

3-parameter generalized quaternions with generalized Tetranacci numbers components

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Abstract

In this study, our aim is to investigate a new special number family, which is called the 3-parameter generalized quaternions with generalized Tetranacci numbers components with both non-negative and negative subscripts. According to this aim, we bring together the 3-parameter generalized quaternions (shortly 3PGQs), which are a general form of the quaternion algebra according to 3-parameters, and generalized Tetranacci numbers, which are also the most general form of all of the fourth-order recurrence sequences. Also, we examine some special cases of them and construct both new and fundamental properties such as recurrence relations, Binet formulas, generating functions, exponential generating

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functions, Poisson generating functions, summation formulas, matrix formulas, and special determinant equations. Furthermore, we give determinants, characteristic polynomials, characteristic equations, eigenvalues, and eigenvectors concerning the matrix representation of 3PGQs with generalized Tetranacci numbers components. Then, we examine a special case with respect to the given arguments and present an example.

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1. Introduction

Numbers and number systems are becoming more attractive and important due to their several applications in lots of workframes such as geometry, combinatorics, architecture, computer sciences, etc. One of the most important concepts of the number system is quaternions. Quaternions, as an extension of the complex numbers were investigated by Hamilton in 1843 [20-22]. Quaternion algebra is non-commutative, associative, and 4-dimensional Clifford algebra. The set of quaternions (real/Hamilton quaternion) is represented by the notation $\mathbb{H}$ and determined as:

$$\mathbb{H} := \{q | q = q_0 + q_1e_1 + q_2e_2 + q_3e_3, q_0, q_1, q_2, q_3 \in \mathbb{R}\},$$

where e_1, e_2, e_3 are quaternionic units and satisfy the following rules: $e_1^2 = -1, e_2^2 = -1, e_3^2 = -1, e_1e_2 = -e_2e_1 = e_3, e_2e_3 = -e_3e_2 = e_1, e_3e_1 = -e_1e_3 = e_2$ [8]. After investigating Hamilton's quaternions, Cockle determined the split quaternions with the rules of quaternionic units: $e_1^2 = -1, e_2^2 = e_3^2 = 1, e_1e_2e_3 = 1$ [12, 14]. Additionally, the generalized quaternions (or 2-parameter generalized quaternions, shortly 2PGQs ([68, 69])) have been studied by several researchers (15, 19, 31-33, 42, 44-46, 71)]. For all $\lambda_1, \lambda_2 \in \mathbb{R}$, the set of the generalized quaternions is denoted by $\mathbb{H}_{\lambda_1\lambda_2}$ and determined as follows:

$$\mathbb{H}_{\lambda_1\lambda_2} := \{q = q_0 + q_1e_1 + q_2e_2 + q_3e_3, q_0, q_1, q_2, q_3, \lambda_1, \lambda_2 \in \mathbb{R}\},$$

where $e_1^2 = -\lambda_1, e_2^2 = -\lambda_2, e_3^2 = -\lambda_1\lambda_2, e_1e_2 = -e_2e_1 = e_3, e_2e_3 = -e_3e_2 = \lambda_2e_1, e_3e_1 = -e_1e_3 = \lambda_1e_2$. For $\lambda_1 = 1, \lambda_2 = 1$; q is a real quaternion, for $\lambda_1 = 1, \lambda_2 = -1$; q is a split quaternion, for $\lambda_1 = 1, \lambda_2 = 0$; q is a semi-quaternion, for $\lambda_1 = -1, \lambda_2 = 0$; q is a split semi-quaternion, for $\lambda_1 = 0, \lambda_2 = 0$; q is a 1/4 quaternion [12, 15, 19, 22, 31-33, 42-46, 71].

Recently, Şentürk and Ünal [68, 69] defined a new type of quite comprehensive quaternion type named 3-parameter generalized quaternions (shortly, 3PGQ). The authors established a new and general aspect for the quaternion algebra depending on the 3-parameters to get a generalization of real, split, and 2PGQs. The set of 3PGQs is denoted by $\mathbb{K}$ and determined as follows:

$$\mathbb{K} := \{q = q_0 + q_1e_1 + q_2e_2 + q_3e_3, q_0, q_1, q_2, q_3, \lambda_1, \lambda_2, \lambda_3 \in \mathbb{R}\},$$

where quaternionic units hold the rules written in Table 1.

Table 1
Multiplication Rules of 3PGQs [68, 69]

.	1	e_1	e_2	e_3
1	1	e_1	e_2	e_3
e_1	e_1	$-\lambda_1\lambda_2$	λ_1e_3	$-\lambda_2e_2$
e_2	e_2	$-\lambda_1e_3$	$-\lambda_1\lambda_3$	λ_3e_1
e_3	e_3	λ_2e_2	$-\lambda_3e_1$	$-\lambda_2\lambda_3$

According to the values $\lambda_1, \lambda_2, \lambda_3$, some special cases of 3PGQs are written in the following Table 2, and one can see that the other special types with respect to the $\lambda_{i \in \{1,2,3\}}$ can be examined [68, 69].

Table 2
Classification of 3PGQs [68, 69]

For	Type of 3PGQs
$\lambda_1 = 1, \lambda_2 = \lambda, \lambda_3 = \mu$	2PGQs [15, 19, 31-33, 42, 44-46, 71]
$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = -1$	Split quaternions [12]
$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 1$	Hamilton quaternions [20-22]
$\lambda_1 = 1, \lambda_2 = 1, \lambda_3 = 0$	Semi-quaternions [43, 45]
$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$	Split semi-quaternions [45]
$\lambda_1 = 1, \lambda_2 = 0, \lambda_3 = 0$	1/4-quaternions [22, 45]

On the other hand, special recurrence sequences (with different orders such as second, third, fourth, and higher than four-order) are also examined in the existing literature in numerous studies by several researchers. Fibonacci numbers (or sequences) come to mind first. Fibonacci sequence is a second-order linear recursively sequence. We can also say that the Jacobsthal, Pell, and Horadam numbers are also second-order linear recursively sequences [23, 24, 35-38, 51]. Then, in literature, the higher-order linear recursive sequences have been studied, such as third-order [53, 73], fourth-order [34, 39, 48, 49, 52, 57, 59-62, 66, 72, 74, 75] (see Table 4), and the others (e.g. [63, 64]). In this study, we care about the fourth-order special sequence is called the generalized Tetranacci sequence (or numbers), which is the most general form of the fourth-order special recurrence sequences. Some special cases of generalized Tetranacci numbers can be examined in Table 3 and Table 4 in Section 2.

The generalized Tetranacci numbers satisfy the following recurrence relation:

$$V_n = rV_{n-1} + sV_{n-2} + tV_{n-3} + uV_{n-4} \text{ for all } n \geq 4, \quad (1.1)$$

with the initial conditions $V_0 = a, V_1 = b, V_2 = c, V_3 = d$ where $a, b, c, d \in \mathbb{R}$ [49, 52, 59-62]. Additionally, the generalized Tetranacci numbers with negative subscripts have the following recurrence relation [61]:

$$V_{-n} = \frac{1}{u}V_{-(n-4)} - \frac{t}{u}V_{-(n-1)} - \frac{s}{u}V_{-(n-2)} - \frac{r}{u}V_{-(n-3)} \text{ for all } n \geq 1, u \neq 0. \quad (1.2)$$

One can see that the recurrence relation (1.1) is valid for all integers [61].

In the existing literature, Tetranacci numbers, quaternions, and also bringing together quaternion types and special recurrence sequences are studied intensively. Taşcı and Acar [67] examined the Gaussian Tetranacci numbers, and then Soykan [55] determined the Gaussian generalized Tetranacci numbers. Furthermore, the generalized Tetranacci hybrid numbers were introduced by Soykan and Taşdemir [56]. Soykan defined the bicomplex Tetranacci and Tetranacci-Lucas quaternions [54], and Soykan and Taşdemir scrutinized the bicomplex generalized Tetranacci quaternions [58]. Additionally, in [65], Tetranacci and Tetranacci-Lucas quaternions were obtained by Soykan. Then, Soykan et al. introduced the quaternions with generalized Tetranacci numbers components [50].

Recently, Bilgici investigated the Fibonacci and Lucas 3PGQs [2] and 3PGQs with Jacobsthal and Jacobsthal-Lucas numbers components [3]. Moreover, Horadam 3PGQs were defined by İşbilir and Gürses in [29]. Then, 3PGQs with generalized Tribonacci numbers components were introduced by İşbilir, Gürses, and Tosun [30]. Also, Chaker and Boua studied the generalized quaternions algebra with generalized Fibonacci quaternions [1]. Then, Kızılateş and Kibar determined the 3PGQs with higher-order generalized Fibonacci numbers components [41]. Moreover, Saçlı defined the 3-parameter generalized quaternions with the Leonardo p -sequence [47]. Bilgici determined the 3PGQs with Gibonacci numbers components [4], with certain generalized Fibonacci numbers components [5], with k -Fibonacci and k -Lucas numbers components [6], and Pell and Pell-Lucas numbers components [7].

