$ -Simplex codes under the Hamming distance for prime power values of r in [5], with subsequent extensions presented in [7], considering any natural number $r \geq 2$, and obtaining $\{A_i\}_{i=1}^n$ for $\mathbb{Z}_r$ -Simplex codes when $r = p^m$ or p is a product of primes and $k = 2$. Durairajan and Mahalakshmi [6] further analyzed the parameters of punctured $\mathbb{Z}_r$ -Simplex codes for $k = 2$ and examined $\{A_i\}_{i=1}^n$ of punctured $\mathbb{Z}_r$ -Simplex codes in the same context. Moreover, in [11, 12], the parameters $[n, k, d]$ and $\{A_i\}_{i=1}^n$ of certain punctured versions of $\mathbb{Z}_r$ -Simplex codes were thoroughly investigated.

Motivated by the existing work, we have obtained the following outcomes. Section 2 contains preliminaries and some essential properties. In Section 3, we determined the $\mathbb{Z}_r$ -Simplex code parameters under the Homogeneous weight metric for rank 2 where $r = p^m$ and extend the same for rank k . Further, its corresponding weight distribution has been evaluated for rank k where $r = p^m$. Section 4 is devoted to finding the parameters of $\mathbb{Z}_r$ -Simplex code under the Lee weight metric for rank k where $r = p^2$, in particular for rank 2 where $r = p^m$. Later, the weight distribution is analysed for rank 2 where $r = p^2$.

2. Main Results

For a prime p and $r = p^m$, we define a code $C \subseteq \mathbb{Z}_r^n$ as $\mathbb{Z}_r$ -linear if it functions as a submodule. This type of code is typically described by the parameters $[n, k, d_{Hom}(d_L)]$, referring to its length, rank, and the minimum Homogeneous (Lee) distance. If each row of a matrix G represents a basis element of the $\mathbb{Z}_r$ -linear code C , then the matrix G is called the generator matrix of C . For a code C with length n , the sequence $(1, A_1, A_2, \dots, A_n)$ describes its weight distribution. Here, A_i represents the total count of codewords that have a Homogeneous (Lee) weight equal to i . An s -weight code is defined as a code C for which there are strictly s non-zero A_i coefficients.

Consider C as a $\mathbb{Z}_r$ -linear code. Let $c_1, c_2 \in C$, then the difference between two codewords $c_1 - c_2 \in C$. This implies that, the minimum Homogeneous distance d_{Hom} of C is

$$\min\{d_{Hom}(c_1, c_2) | c_1, c_2 \in C, c_1 \neq c_2\} = \min\{wt_{Hom}(c) | c \in C, c \neq 0\}.$$

In other words, the Homogeneous distance $d_{Hom}(C)$ is equivalent to the smallest Homogeneous weight found among all codewords within C .

We take $o(x)$ where $x \in \mathbb{Z}_r$ to indicate the additive order of x , everywhere in this paper and $N_a(c)$ as the number of indices where the codeword c in which an additive order equal to a .

2.1 Gray Images

In [16], Greferath and Schmidt introduced the generalized Gray map for finite chain rings relative to the Homogeneous metric. In this context, we follow the definition of Homogeneous weight on $\mathbb{Z}_{p^m}$ provided in [4],

$$wt_{Hom}(x) = \begin{cases} 0 & \text{for } x = 0, \\ p^{m-1} & \text{whenever } x \in p^{m-1}\mathbb{Z}_{p^m} \setminus \{0\}, \\ (p-1)p^{m-2} & \text{in all the other cases} \end{cases}$$

Define $M_i = (1)$ for the case where $i = 1$; for all subsequent values where $m \geq 2$, define $M_i = \begin{bmatrix} \mathbf{0} & \mathbf{1} & \cdots & \mathbf{p} - \mathbf{1} \\ M_{i-1} & M_{i-1} & \cdots & M_{i-1} \end{bmatrix}$. Then the Gray map, denoted by $\psi: \mathbb{Z}_{p^m} \rightarrow \mathbb{F}_p^{p^{m-1}}$, is defined as [15],

$$\begin{aligned} \psi(a) &= \psi(\beta_0 + p\beta_1 + \cdots + p^{m-1}\beta_{m-1}) \\ &= [\beta_0 \ \beta_1 \ \beta_2 \ \cdots \ \beta_{m-1}]M_m \end{aligned}$$

where $\beta_0, \beta_1, \beta_2, \dots, \beta_{m-1} \in \{0, 1, 2, \dots, p-1\} \subset \mathbb{Z}_{p^{m-1}}$. The isometry is interpreted by the mapping ψ from $(\mathbb{Z}_{p^m}, wt_{Hom})$ to $(\mathbb{F}_p^{p^{m-1}}, wt_{Ham})$. That is, $wt_{Hom}(x - y) = wt_{Ham}(\psi(x) - \psi(y))$.