Before the concept 3PGQs, which has just begun to be studied intensively in the literature, many studies have been carried out on 2PGQs and other quaternion types, which are its special cases. Cerda-Morales studied quaternions (especially real and generalized) with third-order recurrence sequences in the studies [8-11]. Dişkaya and Menken [13, 14] defined the (s, t) -Padovan and (s, t) -Perrin real and split quaternions, respectively. Günay [16] and Günay and Taşkara [17] introduced the real quaternions with some generalized third-order recurrence sequences and Padovan quaternions, respectively. Gürses and İşbilir investigated the generalized quaternions with generalized Jacobsthal-Padovan numbers components [18]. Then, generalized quaternions with Padovan and Perrin numbers components [26], Pell-Padovan numbers components [27], and generalized 3-primes numbers components [28] were defined by İşbilir and Gürses.

Our motivation in doing this paper is to combine the 3PGQs, which are a general form of the quaternion algebra according to 3-parameters, and generalized Tetranacci numbers, which are quite a big special number family for fourth-order special recurrence sequences. This paper is divided into three sections, as follows. In Section 2, we remind the general notions and notations about both 3PGQs and generalized Tetranacci numbers components. In Section 3, we determine and scrutinize the 3PGQs with generalized Tetranacci numbers components. Further, we obtain recurrence relation, Binet formula, generating

functions, exponential generating functions, Poisson generating functions, summing formulas, matrix formulas, and determinant equalities, which are the beneficial way for finding n th and $-(n+1)$ th element of this number system, and some other special equalities. Moreover, we define determinants, characteristic polynomials, characteristic equations, eigenvalues, and eigenvectors concerning the matrix representation of 3PGQs with generalized Tetranacci numbers components. Also, we examine a special case with respect to the finding materials.

2. Preliminaries

In this section of this paper, we write the required terminology with respect to the 3PGQs and generalized Tetranacci numbers needed from the beginning to the end of this paper.

2.1 3-parameter generalized quaternions

Let $q = q_0 + q_1e_1 + q_2e_2 + q_3e_3, p = p_0 + p_1e_1 + p_2e_2 + p_3e_3 \in \mathbb{K}$, some basic algebraic properties are provided as follows [68, 69]:

- Equality: $q = p \Leftrightarrow q_0 = p_0, q_1 = p_1, q_2 = p_2, q_3 = p_3$.
- Addition/Subtraction:

$$q \pm p = q_0 \pm p_0 + (q_1 \pm p_1)e_1 + (q_2 \pm p_2)e_2 + (q_3 \pm p_3)e_3.$$
- Multiplication by a scalar: $\tilde{c}q = \tilde{c}q_0 + \tilde{c}q_1e_1 + \tilde{c}q_2e_2 + \tilde{c}q_3e_3, \tilde{c} \in \mathbb{R}$.
- Scalar and vector parts: The scalar part of q is represented as S_q and $S_q = q_0$. The vector part of q is denoted by $\vec{V}_q$ and $\vec{V}_q = q_1e_1 + q_2e_2 + q_3e_3$. Also, $q = S_q + \vec{V}_q$. Then, we get the following equations: $S_{q \pm p} = q_0 \pm p_0 = S_q \pm S_p$ and $\vec{V}_{q \pm p} = \vec{V}_q \pm \vec{V}_p$.
- Multiplication: $qp = S_qS_p - f(\vec{V}_q, \vec{V}_p) + S_q\vec{V}_p + S_p\vec{V}_q + \vec{V}_q \wedge \vec{V}_p$, where $f(\vec{V}_q, \vec{V}_p) = \lambda_1\lambda_2q_1p_1 + \lambda_1\lambda_3q_2p_2 + \lambda_2\lambda_3q_3p_3$ and

$$\vec{V}_q \wedge \vec{V}_p = \begin{vmatrix} \lambda_3e_1 & \lambda_2e_2 & \lambda_1e_3 \\ q_1 & q_2 & q_3 \\ p_1 & p_2 & p_3 \end{vmatrix}$$

$$= \lambda_3(q_2p_3 - q_3p_2)e_1 + \lambda_2(q_3p_1 - q_1p_3)e_2 + \lambda_1(q_1p_2 - q_2p_1)e_3.$$
- Conjugation: $\bar{q} = q_0 - q_1e_1 - q_2e_2 - q_3e_3$.
- Inner product: $\langle q, p \rangle = q_0p_0 + \lambda_1\lambda_2q_1p_1 + \lambda_1\lambda_3q_2p_2 + \lambda_2\lambda_3q_3p_3$.
- Norm: $N_q = q\bar{q} = \bar{q}q = q_0^2 + \lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2$. If $N_q = 1$, then q is a 3-parameter generalized unit quaternion.
- Inverse:

$$q^{-1} = \frac{\bar{q}}{N_q} = \frac{q_0 - q_1e_1 - q_2e_2 - q_3e_3}{q_0^2 + \lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2}, \text{ where } N_q \neq 0.$$

Further, for $N_q > 0$ and $\lambda_1\lambda_2q_1^2 + \lambda_1\lambda_3q_2^2 + \lambda_2\lambda_3q_3^2 \neq 0$, q can be written in a polar form as follows:

$$q = \sqrt{N_q}(\cos \theta + \hat{q} \sin \theta),$$

where

$$\hat{q} = \frac{1}{\sqrt{\lambda_1 \lambda_2 q_1^2 + \lambda_1 \lambda_3 q_2^2 + \lambda_2 \lambda_3 q_3^2}} (q_1, q_2, q_3),$$

and

$$\cos \theta = \frac{q_0}{\sqrt{N_q}}, \quad \sin \theta = \sqrt{\frac{\lambda_1 \lambda_2 q_1^2 + \lambda_1 \lambda_3 q_2^2 + \lambda_2 \lambda_3 q_3^2}{N_q}},$$

and $\hat{q}$ is called 3-parameter generalized unit vector. Moreover, the following fundamental matrix M_q^{-1} is obtained for q :

$$M_q = \begin{pmatrix} q_0 & -\lambda_1 \lambda_2 q_1 & -\lambda_1 \lambda_3 q_2 & -\lambda_2 \lambda_3 q_3 \\ q_1 & q_0 & -\lambda_3 q_3 & \lambda_3 q_2 \\ q_2 & \lambda_2 q_3 & q_0 & -\lambda_2 q_1 \\ q_3 & -\lambda_1 q_2 & \lambda_1 q_1 & q_0 \end{pmatrix}$$

According to the values of $\lambda_{i \in \{1,2,3\}}$, the matrix M_q can be classified. For $\lambda_1 = 1, \lambda_2, \lambda_3 \in \mathbb{R}$, M_q for 2PGQ is constructed. For $\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$, then M_q for split quaternions is obtained. In addition to these, for $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then M_q for Hamilton quaternions is given.

Then, one can write some algebraic calculations for the matrix M_q :

- The determinant of M_q : $\det(M_q) = N_q^2$.
- The characteristic polynomial of M_q :

$$P_{M_q}(u) = (u^2 - 2uq_0 + q_0^2 + \lambda_1 \lambda_2 q_1^2 + \lambda_1 \lambda_3 q_2^2 + \lambda_2 \lambda_3 q_3^2)^2.$$

- The characteristic equation of M_q :

$$\begin{aligned} \det(M_q - uI_4) &= 0 \Leftrightarrow P_{M_q}(u) \\ &= (u^2 - 2uq_0 + q_0^2 + \lambda_1 \lambda_2 q_1^2 + \lambda_1 \lambda_3 q_2^2 + \lambda_2 \lambda_3 q_3^2)^2 = 0. \end{aligned}$$

- The characteristic equation enables to calculate the eigenvalues as follows:

$$\begin{cases} \mathfrak{T}_{1,2} = q_0 + \sqrt{-\lambda_1 \lambda_2 q_1^2 - \lambda_1 \lambda_3 q_2^2 - \lambda_2 \lambda_3 q_3^2}, \\ \mathfrak{T}_{3,4} = q_0 - \sqrt{-\lambda_1 \lambda_2 q_1^2 - \lambda_1 \lambda_3 q_2^2 - \lambda_2 \lambda_3 q_3^2}. \end{cases}$$

- The following equation holds:

$$\mathfrak{T}_{1,2} \mathfrak{T}_{3,4} = q_0^2 + \lambda_1 \lambda_2 q_1^2 + \lambda_1 \lambda_3 q_2^2 + \lambda_2 \lambda_3 q_3^2 = N_q.$$

- The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{1,2}$ are given as follows:

¹ In this study, this representation, which is actually the left matrix representation, will be used due to the non-commutativity of 3PGQs. With the same manner, the right matrix representation can also be considered.