The concept of the Extended Lee weight was first proposed in [13]. For an element $x \in \mathbb{Z}_{p^m}$, this weight is expressed as:

$$wt_L(x) = \begin{cases} x & \text{when } x < p^{m-1}, \\ p^{m-1} & \text{for } p^{m-1} \leq x \leq p^m - p^{m-1}, \\ p^m - x & \text{provided } p^m - p^{m-1} < x \leq p^m - 1 \end{cases}$$

Now, consider the following Gray map $\phi: \mathbb{Z}_p^m$ to $\mathbb{Z}_p^{p^{m-1}}$

$$\begin{aligned} 0 &\rightarrow (0\ 0\ 0\ \cdots\ 0\ 0\ 0) \\ 1 &\rightarrow (1\ 0\ 0\ \cdots\ 0\ 0\ 0) \\ 2 &\rightarrow (1\ 1\ 0\ \cdots\ 0\ 0\ 0) \\ &\vdots \\ p^{m-1} &\rightarrow (1\ 1\ 1\ \cdots\ 1\ 1\ 1) \\ p^{m-1} + 1 &\rightarrow (2\ 1\ 1\ \cdots\ 1\ 1\ 1) \\ &\vdots \\ p^m - 2 &\rightarrow (0\ 0\ 0\ \cdots\ 0\ (p-1)\ (p-1)) \\ p^m - 1 &\rightarrow (0\ 0\ 0\ \cdots\ 0\ 0\ (p-1)) \end{aligned}$$

The mapping ϕ preserves weight during the transition from the Lee metric space $\mathbb{Z}_p^m$ to the Hamming metric space $\mathbb{Z}_p^{(p^{m-1})n}$, satisfying the condition $wt_L(x) = wt_H(\phi(x))$ for all x .

3. Parameters of $\mathbb{Z}_r$ -Simplex code under the Homogeneous weight metric

In this section, we obtained the various parameters of $\mathbb{Z}_r$ -Simplex code under the Homogeneous weight metric.

Let

$$G'_2(r) = \begin{bmatrix} 0 & 1 & 1 & 2 & \cdots & r-1 \\ 1 & 0 & 1 & 1 & \cdots & 1 \end{bmatrix}.$$

Obviously, the rank of $G'_2(r)$ is 2 and the number of columns in the matrix $G'_2(r)$ is $r+1$, which represents the length of the $\mathbb{Z}_r$ -linear code. Let $x = (0\ 1\ 1\ 2\ \cdots\ r-1)$ and $y = (1\ 0\ 1\ 1\ \cdots\ 1)$. Then $C = \{\gamma x + \delta y \mid \gamma, \delta \in \mathbb{Z}_r\}$

Next, we apply an inductive approach to define,

$$G'_2(r) = \begin{bmatrix} 00\cdots 0 & 1 & 11\cdots 1 & 22\cdots 2 & \cdots & r-1\ r-1\cdots r-1 \\ 0 & & & & & \\ G'_{k-1}(r) & \vdots & G'_{k-1}(r) & G'_{k-1}(r) & \cdots & G'_{k-1}(r) \\ 0 & & & & & \end{bmatrix}$$

for $k > 1$.

Evidently, the matrix $G'_k(r)$ generates $S'_k(r)$. This $S'_k(r)$ represents a $\mathbb{Z}_r$ -simplex code of rank k with parameters $\left[n_k = \frac{r^{k-1}-1}{r-1}, k, d_{Hom}\right]$. The code

$$\begin{aligned} S'_k(r) &= \{\delta(1012\cdots \mathbf{r}) + (0cc\cdots c) \mid \delta \in \mathbb{Z}_r, c \in S'_{k-1}(r), \\ &\quad \mathbf{i} = ii\cdots i, \mathbf{0} = 00\cdots 0 \in \mathbb{Z}_q^{n_{k-1}}\}. \end{aligned}$$

For $0 \leq i \leq m$, consider the sets $D_{p^i} = \{y \in \mathbb{Z}_r \mid o(y) = p^i\}$. The sets

$\{D_{p^i} \mid 0 \leq i \leq m\}$, build a partition of $\mathbb{Z}_r$ with the property that $D_{p^i} \cap D_{p^j} = \emptyset$ for $i \neq j$. Furthermore, observe that the set $D_p \cup 0$ with the operation of addition forms a group. Additionally, for $2 \leq i \leq m$, the set D_{p^i} under the additive inverse is closed.

Lemma 3.1: Let $G'_k(r)$ be the generating matrix for an $[n, k, d]$ linear code C over $\mathbb{Z}_r$. For a given codeword $c_1 \in C$, let $\delta \in \mathbb{Z}_r$ have an order $o(\delta)$ equal to p^j , where the index j satisfies $1 \leq j \leq m$. Define $\overline{c}_1 = \delta(1012 \cdots r) + (0c_1c_1c_1 \cdots c_1)$, then

$$wt(\overline{c}_1) = \begin{cases} (np^{2m-1} - np^{2(m-1)}) + p^{m-2}(p-1) & \text{if } 2 \leq j \leq m \\ (np^{2m-1} - np^{2(m-1)}) + p^{m-1} & \text{if } j = 1. \end{cases}$$

The notation $N_l(c_1)$ is the cardinality of the set of indices in c_1 where the entries have order l .