$$\begin{pmatrix} \frac{\lambda_1 q_2 \sqrt{-\lambda_1 \lambda_2 q_1^2 - \lambda_1 \lambda_3 q_2^2 - \lambda_2 \lambda_3 q_3^2 - \lambda_1 \lambda_2 q_1 q_3}}{\lambda_1 q_2^2 + \lambda_2 q_3^2} & \frac{q_3 \sqrt{-\lambda_1 \lambda_2 q_1^2 - \lambda_1 \lambda_3 q_2^2 - \lambda_2 \lambda_3 q_3^2 + \lambda_1 q_1 q_2}}{\lambda_1 q_2^2 + \lambda_2 q_3^2} & 1 & 0 \end{pmatrix}^t$$

and

$$\begin{pmatrix} \frac{\lambda_2 q_3 \sqrt{-\lambda_1 \lambda_2 q_1^2 - \lambda_1 \lambda_3 q_2^2 - \lambda_2 \lambda_3 q_3^2 + \lambda_1 \lambda_2 q_1 q_2}}{\lambda_1 q_2^2 + \lambda_2 q_3^2} & -\frac{q_2 \sqrt{-\lambda_1 \lambda_2 q_1^2 - \lambda_1 \lambda_3 q_2^2 - \lambda_2 \lambda_3 q_3^2 - \lambda_2 q_1 q_3}}{\lambda_1 q_2^2 + \lambda_2 q_3^2} & 0 & 1 \end{pmatrix}^t$$

- The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{3,4}$ are given as follows:

$$\begin{pmatrix} -\frac{\lambda_1 q_2 \sqrt{-\lambda_1 \lambda_2 q_1^2 - \lambda_1 \lambda_3 q_2^2 - \lambda_2 \lambda_3 q_3^2 + \lambda_1 \lambda_2 q_1 q_3}}{\lambda_1 q_2^2 + \lambda_2 q_3^2} & -\frac{q_3 \sqrt{-\lambda_1 \lambda_2 q_1^2 - \lambda_1 \lambda_3 q_2^2 - \lambda_2 \lambda_3 q_3^2 - \lambda_1 q_1 q_2}}{\lambda_1 q_2^2 + \lambda_2 q_3^2} & 1 & 0 \end{pmatrix}^t$$

and

$$\begin{pmatrix} -\frac{\lambda_2 q_3 \sqrt{-\lambda_1 \lambda_2 q_1^2 - \lambda_1 \lambda_3 q_2^2 - \lambda_2 \lambda_3 q_3^2 - \lambda_1 \lambda_2 q_1 q_2}}{\lambda_1 q_2^2 + \lambda_2 q_3^2} & \frac{q_2 \sqrt{-\lambda_1 \lambda_2 q_1^2 - \lambda_1 \lambda_3 q_2^2 - \lambda_2 \lambda_3 q_3^2 + \lambda_2 q_1 q_3}}{\lambda_1 q_2^2 + \lambda_2 q_3^2} & 0 & 1 \end{pmatrix}^t,$$

where the notation $\mathfrak{t}$ represents the transpose of a matrix.

To get more detailed information concerning the 3PGQs, we refer to the studies [68, 69].

2.2 Generalized Tetranacci numbers

The generalized Tetranacci numbers are the largest family for the fourth-order recurrence numbers. Since the received values depend on the V_0, V_1, V_2, V_3 and r, s, t, u , this family includes several special cases.

In the following Table 3 and Table 4, some special cases of generalized Tetranacci numbers are given. In the Table 3, special cases are grouped according to the r, s, t, u values, and in the Table 4, some special cases with respect to the r, s, t, u and initial values are written. For more detailed information about the generalized Tetranacci numbers and their special cases, we refer to the studies [34, 39, 48, 49, 52, 57, 59-62, 66, 72, 74, 75].

Table 3
A Brief Classification of Generalized Tetranacci Numbers

Name	$\{V_n\}$ $= \{V_n(V_0, V_1, V_2, V_3; r, s, t, u)\}$	Recurrence Relation
G. Tetranacci (usual)	$\{\mathcal{A}_n\} = \{V_n(a, b, c, d; 1, 1, 1, 1)\}$	$\mathcal{A}_n = \mathcal{A}_{n-1} + \mathcal{A}_{n-2} + \mathcal{A}_{n-3} + \mathcal{A}_{n-4}$
G. F. Pell	$\{\mathcal{C}_n\} = \{V_n(a, b, c, d; 2, 1, 1, 1)\}$	$\mathcal{C}_n = 2\mathcal{C}_{n-1} + \mathcal{C}_{n-2} + \mathcal{C}_{n-3} + \mathcal{C}_{n-4}$
G. F. Jacobsthal	$\{\mathcal{F}_n\} = \{V_n(a, b, c, d; 1, 1, 1, 2)\}$	$\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} + \mathcal{F}_{n-3} + 2\mathcal{F}_{n-4}$
G. 4-Primes	$\{\mathcal{L}_n\} = \{V_n(a, b, c, d; 2, 3, 5, 7)\}$	$\mathcal{L}_n = 2\mathcal{L}_{n-1} + 3\mathcal{L}_{n-2} + 5\mathcal{L}_{n-3} + 7\mathcal{L}_{n-4}$

*G.: Generalized, F.: Fourth-Order

Table 4
Some Special Cases of Generalized Tetranacci Numbers

Name	$\{V_n\} = \{V_n(V_0, V_1, V_2, V_3; r, s, t, u)\}$	Recurrence Relation
Tetranacci	$\{A_n\} = \{V_n(0, 1, 1, 2; 1, 1, 1, 1)\}$	$A_n = A_{n-1} + A_{n-2} + A_{n-3} + A_{n-4}$
Tetranacci-Lucas	$\{B_n\} = \{V_n(4, 1, 3, 7; 1, 1, 1, 1)\}$	$B_n = B_{n-1} + B_{n-2} + B_{n-3} + B_{n-4}$
F. Pell	$\{C_n\} = \{V_n(0, 1, 2, 5; 2, 1, 1, 1)\}$	$C_n = 2C_{n-1} + C_{n-2} + C_{n-3} + C_{n-4}$
F. Pell-Lucas	$\{D_n\} = \{V_n(4, 2, 6, 17; 2, 1, 1, 1)\}$	$D_n = 2D_{n-1} + D_{n-2} + D_{n-3} + D_{n-4}$
M. F. Pell	$\{E_n\} = \{V_n(0, 1, 1, 3; 2, 1, 1, 1)\}$	$E_n = 2E_{n-1} + E_{n-2} + E_{n-3} + E_{n-4}$
F. Jacobsthal	$\{F_n\} = \{V_n(1, 1, 1, 2; 1, 1, 1, 2)\}$	$F_n = F_{n-1} + F_{n-2} + F_{n-3} + 2F_{n-4}$
F. Jacobsthal-Lucas	$\{G_n\} = \{V_n(2, 1, 5, 10; 1, 1, 1, 2)\}$	$G_n = G_{n-1} + G_{n-2} + G_{n-3} + 2G_{n-4}$
M. F. Jacobsthal	$\{H_n\} = \{V_n(3, 1, 3, 10; 1, 1, 1, 2)\}$	$H_n = H_{n-1} + H_{n-2} + H_{n-3} + 2H_{n-4}$
F. Jacobsthal-Perrin	$\{I_n\} = \{V_n(3, 0, 2, 8; 1, 1, 1, 2)\}$	$I_n = I_{n-1} + I_{n-2} + I_{n-3} + 2I_{n-4}$
A. F. Jacobsthal	$\{J_n\} = \{V_n(0, 1, 1, 2; 1, 1, 1, 2)\}$	$J_n = J_{n-1} + J_{n-2} + J_{n-3} + 2J_{n-4}$
M. F. Jacobsthal-Lucas	$\{K_n\} = \{V_n(4, 1, 3, 7; 1, 1, 1, 2)\}$	$K_n = K_{n-1} + K_{n-2} + K_{n-3} + 2K_{n-4}$
4-Primes	$\{L_n\} = \{V_n(0, 0, 1, 2; 2, 3, 5, 7)\}$	$L_n = 2L_{n-1} + 3L_{n-2} + 5L_{n-3} + 7L_{n-4}$
Lucas 4-Primes	$\{M_n\} = \{V_n(4, 2, 10, 41; 2, 3, 5, 7)\}$	$M_n = 2M_{n-1} + 3M_{n-2} + 5M_{n-3} + 7M_{n-4}$
M. 4-Primes	$\{N_n\} = \{V_n(0, 0, 1, 1; 2, 3, 5, 7)\}$	$N_n = 2N_{n-1} + 3N_{n-2} + 5N_{n-3} + 7N_{n-4}$

*F.: Fourth-Order, M.: Modified, A.: Adjusted

For all $n \in \mathbb{Z}^2$, the Binet formula of the generalized Tetranacci numbers is given [61]:

$$V_n = \frac{\hat{P}x_1^n}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + \frac{\hat{R}x_2^n}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} + \frac{\hat{S}x_3^n}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{\hat{T}x_4^n}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}, \quad (2.1)$$

where

$$\begin{cases} \hat{P} = d - (x_2 + x_3 + x_4)c + (x_2x_3 + x_2x_4 + x_3x_4)b + x_2x_3x_4a, \\ \hat{R} = d - (x_1 + x_3 + x_4)c + (x_1x_3 + x_1x_4 + x_3x_4)b + x_1x_3x_4a, \\ \hat{S} = d - (x_1 + x_2 + x_4)c + (x_1x_2 + x_1x_4 + x_2x_4)b + x_1x_2x_4a, \\ \hat{T} = d - (x_1 + x_2 + x_3)c + (x_1x_2 + x_1x_3 + x_2x_3)b + x_1x_2x_3a, \end{cases} \quad (2.2)$$

and x_1, x_2, x_3, x_4 are the roots of the equation $x^4 - rx^3 - sx^2 - tx - u = 0$ which is called the characteristic equation of generalized Tetranacci numbers, and where

² Binet formula of recurrence sequences is valid for all integers, see [25, 61].

$$\begin{cases} x_1 + x_2 + x_3 + x_4 = r, \\ x_1x_2 + x_1x_3 + x_1x_4 + x_2x_3 + x_2x_4 + x_3x_4 = -s, \\ x_1x_2x_3 + x_1x_2x_4 + x_1x_3x_4 + x_2x_3x_4 = t, \\ x_1x_2x_3x_4 = -u. \end{cases}$$

For all $n \in \mathbb{Z}^+$, by using the Kalman's matrix formulation in [40], Soykan [61] give the following matrix formulation of generalized Tetranacci numbers:

$$\begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} V_{n+3} \\ V_{n+2} \\ V_{n+1} \\ V_n \end{pmatrix} = \begin{pmatrix} V_3 \\ V_2 \\ V_1 \\ V_0 \end{pmatrix}. \quad (2.3)$$

3. 3-Parameter Generalized Quaternions with Generalized Tetranacci Numbers Components

The main purpose of this section is to introduce and examine the 3-parameter generalized quaternions with generalized Tetranacci numbers components (3PGQs with generalized Tetranacci numbers components) with non-negative and negative subscripts, and some special cases of them. We construct some fundamental and new equations and properties with respect to this special new number system.

Definition 3.1: Let V_n and V_{-n} denote the n th generalized Tetranacci number with non-negative and negative subscripts, respectively. The n th 3PGQ with generalized Tetranacci numbers components is defined as follows:

$$\mathcal{V}_n = V_n + V_{n+1}e_1 + V_{n+2}e_2 + V_{n+3}e_3 \text{ for all } n \geq 0. \quad (3.1)$$

Also, the $-n$ th 3PGQ with generalized Tetranacci numbers components with negative subscript is defined as follows:

$$\mathcal{V}_{-n} = V_{-n} + V_{-n+1}e_1 + V_{-n+2}e_2 + V_{-n+3}e_3 \text{ for all } n \geq 1. \quad (3.2)$$

Here, e_1, e_2, e_3 satisfy the rules in the Table 1.

It also should be noted that the equation (3.1) is valid for all integers.

Throughout this article, for the sake of brevity while some equations and properties are expressed separately for 3PGQs with generalized Tetranacci numbers components with both non-negative subscripted and negative subscripted, some of them are written for all integers.

The classification of the 3PGQs with generalized Tetranacci numbers components can be examined in the following Table 5.

Table 5
Classification of 3PGQs with Generalized Tetranacci Numbers Components

For	Type of 3PGQs with Generalized Tetranacci N.C.
$\lambda_1 = 1, \lambda_2, \lambda_3 \in \mathbb{R}$	2PGQs with generalized Tetranacci N.C. [50, 65]
$\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$	Split quaternion with generalized Tetranacci N.C.

$\lambda_1 = \lambda_2 = \lambda_3 = 1$	Hamilton quaternions with generalized Tetranacci N.C. [50 65]
$\lambda_1 = \lambda_2 = 1, \lambda_3 = 0$	Semi-quaternions with generalized Tetranacci N.C.
$\lambda_1 = 1, \lambda_2 = -1, \lambda_3 = 0$	Split semi-quaternions with generalized Tetranacci N.C.
$\lambda_1 = 1, \lambda_2 = \lambda_3 = 0$	1/4-quaternions with generalized Tetranacci N.C.

*N.C.: Numbers Components

Now, let us write some values of 3PGQs with generalized Tetranacci numbers components.

$$\begin{cases} \vdots \\ \mathcal{V}_3 = d + (rd + sc + tb + ua)e_1 + (r^2d + rsc + rtb + rua + sd + tc + ub)e_2 \\ \quad + ((r^2 + s)(rd + sc + tb + ua) + r(sd + tc + ub) + td + uc)e_3 \\ \mathcal{V}_2 = c + de_1 + (rd + sc + tb + ua)e_2 + (r^2d + rsc + rtb + rua + sd + tc + ub)e_3 \\ \mathcal{V}_1 = b + ce_1 + de_2 + (rd + sc + tb + ua)e_3 \\ \mathcal{V}_0 = a + be_1 + ce_2 + de_3 \\ \mathcal{V}_{-1} = \frac{1}{u}(d - rc - sb - ta) + ae_1 + be_2 + ce_3 \\ \mathcal{V}_{-2} = \frac{1}{u}(c - rb - sa) - \frac{t}{u^2}(d - rc - sb - ta) + \frac{1}{u}(d - rc - sb - ta)e_1 + ae_2 + be_3 \\ \mathcal{V}_{-3} = \frac{t^2}{u^3}(d - rc - sb - ta) - \frac{t}{u^2}(c - rb - sa) - \frac{s}{u^2}(d - rc - sb - ta) + \frac{1}{u}(b - ra) \\ \quad + \left[\frac{1}{u}(c - rb - sa) - \frac{t}{u^2}(d - rc - sb - ta) \right] e_1 + \frac{1}{u}(d - rc - sb - ta)e_2 + ae_3 \\ \vdots \end{cases}$$

Algebraic Properties

We shall give the following algebraic properties of 3PGQ with generalized Tetranacci number components:

Let for all $n, m \in \mathbb{Z}$, $\mathcal{V}_n$ and $\mathcal{V}_m$ be the n th and m th 3PGQ with generalized Tetranacci numbers components, respectively. We begin with equality, addition/subtraction, multiplication by a scalar, scalar and vector parts, multiplication, conjugation, inner product, norm, and inverse.

- Equality: $\mathcal{V}_n = \mathcal{V}_m \Leftrightarrow V_n = V_m, V_{n+1} = V_{m+1}, V_{n+2} = V_{m+2}, V_{n+3} = V_{m+3}$.

- Addition/Subtraction:

$$\begin{aligned} \mathcal{V}_n \pm \mathcal{V}_m = & (V_n \pm V_m) + (V_{n+1} \pm V_{m+1})e_1 + (V_{n+2} \pm V_{m+2})e_2 \\ & + (V_{n+3} \pm V_{m+3})e_3. \end{aligned}$$

- Multiplication by a scalar:

$$\tilde{c} \mathcal{V}_n = \tilde{c} V_n + \tilde{c} V_{n+1}e_1 + \tilde{c} V_{n+2}e_2 + \tilde{c} V_{n+3}e_3, \tilde{c} \in \mathbb{R}.$$

- Scalar and vector parts: The scalar part of $\mathcal{V}_n$ is represented as $S_{\mathcal{V}_n}$ and $S_{\mathcal{V}_n} = V_n$. The vector part of $\mathcal{V}_n$ is denoted by $\vec{V}_{\mathcal{V}_n}$ and $\vec{V}_{\mathcal{V}_n} = V_{n+1}e_1 + V_{n+2}e_2 + V_{n+3}e_3$. Also, $\mathcal{V}_n = S_{\mathcal{V}_n} + \vec{V}_{\mathcal{V}_n}$. Then, we have the properties: $S_{\mathcal{V}_n \pm \mathcal{V}_m} = V_n \pm V_m = S_{\mathcal{V}_n} \pm S_{\mathcal{V}_m}$ and $\vec{V}_{\mathcal{V}_n \pm \mathcal{V}_m} = \vec{V}_{\mathcal{V}_n} \pm \vec{V}_{\mathcal{V}_m}$.