Proof: If $2 \leq j \leq m$, then

$$\begin{aligned} wt_{Hom}(\delta(0112 \cdots r-1) + (c0c \cdots c)) &= (\sum_{j+1 \leq i \leq m} N_{p^i}(c))(p-1)p^m p^{m-2} \\ &+ p^{m-2}(p-1)(p^j - p)(n - \sum_{j+1 \leq i \leq m} N_{p^i}(c)) \left(\frac{r}{o(\delta)}\right) \\ &+ p^{m-1}(p-1)(n - \sum_{j+1 \leq i \leq m} N_{p^i}(c)) \left(\frac{r}{o(\delta)}\right) + (p-1)p^{m-2} \end{aligned}$$

By the simplifying the summation terms, we get

$$\begin{aligned} &= (p^{2m-1} - p^{2(m-1)})(\sum_{j+1 \leq i \leq m} N_{p^i}(c)) \\ &+ n(p^{2m-1} - p^{2m-j} - p^{2(m-1)} + p^{2m-j-1}) \\ &- (p^{2m-1} - p^{2m-j} - p^{2(m-1)} + p^{2m-j-1})(\sum_{j+1 \leq i \leq m} N_{p^i}(c)) \\ &+ n(p^{2m-j} - p^{2m-j-1}) \\ &- (p^{2m-j} - p^{2m-j-1})(\sum_{j+1 \leq i \leq m} N_{p^i}(c)) + p^{m-2}(p-1) \end{aligned}$$

Consequently, the final weight is expressed as,

$$= n(p^{2m-1} - p^{2(m-1)}) + p^{m-2}(p-1)$$

If $\delta \in B_p$, then to compute the total homogeneous weight by summing across the relevant indices $2 \leq i \leq m$, which gives

$$\begin{aligned} wt_{Hom}(\delta(0112 \cdots r-1) + (c0c \cdots c)) &= (\sum_{2 \leq i \leq m} N_{p^i}(c))(p-1)p^m p^{m-2} \\ &+ \left(\frac{r}{o(\delta)}\right)(n - \sum_{2 \leq i \leq m} N_{p^i}(c))p^{m-1}(p-1) + p^{m-1} \end{aligned}$$

After incorporating the fractional term $(r/o(\delta))$ and expanding the brackets, the equation becomes

$$\begin{aligned} &= (\sum_{2 \leq i \leq m} N_{p^i}(c))(p^{2m-1} - p^{2(m-1)}) \\ &- (\sum_{2 \leq i \leq m} N_{p^i}(c))(p^{2m-1} - p^{2(m-1)}) \\ &+ n(p^{2m-1} - p^{2(m-1)}) + p^{m-1} \end{aligned}$$

Grouping the terms containing powers of p allows us to simplify the expression to

$$= n(p^{2m-1} - p^{2(m-1)}) + p^{m-1}$$

This result indicates that for the codeword $\bar{c}_1$, the homogeneous weight is given by the expression $(np^{2m-1} - np^{2(m-1)}) + p^{m-1}$.

$$\text{Consider } S'_2(r) = \gamma(0 \ 1 \ 1 \ 2 \ 3 \ \dots \ r-1) + \delta(1 \ 0 \ 1 \ 1 \ \dots \ 1) | \gamma, \delta \in \mathbb{Z}_r\}.$$

Case i: Suppose both γ and δ are zero. Then

$$wt_{Hom}(\gamma x + \delta y) = 0$$

Case ii: Alternatively, if $\gamma = 0$ while δ remains non-zero, then

$$\begin{aligned} \min_{\delta \in \mathbb{Z}_r} \{wt_{Hom}(\delta y)\} &= wt_{Hom}(\delta y), \forall \delta \notin p^{m-1}\mathbb{Z}_r \\ &= p^m((p-1)p^{m-2}) \\ &= p^{2m-1} - p^{2(m-1)} \end{aligned}$$

Case iii: $\gamma \neq 0$ and $\delta = 0$ then

$$\min_{\gamma \in \mathbb{Z}_r} \{wt_{Hom}(\gamma x)\} = wt_{Hom}(\gamma y), \forall \gamma \notin p^{m-1}\mathbb{Z}_r$$

Let $o(\gamma') = p^j$, $2 \leq j \leq m$, then to evaluate the Homogeneous weight of $\gamma'y$, we utilize the structure of the mapping and sum over the appropriate indices. That is,

$$wt_{Hom}(\gamma'y) = wt_{Hom}(\gamma') + \frac{r}{o(\gamma')} \left[\sum_{1 \leq i \leq j} \sum_{s \in B_{p^i}} wt_{Hom}(s) \right]$$

By substituting the specific values for the Homogeneous weights into the summation brackets, the equation expands to

$$= p^{m-2}(p-1) + [p^{m-1}(p-1) + (p^m - p)(p^{m-2}(p-1))]$$

After performing the necessary algebraic reductions on the powers of p , the expression simplifies to its final form

$$= p^{m-2}(p-1) + p^{2m-1} - p^{2(m-1)}$$

Case iv: $\gamma \neq 0$ and $\delta \neq 0$, then

$$\min_{\gamma, \delta \in \mathbb{Z}_r} \{wt_{Hom}(\gamma x + \delta y)\} = wt_{Hom}(\gamma'x + \delta y), \forall \gamma' \notin p^{m-1}\mathbb{Z}_r$$

Let $o(\gamma') = p^i$ for $2 \leq i \leq m$ and $o(\delta) = p^j$ for $1 \leq j \leq i$, therefore

$$wt_{Hom}(\gamma'x + \delta y) = wt_{Hom}(\gamma') + \frac{r}{o(\gamma')} \left[\sum_{1 \leq k \leq i} \sum_{s \in \mathbb{B}_{p^k}} wt_{Hom}(s) \right]$$