- Multiplication:

$$\mathcal{V}_n \mathcal{V}_m = S_{\mathcal{V}_n} S_{\mathcal{V}_m} - f(\vec{\mathcal{V}}_{\mathcal{V}_n}, \vec{\mathcal{V}}_{\mathcal{V}_m}) + S_{\mathcal{V}_n} \vec{\mathcal{V}}_{\mathcal{V}_m} + S_{\mathcal{V}_m} \vec{\mathcal{V}}_{\mathcal{V}_n} + \vec{\mathcal{V}}_{\mathcal{V}_n} \wedge \vec{\mathcal{V}}_{\mathcal{V}_m},$$

$$\text{where } f(\vec{\mathcal{V}}_{\mathcal{V}_n}, \vec{\mathcal{V}}_{\mathcal{V}_m}) = \lambda_1 \lambda_2 V_{n+1} V_{m+1} + \lambda_1 \lambda_3 V_{n+2} V_{m+2} + \lambda_2 \lambda_3 V_{n+3} V_{m+3}$$

and

$$\begin{aligned} \vec{\mathcal{V}}_{\mathcal{V}_n} \wedge \vec{\mathcal{V}}_{\mathcal{V}_m} &= \begin{vmatrix} \lambda_3 e_1 & \lambda_2 e_2 & \lambda_1 e_3 \\ V_{n+1} & V_{n+2} & V_{n+3} \\ V_{m+1} & V_{m+2} & V_{m+3} \end{vmatrix} \\ &= \lambda_3 (V_{n+2} V_{m+3} - V_{n+3} V_{m+2}) e_1 + \lambda_2 (V_{n+3} V_{m+1} - V_{n+1} V_{m+3}) e_2 \\ &\quad + \lambda_1 (V_{n+1} V_{m+2} - V_{n+2} V_{m+1}) e_3. \end{aligned}$$

According to the Table 1, the multiplication can be expressed as follows:

$$\begin{aligned} \mathcal{V}_n \mathcal{V}_m &= V_n V_m - \lambda_1 \lambda_2 V_{n+1} V_{m+1} - \lambda_1 \lambda_3 V_{n+2} V_{m+2} - \lambda_2 \lambda_3 V_{n+3} V_{m+3} \\ &\quad + (V_n V_{m+1} + V_m V_{n+1} + \lambda_3 (V_{n+2} V_{m+3} - V_{n+3} V_{m+2})) e_1 \\ &\quad + (V_n V_{m+2} + V_m V_{n+2} + \lambda_2 (V_{n+3} V_{m+1} - V_{n+1} V_{m+3})) e_2 \\ &\quad + (V_n V_{m+3} + V_m V_{n+3} + \lambda_1 (V_{n+1} V_{m+2} - V_{n+2} V_{m+1})) e_3. \end{aligned}$$

- Conjugation: $\overline{\mathcal{V}}_n = V_n - V_{n+1} e_1 - V_{n+2} e_2 - V_{n+3} e_3$ in which $\overline{\mathcal{V}}_n$ represents the conjugation of $\mathcal{V}_n$.

- Inner product:

$$\langle \mathcal{V}_n, \mathcal{V}_m \rangle = V_n V_m + \lambda_1 \lambda_2 V_{n+1} V_{m+1} + \lambda_1 \lambda_3 V_{n+2} V_{m+2} + \lambda_2 \lambda_3 V_{n+3} V_{m+3}.$$

- Norm:

$$N_{\mathcal{V}_n} = \mathcal{V}_n \overline{\mathcal{V}}_n = \overline{\mathcal{V}}_n \mathcal{V}_n = V_n^2 + \lambda_1 \lambda_2 V_{n+1}^2 + \lambda_1 \lambda_3 V_{n+2}^2 + \lambda_2 \lambda_3 V_{n+3}^2.$$

- Inverse:

$$\mathcal{V}_n^{-1} = \frac{\overline{\mathcal{V}}_n}{N_{\mathcal{V}_n}} = \frac{V_n - V_{n+1} e_1 - V_{n+2} e_2 - V_{n+3} e_3}{V_n^2 + \lambda_1 \lambda_2 V_{n+1}^2 + \lambda_1 \lambda_3 V_{n+2}^2 + \lambda_2 \lambda_3 V_{n+3}^2}, \quad N_{\mathcal{V}_n} \neq 0.$$

Theorem 3.1: Let $\mathcal{V}_n$ be the n th 3PGQ with generalized Tetranacci number components with non-negative subscripts. The following recurrence relation holds:

$$\mathcal{V}_n = r \mathcal{V}_{n-1} + s \mathcal{V}_{n-2} + t \mathcal{V}_{n-3} + u \mathcal{V}_{n-4} \text{ for all } n \geq 4. \quad (3.3)$$

Also, $\mathcal{V}_{-n}$ be the $-n$ th 3PGQ with generalized Tetranacci number components with negative subscripts. The below recurrence relation holds for it:

$$\mathcal{V}_{-n} = \frac{1}{u} \mathcal{V}_{-(n-4)} - \frac{t}{u} \mathcal{V}_{-(n-1)} - \frac{s}{u} \mathcal{V}_{-(n-2)} - \frac{r}{u} \mathcal{V}_{-(n-3)} \text{ for all } n \geq 1. \quad (3.4)$$

Proof: By the equations (1.1) and (3.1), we get the equation (3.3). Also, the equation (3.4) can be shown with the help of the equations (1.2) and (3.2).

Indeed, one can see that the recurrence relation (3.3) is valid for all integers.

Now, we give some special classifications in Table 6 and Table 7 for 3PGQs with generalized Tetranacci numbers components in the light of Table 3 and Table 4. The following properties are valid for all $n \in \mathbb{Z}$, because we have said the (3.1) and (3.3) are valid for all $n \in \mathbb{Z}$.

Table 6
A Brief Classification of 3PGQs with Generalized Tetranacci Numbers Components

Name	Definition	Recurrence Relation
3PGQs with G. Tetranacci N.C.	$\check{\mathcal{A}}_n = \mathcal{A}_n + \mathcal{A}_{n+1}e_1 + \mathcal{A}_{n+2}e_2 + \mathcal{A}_{n+3}e_3$	$\check{\mathcal{A}}_n = \check{\mathcal{A}}_{n-1} + \check{\mathcal{A}}_{n-2} + \check{\mathcal{A}}_{n-3} + \check{\mathcal{A}}_{n-4}$
3PGQs with G. F. Pell N.C.	$\check{\mathcal{C}}_n = \mathcal{C}_n + \mathcal{C}_{n+1}e_1 + \mathcal{C}_{n+2}e_2 + \mathcal{C}_{n+3}e_3$	$\check{\mathcal{C}}_n = 2\check{\mathcal{C}}_{n-1} + \check{\mathcal{C}}_{n-2} + \check{\mathcal{C}}_{n-3} + \check{\mathcal{C}}_{n-4}$
3PGQs with G. F. Jacobsthal N.C.	$\check{\mathcal{F}}_n = \mathcal{F}_n + \mathcal{F}_{n+1}e_1 + \mathcal{F}_{n+2}e_2 + \mathcal{F}_{n+3}e_3$	$\check{\mathcal{F}}_n = \check{\mathcal{F}}_{n-1} + \check{\mathcal{F}}_{n-2} + \check{\mathcal{F}}_{n-3} + 2\check{\mathcal{F}}_{n-4}$
3PGQs with G. 4-Primes N.C.	$\check{\mathcal{L}}_n = \mathcal{L}_n + \mathcal{L}_{n+1}e_1 + \mathcal{L}_{n+2}e_2 + \mathcal{L}_{n+3}e_3$	$\check{\mathcal{L}}_n = 2\check{\mathcal{L}}_{n-1} + 3\check{\mathcal{L}}_{n-2} + 5\check{\mathcal{L}}_{n-3} + 7\check{\mathcal{L}}_{n-4}$

*G.: Generalized, F.: Fourth-Order, N.C.: Numbers Components

Table 7
Some Special Cases of 3PGQs with Generalized Tetranacci Numbers Components