By substituting the weight parameters for the specific indices, the above can be written as

$$= p^{m-2}(p-1) + p^{m-i}(p^{m+i-1} - p^{m+i-2})$$

Simplifying further,

$$= p^{m-2}(p-1) + p^{2m-1} - p^{2(m-1)}$$

Let $o(\gamma') = p^i$ for $2 \leq i \leq m-1$ and $o(\delta) = p^j$ for $i < j \leq m$, therefore

$$\begin{aligned} wt_{Hom}(\gamma'x + \delta y) &= wt_{Hom}(\gamma') + p^m wt_{Hom}(\delta) \\ &= p^{m-2}(p-1) + p^{2m-1} - p^{2(m-1)} \end{aligned}$$

By the above four cases,

$$\begin{aligned} d_{Hom}(S'_2(r)) &= \min\{p^{2m-1} - p^{2(m-1)}, (p-1)p^{m-2} + p^{2m-1} - p^{2(m-1)}\} \\ &= p^{2m-1} - p^{2(m-1)} \end{aligned}$$

Consequently, we establish the following theorem,

Theorem 3.2: For a prime integer p , let $r = p^m$. Then the matrix $G'_2(r)$ generates $[p^m + 1, 2, p^{2m-1} - p^{2(m-1)}] \mathbb{Z}_r$ -linear code.

Remark 3.3: Under the Gray map, $S'_2(r)$ is mapped to codes that are typically non-linear codes in Hamming space with $(n = p^{(m-1)}(p^m + 1), p^{2m}, d = p^{2m-1} - p^{2(m-1)})_p$.

Example 3.4: Table 1 provides the parameters of the code $S'_2(r)$ according to Theorem 3.2.

Table 1
Parameters of $S'_2(r)$

$S'_2(r)$	$[n = p^m + 1, 2, d = p^{2m-1} - p^{2(m-1)}]$		
	$p = 2$	$p = 3$	$p = 5$
$m = 2$	[5,2,4]	[10,2,18]	[26,2,100]
$m = 3$	[9,2,16]	[28,2,162]	[126,2,2500]
$m = 4$	[17,2,64]	[82,2,1458]	[626,2,62500]
$m = 5$	[33,2,256]	[244,2,13122]	[3126,2,1562500]

Theorem 3.5: Let $r = p^m$, where p is a prime number. Then $S'_k(r)$ is

$$a \left[n_k = \frac{r^k - 1}{r - 1}, k, r^{k-2}(p^{2m-1} - p^{2(m-1)}) \right] \mathbb{Z}_r\text{-linear code for } k \geq 2.$$

Proof: We establish this result through mathematical induction on the parameter k .

In the base case where $k = 2$, $S'_2(r) = \gamma(0 \ 1 \ 1 \ 2 \ 3 \cdots r-1) + \delta(1 \ 0 \ 1 \ 1 \cdots 1) | \gamma, \delta \in \mathbb{Z}_r \}$ is a code of length $r+1$. Let $x = (0 \ 1 \ 1 \ 2 \cdots r-1)$ and $y = (1 \ 0 \ 1 \ 1 \cdots 1)$. Then

$$d_{Hom}(S'_2(r)) = \min\{wt_{Hom}(\gamma x), wt_{Hom}(\delta y), wt_{Hom}(\gamma x + \delta y)\}$$

By Theorem 3.2, we have

$$d_{Hom}(S'_2(r)) = p^{2m-1} - p^{2(m-1)}.$$

Proceeding by induction, we take the claim as given for all indices $i < k$ and focus now on demonstrating its validity for the case where $i = k$.

By the assumption, $S'_i(r)$ is a $\left[\frac{r^{i-1}}{r-1}, i, r^{i-2}(p^{2m-1} - p^{2(m-1)})\right] \mathbb{Z}_r$ -linear code.

Let $S'_{k-1}(r)$ be an $[n, k-1, d]$ code and

$$S'_k(r) = \{\delta (\mathbf{0} \ 1 \ 1 \ 2 \dots \mathbf{r} - \mathbf{1}) + (c \ 0 \ c \ c \dots c) | \delta \in \mathbb{Z}_r, c \in S'_{k-1}(r)\}.$$

Then

$$d_{Hom}(S'_k(r)) = \min\{wt_{Hom}(\delta (\mathbf{0} \ 1 \ 1 \ 2 \dots \mathbf{r} - \mathbf{1}) + (c \ 0 \ c \ c \dots c) | \delta \in \mathbb{Z}_r, c \in S'_{k-1}(r)\}.$$

Case i: If $\delta = 0$, then

$$\min\{wt_{Hom}(c \ 0 \ c \ c \dots c) | c \in C, c \neq 0\} = rd$$

Case ii: Fix $c = 0$,

Subcase i: For $2 \leq j \leq m$. If $\delta \in B_{pj}$, then

$$\begin{aligned} wt_{Hom}(\delta(\mathbf{0} \ 1 \ 1 \ 2 \dots \mathbf{r} - \mathbf{1})) &= \left(\frac{r}{o(\delta)} \left[\sum_{i=1}^j \sum_{s \in B_{pi}} wt_{Hom}(s)\right]\right) n \\ &\quad + p^{m-2}(p-1) \\ &= n(p^{m-j}[p^{m+j-1} - p^{m+j-2}]) \\ &\quad + p^{m-2}(p-1) \\ &= n(p^{2m-1} - p^{2(m-1)}) + p^{m-2}(p-1) \end{aligned}$$