Name	Definition	Recurrence Relation
3PGQs with Tetranacci N.C.	$\mathcal{A}_n = \mathcal{A}_n + \mathcal{A}_{n+1}e_1 + \mathcal{A}_{n+2}e_2 + \mathcal{A}_{n+3}e_3$	$\mathcal{A}_n = \mathcal{A}_{n-1} + \mathcal{A}_{n-2} + \mathcal{A}_{n-3} + \mathcal{A}_{n-4}$
3PGQs with Tetranacci-Lucas N.C.	$\mathcal{B}_n = \mathcal{B}_n + \mathcal{B}_{n+1}e_1 + \mathcal{B}_{n+2}e_2 + \mathcal{B}_{n+3}e_3$	$\mathcal{B}_n = \mathcal{B}_{n-1} + \mathcal{B}_{n-2} + \mathcal{B}_{n-3} + \mathcal{B}_{n-4}$
3PGQs with F. Pell N.C.	$\mathcal{C}_n = \mathcal{C}_n + \mathcal{C}_{n+1}e_1 + \mathcal{C}_{n+2}e_2 + \mathcal{C}_{n+3}e_3$	$\mathcal{C}_n = 2\mathcal{C}_{n-1} + \mathcal{C}_{n-2} + \mathcal{C}_{n-3} + \mathcal{C}_{n-4}$
3PGQs with F. Pell-Lucas N.C.	$\mathcal{D}_n = \mathcal{D}_n + \mathcal{D}_{n+1}e_1 + \mathcal{D}_{n+2}e_2 + \mathcal{D}_{n+3}e_3$	$\mathcal{D}_n = 2\mathcal{D}_{n-1} + \mathcal{D}_{n-2} + \mathcal{D}_{n-3} + \mathcal{D}_{n-4}$
3PGQs with M. F. Pell N.C.	$\mathcal{E}_n = \mathcal{E}_n + \mathcal{E}_{n+1}e_1 + \mathcal{E}_{n+2}e_2 + \mathcal{E}_{n+3}e_3$	$\mathcal{E}_n = 2\mathcal{E}_{n-1} + \mathcal{E}_{n-2} + \mathcal{E}_{n-3} + \mathcal{E}_{n-4}$
3PGQs with F. Jacobsthal N.C.	$\mathcal{F}_n = \mathcal{F}_n + \mathcal{F}_{n+1}e_1 + \mathcal{F}_{n+2}e_2 + \mathcal{F}_{n+3}e_3$	$\mathcal{F}_n = \mathcal{F}_{n-1} + \mathcal{F}_{n-2} + \mathcal{F}_{n-3} + 2\mathcal{F}_{n-4}$
3PGQs with F. Jacobsthal-Lucas N.C.	$\mathcal{G}_n = \mathcal{G}_n + \mathcal{G}_{n+1}e_1 + \mathcal{G}_{n+2}e_2 + \mathcal{G}_{n+3}e_3$	$\mathcal{G}_n = \mathcal{G}_{n-1} + \mathcal{G}_{n-2} + \mathcal{G}_{n-3} + 2\mathcal{G}_{n-4}$
3PGQs with M. F. Jacobsthal N.C.	$\mathcal{H}_n = \mathcal{H}_n + \mathcal{H}_{n+1}e_1 + \mathcal{H}_{n+2}e_2 + \mathcal{H}_{n+3}e_3$	$\mathcal{H}_n = \mathcal{H}_{n-1} + \mathcal{H}_{n-2} + \mathcal{H}_{n-3} + 2\mathcal{H}_{n-4}$
3PGQs with F. Jacobsthal-Perrin N.C.	$\mathcal{I}_n = \mathcal{I}_n + \mathcal{I}_{n+1}e_1 + \mathcal{I}_{n+2}e_2 + \mathcal{I}_{n+3}e_3$	$\mathcal{I}_n = \mathcal{I}_{n-1} + \mathcal{I}_{n-2} + \mathcal{I}_{n-3} + 2\mathcal{I}_{n-4}$
3PGQs with A. F. Jacobsthal N.C.	$\mathcal{J}_n = \mathcal{J}_n + \mathcal{J}_{n+1}e_1 + \mathcal{J}_{n+2}e_2 + \mathcal{J}_{n+3}e_3$	$\mathcal{J}_n = \mathcal{J}_{n-1} + \mathcal{J}_{n-2} + \mathcal{J}_{n-3} + 2\mathcal{J}_{n-4}$
3PGQs with M. F. Jacobsthal-Lucas N.C.	$\mathcal{K}_n = \mathcal{K}_n + \mathcal{K}_{n+1}e_1 + \mathcal{K}_{n+2}e_2 + \mathcal{K}_{n+3}e_3$	$\mathcal{K}_n = \mathcal{K}_{n-1} + \mathcal{K}_{n-2} + \mathcal{K}_{n-3} + 2\mathcal{K}_{n-4}$
3PGQs with 4-Primes N.C.	$\mathcal{L}_n = \mathcal{L}_n + \mathcal{L}_{n+1}e_1 + \mathcal{L}_{n+2}e_2 + \mathcal{L}_{n+3}e_3$	$\mathcal{L}_n = 2\mathcal{L}_{n-1} + 3\mathcal{L}_{n-2} + 5\mathcal{L}_{n-3} + 7\mathcal{L}_{n-4}$
3PGQs with Lucas 4-Primes N.C.	$\mathcal{M}_n = \mathcal{M}_n + \mathcal{M}_{n+1}e_1 + \mathcal{M}_{n+2}e_2 + \mathcal{M}_{n+3}e_3$	$\mathcal{M}_n = 2\mathcal{M}_{n-1} + 3\mathcal{M}_{n-2} + 5\mathcal{M}_{n-3} + 7\mathcal{M}_{n-4}$
3PGQs with M. 4-Primes N.C.	$\mathcal{N}_n = \mathcal{N}_n + \mathcal{N}_{n+1}e_1 + \mathcal{N}_{n+2}e_2 + \mathcal{N}_{n+3}e_3$	$\mathcal{N}_n = 2\mathcal{N}_{n-1} + 3\mathcal{N}_{n-2} + 5\mathcal{N}_{n-3} + 7\mathcal{N}_{n-4}$

*F.: Fourth-Order, M.: Modified, A.: Adjusted, N.C.: Numbers Components

Theorem 3.2: For all $n \in \mathbb{Z}$, Binet formula of the 3PGQs with generalized Tetranacci number components is as:

$$\mathcal{V}_n = \frac{\hat{P}x_1^n x_1}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + \frac{\hat{R}x_2^n x_2}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} + \frac{\hat{S}x_3^n x_3}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{\hat{T}x_4^n x_4}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}, \quad (3.5)$$

where $\hat{P}, \hat{R}, \hat{S}, \hat{T}$ are given in the equation (2.2) and

$$\begin{cases} \underline{x_1} = 1 + x_1 e_1 + x_1^2 e_2 + x_1^3 e_3, \underline{x_2} = 1 + x_2 e_1 + x_2^2 e_2 + x_2^3 e_3, \\ \underline{x_3} = 1 + x_3 e_1 + x_3^2 e_2 + x_3^3 e_3, \underline{x_4} = 1 + x_4 e_1 + x_4^2 e_2 + x_4^3 e_3. \end{cases}$$

Proof: With the help of the equations (2.1) and (3.1), we have:

$$\begin{aligned} \mathcal{V}_n &= V_n + V_{n+1}e_1 + V_{n+2}e_2 + V_{n+3}e_3 \\ &= \frac{\hat{P}x_1^n}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + \frac{\hat{R}x_2^n}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\ &\quad + \frac{\hat{S}x_3^n}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{\hat{T}x_4^n}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \\ &\quad + \left(\frac{\hat{P}x_1^{n+1}}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + \frac{\hat{R}x_2^{n+1}}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \right. \\ &\quad \left. + \frac{\hat{S}x_3^{n+1}}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{\hat{T}x_4^{n+1}}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \right) e_1 \\ &\quad + \left(\frac{\hat{P}x_1^{n+2}}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + \frac{\hat{R}x_2^{n+2}}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \right. \\ &\quad \left. + \frac{\hat{S}x_3^{n+2}}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{\hat{T}x_4^{n+2}}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \right) e_2 \\ &\quad + \left(\frac{\hat{P}x_1^{n+3}}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + \frac{\hat{R}x_2^{n+3}}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \right. \\ &\quad \left. + \frac{\hat{S}x_3^{n+3}}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{\hat{T}x_4^{n+3}}{(x_4-x_1)(x_4-x_2)(x_4-x_3)} \right) e_3 \\ &= \frac{\hat{P}x_1^n x_1}{(x_1-x_2)(x_1-x_3)(x_1-x_4)} + \frac{\hat{R}x_2^n x_2}{(x_2-x_1)(x_2-x_3)(x_2-x_4)} \\ &\quad + \frac{\hat{S}x_3^n x_3}{(x_3-x_1)(x_3-x_2)(x_3-x_4)} + \frac{\hat{T}x_4^n x_4}{(x_4-x_1)(x_4-x_2)(x_4-x_3)}. \end{aligned}$$

This finishes the proof.