Subcase ii: If $\delta \in B_p$, then

$$\begin{aligned} wt_{Hom}(\delta(\mathbf{0} \ 1 \ 1 \ 2 \dots \mathbf{r} - \mathbf{1})) &= \left(\frac{r}{o(\delta)} \left[\sum_{s \in B_p} wt_{Hom}(s)\right]\right) n + p^{m-1} \\ &= n(p^{m-1}[(p-1)p^{m-1}]) + p^{m-1} \\ &= n(p^{2m-1} - p^{2(m-1)}) + p^{m-1} \end{aligned}$$

Case iii: Fix $\delta \neq 0, c \neq 0$,

For $1 \leq j \leq m$. If $\delta \in B_{pj}$, then By Lemma 3.1

$$wt_{Hom}(\delta (\mathbf{0} \ 1 \ 1 \ 2 \dots \mathbf{r} - \mathbf{1}) + (c \ 0 \ c \ c \dots c)) = n(p^{2m-1} - p^{2(m-1)}) + p^{m-2}(p-1)$$

Thus,

$$d(S'_k(r)) = \min\{rd, n(p^{2m-1} - p^{2(m-1)}) + p^{m-2}(p-1), (p^{2m-1} - p^{2(m-1)}) + p^{m-1}\}$$

Therefore, $d(S'_k(r)) = r^{k-2}(p^{2m-1} - p^{2(m-1)})$.

And hence, $S'_k(r)$ is an $\left[\frac{r^{k-1}}{r-1}, k, r^{k-2}(p^{2m-1} - p^{2(m-1)})\right] \mathbb{Z}_r$ -linear code.

Remark 3.6: The Gray map transforms $S'_k(r)$ into generally non-linear codes in the Hamming space with parameters $\left(\frac{p^{m-1}(r^k-1)}{r-1}, p^{mk}, r^{k-2}(p^{2m-1} - p^{2(m-1)})\right)_p$.

Example 3.7: With $r = 9$, Table 2 presents the code parameters for $S'_k(r)$ (where $k \in \{2, 3, 4, 5\}$) as derived from Theorem 3.5.

Table 2
Parameters of $S'_k(r)$

$G'_k(9)$	$\left[\frac{r^k - 1}{r - 1}, k, r^{k-2}(p^{2m-1} - p^{2(m-1)})\right]$
$G'_2(9)$	[10,2,18]
$G'_3(9)$	[91,3,162]
$G'_4(9)$	[820,4,1458]
$G'_5(9)$	[7381,5,13122]

3.1 Weight Distribution of $S'_k(r)$ under the Homogeneous metric where $r = p^m$

In this subsection, we find the weight distribution of $S'_k(r)$ under the Homogeneous weight metric. Consider the generator matrix $G'_2(r)$. Here, the matrix $G'_2(r)$ has x as its first row and y as its second. Then

$$S'_2(r) = \{\gamma x + \delta y | \gamma, \delta \in \mathbb{Z}_r\}$$

Case i: If $\gamma = 0$ and $\delta = 0$, then $wt_{Hom}(\gamma x + \delta y) = 0$. Therefore, $A_0 = 1$

Case ii: If $\gamma = 0$ and $\delta \neq 0$, then

Subcase i: For $2 \leq j \leq m$. If $\delta \in B_{p^j}$, then

$$\begin{aligned} wt_{Hom}(\delta y) &= rwt_{Hom}(\delta) \\ &= p^m((p-1)p^{m-2}) \\ &= p^{2m-1} - p^{2(m-1)} \end{aligned}$$

Subcase ii: If $\delta \in B_p$, then

$$\begin{aligned} wt_{Hom}(\delta y) &= rwt_{Hom}(\delta) \\ &= p^m(p^{m-1}) \\ &= p^{2m-1} \end{aligned}$$

Therefore, a Homogeneous weight of $(p^{2m-1} - p^{2(m-1)})$ is observed in exactly $p^m - p$ codewords. That is, $A_{p^{2m-1}-p^{2(m-1)}} = p^m - p$. Further, $p-1$ codewords with homogeneous weight p^{2m-1} exist. That is, $A_{p^{2m-1}} = p-1$.

Case iii: If $\gamma \neq 0$ and $\delta = 0$,

Subcase i: For $2 \leq j \leq m$. If $\gamma \in B_{p^j}$, then

$$wt_{Hom}(\gamma x) = wt_{Hom}(\gamma) + \frac{r}{p^j} \left[\sum_{1 \leq i \leq j} \sum_{s \in B_{p^i}} wt_{Hom}(s) \right]$$

By substituting the weight parameters for the specific indices, the above can be written as

$$= p^{m-2}(p-1) + p^{m-j}[p^{m+j-1} - p^{m+j-2}]$$

Distributing the p^{m-j} term throughout the brackets, so that

$$wt_{Hom}(\gamma x) = p^{m-2}(p-1) + p^{2m-1} - p^{2(m-1)}$$

Subcase ii: If $\gamma \in B_p$, then

$$wt_{Hom}(\gamma x) = wt_{Hom}(\gamma) + r/p \left[\sum_{s \in \mathbb{B}_p} wt_{Hom}(i) \right]$$

By substituting the known values in the above summation, thus

$$wt_{Hom}(\gamma x) = p^{m-1} + p^{2m-1} - p^{2(m-1)}$$

Therefore, there are $p^m - p$ codewords of Homogeneous weight $p^{m-2}(p-1) + p^{2m-1} - p^{2(m-1)}$. That is, $A_{p^{m-2}(p-1)+p^{2m-1}-p^{2(m-1)}} = p^m - p$. Further, $p-1$ codewords with Homogeneous weight $p^{2m-1} - p^{2(m-1)} + p^{m-1}$ exist. Thus, $A_{p^{2m-1}-p^{2(m-1)}+p^{m-1}} = p-1$.

Case iv: Fix $\gamma \neq 0$ and $\delta \neq 0$,

Subcase i: If $o(\gamma) = p^i$ for $2 \leq i \leq m$ and $o(\delta) = p^j$ for $1 \leq j \leq i$, then

$$\begin{aligned} wt_{Hom}(\gamma x + \delta y) &= wt_{Hom}(\gamma) + r/o(\gamma) \left[\sum_{1 \leq k \leq i} \sum_{s \in \mathbb{B}_{p^k}} wt_{Hom}(s) \right] \\ &= p^{m-2}(p-1) + p^{m-i}(p^{m+i-1} - p^{m+i-2}) \\ &= p^{m-2}(p-1) + p^{2m-1} - p^{2(m-1)} \end{aligned}$$

Subcase ii: If $o(\gamma) = p^i$ for $2 \leq i \leq m-1$ and $o(\delta) = p^j$ for $i < j \leq m$, then

$$\begin{aligned} wt_{Hom}(\gamma x + \delta y) &= wt_{Hom}(\gamma) + p^m wt_{Hom}(\delta) \\ &= p^{m-2}(p-1) + p^{2m-1} - p^{2(m-1)} \end{aligned}$$

Subcase iii: If $o(\gamma) = p$ and $o(\delta) = p^i$ for $1 < i \leq m$, then

$$\begin{aligned} wt_{Hom}(\gamma x + \delta y) &= wt_{Hom}(\gamma) + p^m wt_{Hom}(\delta) \\ &= p^{m-1} + p^{2m-1} - p^{2(m-1)} \end{aligned}$$

Subcase iv: If $o(\gamma) = p$ and $o(\delta) = p$ then

$$\begin{aligned} wt_{Hom}(\gamma x + \delta y) &= wt_{Hom}(\gamma) + r/o(\gamma) \left[\sum_{s \in \mathbb{B}_p} wt_{Hom}(x) \right] \\ &= p^{m-1} + p^{m-1}(p^m - p^{m-1}) \\ &= p^{m-1} + p^{2m-1} - p^{2(m-1)} \end{aligned}$$

Therefore, there are $(p^m - 1)(p^m - p)$ codewords of Homogeneous weight $p^{2m-1} - p^{2(m-1)} + p^{m-2}(p - 1)$. That is, $A_{p^{2m-1}-p^{2(m-1)}+p^{m-2}(p-1)} = (p^m - 1)(p^m - p)$. Further, $(p^m - 1)(p - 1)$ codewords with Homogeneous weight $p^{2m-1} - p^{2(m-1)} + p^{m-1}$ exist. Thus, $A_{p^{2m-1}-p^{2(m-1)}+p^{m-1}} = (p^m - 1)(p - 1)$.

The result of the discussion above is as follows:

Theorem 3.8: *Let $r = p^m$, where p is a prime, then $S'_2(r)$ has the weight distribution as,*

$$\begin{aligned} A_0 &= 1 \\ A_{p^{2m-1}-p^{2(m-1)}} &= p^m - p \\ A_{p^{2m-1}} &= p - 1 \\ A_{p^{m-2}(p-1)+p^{2m-1}-p^{2(m-1)}} &= (p^m - p)p^m \\ A_{p^{m-1}+p^{2m-1}-p^{2(m-1)}} &= (p - 1)p^m \end{aligned}$$

Theorem 3.9: *For a prime integer p and $r = p^m$, the weight distribution of $S'_k(r)$ is given by,*

$$\begin{aligned} A_0 &= 1 \\ A_{p^{m(k-2)}(p^{2m-1}-p^{2(m-1)})} &= p^m - p \\ A_{p^{m(k-2)}(p^{2m-1})} &= p - 1 \\ A_{p^{m(k-2)}((p-1)p^{m-2}+p^{2m-1}-p^{2(m-1)})} &= (p^m - p)p^m \\ A_{p^{m(k-2)}(p^{m-1}+p^{2m-1}-p^{2(m-1)})} &= (p - 1)p^m \\ A_{p^{m(k-i)}\left[\left(\frac{r^{i-1}-1}{r-1}\right)(p^{2m-1}-p^{2(m-1)})+p^{m-2}(p-1)\right]} &= (p^m - p)p^{(i-1)m} \\ A_{p^{m(k-i)}\left[\left(\frac{r^{i-1}-1}{r-1}\right)(p^{2m-1}-p^{2(m-1)})+p^{m-1}\right]} &= (p - 1)p^{(i-1)m} \end{aligned}$$

for $2 \leq i \leq k$

Proof: We establish the proof of this theorem by applying the results of Lemma 3.1.

Example 3.10: The weight distributions of $S'_4(8)$, $S'_4(9)$, and $S'_2(25)$ are presented in Table 3, derived from Theorems 3.8, 3.9 and analyzed using MATLAB.

Table 3
Weight distribution of $S'_k(r)$.