Theorem 3.3: The generating functions are written for 3PGQs with generalized Tetranacci numbers components with non-negative and negative subscripts, respectively.

$$\sum_{n=0}^{\infty} \mathcal{V}_n x^n = \frac{v_0 + (v_1 - r v_0)x + (v_2 - r v_1 - s v_0)x^2 + (v_3 - r v_2 - s v_1 - t v_0)x^3}{1 - rx - sx^2 - tx^3 - ux^4} \quad (3.6)$$

$$\sum_{n=0}^{\infty} \mathcal{V}_{-n} x^n = \frac{v_0 + (v_{-1} + \frac{t}{u} v_0)x + (v_{-2} + \frac{t}{u} v_{-1} + \frac{s}{u} v_0)x^2 + (v_{-3} + \frac{t}{u} v_{-2} + \frac{s}{u} v_{-1} + \frac{r}{u} v_0)x^3}{1 + \frac{t}{u}x + \frac{s}{u}x^2 + \frac{r}{u}x^3 - \frac{1}{u}x^4} \quad (3.7)$$

Proof: Assume that $\sum_{n=0}^{\infty} \mathcal{V}_n x^n = \mathcal{V}_0 + \mathcal{V}_1 x + \mathcal{V}_2 x^2 + \dots + \mathcal{V}_n x^n + \dots$ be generating function of $\mathcal{V}_n$. Multiply both sides of the equality by rx, sx^2, tx^3, ux^4 , and then some required calculations are done, and by using the equation (3.3), we get:

$$(1 - rx - sx^2 - tx^3 - ux^4) \sum_{n=0}^{\infty} \mathcal{V}_n x^n = \mathcal{V}_0 + (\mathcal{V}_1 - r\mathcal{V}_0)x + (\mathcal{V}_2 - r\mathcal{V}_1 - s\mathcal{V}_0)x^2 + (\mathcal{V}_3 - r\mathcal{V}_2 - s\mathcal{V}_1 - t\mathcal{V}_0)x^3.$$

Consequently, we have the equation (3.6). By using the equation (3.4), then we get the equation (3.7).

Theorem 3.4: *The exponential generating functions are obtained for 3PGQs with generalized Tetranacci numbers components with non-negative and negative subscripts, respectively.*

$$\sum_{n=0}^{\infty} \mathcal{V}_n \frac{y^n}{n!} = \frac{\hat{P}x_1 e^{x_1 y}}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} + \frac{\hat{R}x_2 e^{x_2 y}}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} + \frac{\hat{S}x_3 e^{x_3 y}}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + \frac{\hat{T}x_4 e^{x_4 y}}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} \quad (3.8)$$

$$\sum_{n=0}^{\infty} \mathcal{V}_{-n} \frac{y^n}{n!} = \frac{\hat{P}x_1 e^{\frac{y}{x_1}}}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} + \frac{\hat{R}x_2 e^{\frac{y}{x_2}}}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} + \frac{\hat{S}x_3 e^{\frac{y}{x_3}}}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + \frac{\hat{T}x_4 e^{\frac{y}{x_4}}}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} \quad (3.9)$$

Proof: With the help of the equation (3.5), we obtain:

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{V}_n \frac{y^n}{n!} &= \sum_{n=0}^{\infty} \left(\frac{\hat{P}x_1^n x_1}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} + \frac{\hat{R}x_2^n x_2}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} \right. \\ &\quad \left. + \frac{\hat{S}x_3^n x_3}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + \frac{\hat{T}x_4^n x_4}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} \right) \frac{y^n}{n!} \\ &= \frac{\hat{P}x_1 e^{x_1 y}}{(x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} + \frac{\hat{R}x_2 e^{x_2 y}}{(x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} \\ &\quad + \frac{\hat{S}x_3 e^{x_3 y}}{(x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + \frac{\hat{T}x_4 e^{x_4 y}}{(x_4 - x_1)(x_4 - x_2)(x_4 - x_3)}. \end{aligned}$$

The equation (3.8) is proved, and the equation (3.9) can be demonstrated by the equation (3.5) substituting $n \rightarrow -n$ in (3.5) since it is valid for all $n \in \mathbb{Z}$.

Theorem 3.5: *The Poisson generating functions are satisfied for 3PGQs with generalized Tetranacci numbers components with non-negative and negative subscripts, respectively.*

$$\begin{aligned} e^{-y} \sum_{n=0}^{\infty} \mathcal{V}_n \frac{y^n}{n!} &= \frac{\hat{P}x_1 e^{x_1 y}}{e^y (x_1 - x_2)(x_1 - x_3)(x_1 - x_4)} + \frac{\hat{R}x_2 e^{x_2 y}}{e^y (x_2 - x_1)(x_2 - x_3)(x_2 - x_4)} \\ &\quad + \frac{\hat{S}x_3 e^{x_3 y}}{e^y (x_3 - x_1)(x_3 - x_2)(x_3 - x_4)} + \frac{\hat{T}x_4 e^{x_4 y}}{e^y (x_4 - x_1)(x_4 - x_2)(x_4 - x_3)} \end{aligned}$$

$$e^{-y} \sum_{n=0}^{\infty} \mathcal{V}_{-n} \frac{y^n}{n!} = \frac{\underline{p}x_1 e^{\frac{y}{x_1}}}{e^{y(x_1-x_2)(x_1-x_3)(x_1-x_4)}} + \frac{\underline{R}x_2 e^{\frac{y}{x_2}}}{e^{y(x_2-x_1)(x_2-x_3)(x_2-x_4)}} \\ + \frac{\underline{s}x_3 e^{\frac{y}{x_3}}}{e^{y(x_3-x_1)(x_3-x_2)(x_3-x_4)}} + \frac{\underline{T}x_4 e^{\frac{y}{x_4}}}{e^{y(x_4-x_1)(x_4-x_2)(x_4-x_3)}}$$

Proof: With the help of the equations (3.8) and (3.9), we get the desired results, since Poisson generating function is obtained as multiplying the exponential generating function by e^{-y} (see [70]).

By utilizing the [62], we get the following summation formulas in the following Theorem 3.6 and Theorem 3.7, because of the fact that all of them can be shown by using the mathematical induction, we omit the proofs.

Theorem 3.6: *The following summation formulas hold for 3PGQs with generalized Tetranacci numbers components with non-negative subscripts:*

$$\begin{aligned} \bullet \quad \sum_{n=0}^m \mathcal{V}_n &= \frac{\mathcal{V}_{m+4} + (1-r)\mathcal{V}_{m+3} + (1-r-s)\mathcal{V}_{m+2} + (1-r-s-t)\mathcal{V}_{m+1} - \mathcal{V}_3}{r+s+t+u-1}, \\ \bullet \quad \sum_{n=0}^m \mathcal{V}_{2n} &= \frac{(1-s-u)\mathcal{V}_{2m+2} + (t+rs+ru)\mathcal{V}_{2m+1} + (t^2-u^2+rt-su+u)\mathcal{V}_{2m} \\ &\quad + (ru+tu)\mathcal{V}_{2m-1} - (r+t)\mathcal{V}_3 + (s+u+rt+r^2-1)\mathcal{V}_2 \\ &\quad + (st-ru-t)\mathcal{V}_1 + (r^2-s^2+t^2+2s+u+2rt-su-1)\mathcal{V}_0}{(r+s+t+u-1)(r-s+t-u+1)}, \\ \bullet \quad \sum_{n=0}^m \mathcal{V}_{2n+1} &= \frac{(r+t)\mathcal{V}_{2m+2} + (-s^2+t^2-u^2+rt-2su+s+u)\mathcal{V}_{2m+1} \\ &\quad + (t+ru-st)\mathcal{V}_{2m-u}(s+u-1)\mathcal{V}_{2m-1} + (s+u-1)\mathcal{V}_3 \\ &\quad - (t+rs+ru)\mathcal{V}_2 + (r^2-s^2+rt-su+2s+u-1)\mathcal{V}_1 - u(r+t)\mathcal{V}_0}{(r+s+t+u-1)(r-s+t-u+1)}, \end{aligned}$$

where denominators are not equal to zero.

Particular Cases 3.10: If $s + u = 1$, we get the following summation formulas:

$$\begin{aligned} \bullet \quad \sum_{n=0}^m \mathcal{V}_{2n} &= \frac{\mathcal{V}_{2m+1} + t\mathcal{V}_{2m} + u\mathcal{V}_{2m-1} - \mathcal{V}_3 + r\mathcal{V}_2 - u\mathcal{V}_1 + (r+t)\mathcal{V}_0}{r+t}, \\ \bullet \quad \sum_{n=0}^m \mathcal{V}_{2n+1} &= \frac{\mathcal{V}_{2m+2} + t\mathcal{V}_{2m+1} + u\mathcal{V}_{2m} - \mathcal{V}_2 + r\mathcal{V}_1 - u\mathcal{V}_0}{r+t}, \end{aligned}$$

where $r + t \neq 0$.