$S'_4(8)$	$S'_4(9)$	$S'_2(25)$
$A_0 = 1, A_{1024} = 6$	$A_0 = 1, A_{1458} = 6$	$A_0 = 1$
$A_{1152} = 48, A_{1168} = 384$	$A_{1620} = 54, A_{1638} = 486$	$A_{100} = 20$
$A_{1170} = 3072, A_{1172} = 512$	$A_{1640} = 4374, A_{1641} = 1458$	$A_{125} = 4$
$A_{1184} = 64, A_{1280} = 8$	$A_{1647} = 162, A_{1701} = 18$	$A_{104} = 500$
$A_{2048} = 1$	$A_{2187} = 2$	$A_{105} = 100$

4. Parameters of $\mathbb{Z}_r$ -Simplex code under the Lee weight metric

In this section, we obtained the various parameters of $\mathbb{Z}_r$ -Simplex code under the Lee weight metric.

$$\text{Consider } S'_2(r) = \gamma(0 \ 1 \ 1 \ 2 \ 3 \ \dots \ r-1) + \delta(1 \ 0 \ 1 \ 1 \ \dots \ 1) | \gamma, \delta \in \mathbb{Z}_r\}.$$

Case i: Assuming that $\gamma = 0$ and $\delta = 0$ for Case i , it follows that,

$$wt_L(\gamma x + \delta y) = 0.$$

Case ii: Let $\gamma = 0$ and suppose that $\delta \neq 0$. Then

$$\min_{\delta \in \mathbb{Z}_r} \{wt_L(\delta y)\} = wt_L(1 \cdot y) = r = p^m$$

Case iii: $\gamma \neq 0$ and $\delta = 0$ then

$$\begin{aligned} \min_{\gamma \in \mathbb{Z}_r} \{wt_L(\gamma x)\} &= wt_L(1 \cdot x) \\ &= 1 + p^{2m-1} - p^{2(m-1)} \end{aligned}$$

Case iv: $\gamma \neq 0$ and $\delta \neq 0$ then

$$\begin{aligned} \min_{\gamma, \delta \in \mathbb{Z}_r} \{wt_L(\gamma x + \delta y)\} &= wt_L(1 \cdot x + 1 \cdot y) \\ &= 1 + p^{2m-1} - p^{2(m-1)} \end{aligned}$$

By the above four cases,

$$d_L(S'_2(r)) = \min\{p^m, 1 + p^{2m-1} - p^{2(m-1)}\} = p^m$$

Thus, we conclude the following result

Theorem 4.1: Let p be a prime integer and let $r = p^m$. It follows that the matrix $G'_2(r)$ acts as a generator for a $\mathbb{Z}_r$ -linear code with the parameters $[p^m + 1, 2, p^m]$.

Remark 4.2: Under the Gray map, $S'_2(r)$ is mapped to codes that are typically non-linear codes in Hamming space with $(p^{(m-1)}(p^m + 1), p^{2m}, p^m)_p$.

Example 4.3: By Theorem 4.4, $[n, k, d]$ of the code generated by $G'_2(r)$ when $p = 2, 3$ and 5 and $m = 2, 3, 4$ and 5 are given in Table 4.

Table 4
Parameters of $S'_2(r)$

$S'_2(r)$	Parameters $[p^m + 1, 2, p^m]$		
	$p = 2$	$p = 3$	$p = 5$
$m = 2$	[5,2,4]	[10,2,9]	[26,2,25]
$m = 3$	[9,2,8]	[28,2,27]	[126,2,125]
$m = 4$	[17,2,16]	[82,2,81]	[626,2,625]
$m = 5$	[33,2,32]	[244,2,243]	[3126,2,3125]

Theorem 4.4: *Let $r = p^2$, where p is a prime number. Then $S'_k(r)$ is a $\left[n_k = \frac{r^{k-1}}{r-1}, k, r^{k-1}\right] \mathbb{Z}_r$ -linear code for $k > 2$.*

Proof: We establish this result using induction on k . Considering the base case where $k = 2$, the set $S'_2(r) = \gamma(0 \ 1 \ 1 \ 2 \ 3 \cdots r-1) + \delta(1 \ 0 \ 1 \ 1 \cdots 1) | \gamma, \delta \in \mathbb{Z}_r$ is a code of length $r+1$. Let $x = (0 \ 1 \ 1 \ 2 \cdots r-1)$ and $y = (1 \ 0 \ 1 \ 1 \cdots 1)$. Then

$$d_L(S'_2(r)) = \min\{wt_L(\gamma x), wt_L(\delta y), wt_L(\gamma x + \delta y)\}$$

From Theorem 4.1, we have $\min\{wt_L(\gamma x) | \gamma \neq 0\} = 1 + p^3 - p^2$, $\min\{wt_L(\delta y) | \delta \neq 0\} = p^2$ and $\min\{wt_L(\gamma x + \delta y)\} = 1 + p^3 - p^2$.

Therefore, $d_L(S'_2(r)) = \min\{1 + p^3 - p^2, p^2\} = p^2$.

Assuming the result holds for all i such that $i < k$, we must now establish its truth for k .

By the assumption, $S'_i(r)$ is a $\left[\frac{r^{i-1}}{r-1}, i, r^{i-1}\right] \mathbb{Z}_r$ -linear code.

Let $S'_{k-1}(r)$ be an $[n, k-1, d]$ code and

$$S'_k(r) = \{\delta (\mathbf{0} \ 1 \ 1 \ 2 \cdots \mathbf{r} - 1) + (c \ 0 \ c \ c \cdots c) | \delta \in \mathbb{Z}_r, c \in S'_{k-1}(r)\}.$$

Then

$$d_L(S'_k(r)) = \min\{wt_L(\delta (\mathbf{0} \ 1 \ 1 \ 2 \cdots \mathbf{r} - 1) + (c \ 0 \ c \ c \cdots c) | \delta \in \mathbb{Z}_r, c \in S'_{k-1}(r)\}.$$

Case i: $\delta = 0$ then $\min\{wt_L(c \ 0 \ c \ c \cdots c) | c \in C, c \neq 0\} = rd$.

Case ii: $c = 0$ then

Subcase i: δ is a unit then

$$\min\{wt_L(\delta (\mathbf{0} \ 1 \ 1 \ 2 \cdots \mathbf{r} - 1) | \delta \in \mathbb{Z}_r, \delta \neq 0\} = (p^3 - p^2) \left\lceil \frac{r^{k-1}-1}{r-1} \right\rceil + 1$$

Subcase ii: δ is a zero divisor, then

$$\begin{aligned} \min\{wt_L(\delta (\mathbf{0} \ 1 \ 1 \ 2 \cdots \mathbf{r} - 1) | \delta \in \mathbb{Z}_r, \delta \neq 0\} &= \\ &= \left(\frac{r}{p} \left[\sum_{i \in Z(\mathbb{Z}_r)} wt_L(i) \right] \right) \left\lceil \frac{r^{k-1}-1}{r-1} \right\rceil + p \\ &= (p^3 - p^2) \left\lceil \frac{r^{k-1}-1}{r-1} \right\rceil + p \end{aligned}$$

where $Z(\mathbb{Z}_r)$ denote the set of all zero divisors in $\mathbb{Z}_{p^2}$.

Case iii: $\delta \neq 0$ and $c \neq \mathbf{0}$ then

$$\begin{aligned} \min\{wt_L(\delta (\mathbf{0} \ 1 \ 1 \ 2 \cdots \mathbf{r} - 1) \\ + (c \ 0 \ c \ c \cdots c) | \delta \in \mathbb{Z}_r, c \in S'_{k-1}(r)\} &= r(p-1) \left\lceil \frac{r^{k-1}-1}{r-1} \right\rceil + 1 \\ &= (p^3 - p^2) \left\lceil \frac{r^{k-1}-1}{r-1} \right\rceil + 1 \end{aligned}$$

Thus,

$$\begin{aligned}
d(S'_k(r)) &= \min \left\{ rd, (p^3 - p^2) \left\lceil \frac{r^{k-1}-1}{r-1} \right\rceil + 1, (p^3 - p^2) \left\lceil \frac{r^{k-1}-1}{r-1} \right\rceil + p \right\} \\
&= \min \left\{ r^{k-1}, (p^3 - p^2) \left\lceil \frac{r^{k-1}-1}{r-1} \right\rceil + 1, (p^3 - p^2) \left\lceil \frac{r^{k-1}-1}{r-1} \right\rceil + p \right\}
\end{aligned}$$

$$\begin{aligned}
\text{Since, } (p^3 - p^2) \left\lceil \frac{r^{k-1}-1}{r-1} \right\rceil + 1 &= (p^3 - p^2)[r^{k-2} + r^{k-3} + \dots + 1] + 1 \\
&= (p - 1)[r^{k-1} + r^{k-2} + \dots + r] + 1 \\
&> r^{k-1}.
\end{aligned}$$

Therefore, $d(S'_k(r)) = r^{k-1}$.

And hence, $S'_k(r)$ is an $\left\lceil \frac{r^k-1}{r-1} \right\rceil, k, r^{k-1} \mathbb{Z}_r$ -linear code.

Remark 4.5: Under the Gray map, $S'_k(r)$ is mapped to codes that are typically non-linear codes in Hamming space with $\left(\frac{p(r^k-1)}{r-1}, p^{2k}, r^{k-1} \right)_p$.

Example 4.6: Let $r = 9$. Then, by Theorem 4.4, the parameters of the code $S'_k(r)$ are given in Table 5.

Table 5
Parameters of $S'_k(r)$

$G'_k(9)$	$\left\lceil \frac{r^k-1}{r-1} \right\rceil, k, r^{k-1}$
$G'_2(9)$	[10,2,9]
$G'_3(9)$	[91,3,81]
$G'_4(9)$	[820,4,729]
$G'_5(9)$	[7381,5,6561]

5. Conclusion

In this article, we obtained the various parameters of $\mathbb{Z}_r$ -Simplex code under the Homogeneous and Lee weight metric for some particular cases. Generally, finding weight distribution of higher rank code is not an easy task. Here, we examined the weight distribution of $\mathbb{Z}_r$ -Simplex code under the Homogeneous weight metric for rank k where $r = p^m$ under the Lee metric. In the last decade, the construction of nonlinear codes has been increased because of the existence of BTL (Better than Linear) and BTKL (Better than Known Linear) codes [10]. Hence, this work establishes a class of nonlinear codes over $\mathbb{F}_p$. Based on the puncturing technique in [6, 12], one can obtain new families of nonlinear code with better parameters.

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