Theorem 3.7: *The following summation formulas hold for 3PGQs with generalized Tetranacci numbers components with negative subscripts:*

$$\begin{aligned} \bullet \quad \sum_{n=1}^m \mathcal{V}_{-n} &= \frac{-\mathcal{V}_{-m+3} + (r-1)\mathcal{V}_{-m+2} + (r+s-1)\mathcal{V}_{-m+1} + (r+s+t-1)\mathcal{V}_{-m} \\ &\quad + \mathcal{V}_3 + (1-r)\mathcal{V}_2 + (1-r-s)\mathcal{V}_1 + (1-s-r-t)\mathcal{V}_0}{r+s+t+u-1}, \\ \bullet \quad \sum_{n=1}^m \mathcal{V}_{-2n} &= \frac{(s+u-1)\mathcal{V}_{-2m+2} - (t+rs+ru)\mathcal{V}_{-2m+1} \\ &\quad + (r^2-s^2+rt-su+2s+u-1)\mathcal{V}_{-2m} - (ru+tu)\mathcal{V}_{-2m-1} \\ &\quad + (r+t)\mathcal{V}_3 + (1-r^2-rt-s-u)\mathcal{V}_2 + (t+ru-st)\mathcal{V}_1 \\ &\quad + (1-r^2+s^2-t^2-2rt+su-2s-u)\mathcal{V}_0}{(r-s+t-u+1)(r+s+t+u-1)}, \\ \bullet \quad \sum_{n=1}^m \mathcal{V}_{-2n+1} &= \frac{-(r+t)\mathcal{V}_{-2m+2} + (r^2+rt+s+u-1)\mathcal{V}_{-2m+1} \\ &\quad + (st-t-ru)\mathcal{V}_{-2m} + (u^2+su-u)\mathcal{V}_{-2m-1} \\ &\quad + (1-s-u)\mathcal{V}_3 + (t+ru+rs)\mathcal{V}_2 \\ &\quad + (1-r^2+s^2-rt+su-2s-u)\mathcal{V}_1 + u(r+t)\mathcal{V}_0}{(r-s+t-u+1)(r+s+t+u-1)}, \end{aligned}$$

where denominators are not equal to zero.

Particular Cases 3.11: If $s + u = 1$, we can get the following summation formulas:

- $\sum_{n=1}^m \mathcal{V}_{-2n} = \frac{-\mathcal{V}_{-2m+1} + r\mathcal{V}_{-2m} - u\mathcal{V}_{-2m-1} + \mathcal{V}_3 - r\mathcal{V}_2 + u\mathcal{V}_1 - (r+t)\mathcal{V}_0}{r+t},$
- $\sum_{n=1}^m \mathcal{V}_{-2n+1} = \frac{-\mathcal{V}_{-2m+2} + r\mathcal{V}_{-2m+1} - u\mathcal{V}_{-2m} + \mathcal{V}_2 - r\mathcal{V}_1 + u\mathcal{V}_0}{r+t},$

where denominators are not equal to zero.

Theorem 3.8: For all $n \in \mathbb{Z}^+$, the following matrix formulations are satisfied.

- $\begin{pmatrix} \mathcal{V}_{n+3} \\ \mathcal{V}_{n+2} \\ \mathcal{V}_{n+1} \\ \mathcal{V}_n \end{pmatrix} = \begin{pmatrix} r & s & t & u \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{V}_3 \\ \mathcal{V}_2 \\ \mathcal{V}_1 \\ \mathcal{V}_0 \end{pmatrix}$
- $\begin{pmatrix} \mathcal{V}_{-n} \\ \mathcal{V}_{-n-1} \\ \mathcal{V}_{-n-2} \\ \mathcal{V}_{-n-3} \end{pmatrix} = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{u} & \frac{-r}{u} & \frac{-s}{u} & \frac{-t}{u} \end{pmatrix}^n \begin{pmatrix} \mathcal{V}_0 \\ \mathcal{V}_{-1} \\ \mathcal{V}_{-2} \\ \mathcal{V}_{-3} \end{pmatrix}$

Proof: The proofs are straightforward by using mathematical induction.

By inspired the studies [54, 58], we get the following theorem which includes a different method for finding the n th and $-(n+1)$ th 3PGQs with generalized Tetranacci numbers components, respectively.

Theorem 3.9: For all $n \geq 0$, the following determinant equations³ are satisfied.

$$\mathcal{V}_n = \begin{vmatrix} \mathcal{V}_0 & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathcal{V}_1 & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathcal{V}_2 & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathcal{V}_3 & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & u & t & s & r & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & u & t & s & r & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & s & r & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & t & s & r & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & u & t & s & r \end{vmatrix}_{(n+1) \times (n+1)} \quad (3.12)$$

³ The Laplace expansion along the last column is used to calculate the determinant, bearing in mind the non-commutativity of 3PGQs, namely, for all $[a_{ij}]_{n \times n}$, $\det([a_{ij}]_{n \times n}) = \sum_{i=1}^n b_{in} a_{in}$, with $b_{in} = (-1)^{i+n} \det C_{in}$ where $\det C_{in}$ is the i, n minor of $[a_{ij}]_{n \times n}$.

$$\mathcal{V}_{-(n+1)} = \begin{vmatrix} \mathcal{V}_{-1} & -1 & 0 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathcal{V}_{-2} & 0 & -1 & 0 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathcal{V}_{-3} & 0 & 0 & -1 & 0 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ \mathcal{V}_{-4} & 0 & 0 & 0 & -1 & 0 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & \frac{1}{u} & -\frac{r}{u} & -\frac{s}{u} & -\frac{t}{u} & -1 & 0 & \cdots & 0 & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{u} & -\frac{r}{u} & -\frac{s}{u} & -\frac{t}{u} & -1 & \cdots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -\frac{s}{u} & -\frac{t}{u} & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -\frac{r}{u} & -\frac{s}{u} & -\frac{t}{u} & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & \frac{1}{u} & -\frac{r}{u} & -\frac{s}{u} & -\frac{t}{u} \end{vmatrix}_{(n+1) \times (n+1)} \quad (3.13)$$

Proof: With the help of Theorem 3.1 on page 173 in the study [58] (or Theorem 10 on page 107 in the study [54]), the proof can be completed for both parts.

Definition 3.2: For $N_{\mathcal{V}_n} > 0$ and $\lambda_1\lambda_2V_{n+1}^2 + \lambda_1\lambda_3V_{n+2}^2 + \lambda_2\lambda_3V_{n+3}^2 \neq 0$, polar representation of 3PGQs with generalized Tetranacci numbers components is expressed as follows:

$$\mathcal{V}_n = \sqrt{N_{\mathcal{V}_n}}(\cos\theta + \hat{\mathcal{V}}_n\sin\theta),$$

where

$$\hat{\mathcal{V}}_n = \frac{1}{\sqrt{\lambda_1\lambda_2V_{n+1}^2 + \lambda_1\lambda_3V_{n+2}^2 + \lambda_2\lambda_3V_{n+3}^2}}(V_{n+1}, V_{n+2}, V_{n+3})$$

and

$$\cos\theta = \frac{V_n}{\sqrt{N_{\mathcal{V}_n}}}, \sin\theta = \sqrt{\frac{\lambda_1\lambda_2V_{n+1}^2 + \lambda_1\lambda_3V_{n+2}^2 + \lambda_2\lambda_3V_{n+3}^2}{N_{\mathcal{V}_n}}},$$

where $\hat{\mathcal{V}}_n$ is named generalized unit vector of 3PGQs with generalized Tetranacci numbers components.

Theorem 3.10: The matrix representation of $\mathcal{V}_n$ can be expressed as follows:

$$\mathcal{M}_{\mathcal{V}_n} = \begin{pmatrix} V_n & -\lambda_1\lambda_2V_{n+1} & -\lambda_1\lambda_3V_{n+2} & -\lambda_2\lambda_3V_{n+3} \\ V_{n+1} & V_n & -\lambda_3V_{n+3} & \lambda_3V_{n+2} \\ V_{n+2} & \lambda_2V_{n+3} & V_n & -\lambda_2V_{n+1} \\ V_{n+3} & -\lambda_1V_{n+2} & \lambda_1V_{n+1} & V_n \end{pmatrix},$$

where the matrix $\mathcal{M}_{\mathcal{V}_n}$ is called fundamental matrix for 3PGQs with generalized Tetranacci numbers components.

Proof: Multiplying $\mathcal{V}_n = V_n + V_{n+1}e_1 + V_{n+2}e_2 + V_{n+3}e_3$ with $1, e_1, e_2, e_3$ from the left side and with the help of the Table 1, we have:

$$\begin{aligned}
\mathcal{V}_n 1 &= V_n + V_{n+1}e_1 + V_{n+2}e_2 + V_{n+3}e_3, \\
\mathcal{V}_n e_1 &= -\lambda_1\lambda_2V_{n+1} + V_n e_1 + \lambda_2V_{n+3}e_2 - \lambda_1V_{n+2}e_3, \\
\mathcal{V}_n e_2 &= -\lambda_1\lambda_3V_{n+2} - \lambda_3V_{n+3}e_1 + V_n e_2 + \lambda_1V_{n+1}e_3, \\
\mathcal{V}_n e_3 &= -\lambda_2\lambda_3V_{n+3} + \lambda_3V_{n+2}e_1 - \lambda_2V_{n+1}e_2 + V_n e_3.
\end{aligned}$$

Then, writing the coefficients of $\{1, e_1, e_2, e_3\}$ of the above equations as columns establishes the matrix $\mathcal{M}_{\mathcal{V}_n}$.

Then, according to the values of $\lambda_{i \in \{1,2,3\}}$, $\mathcal{M}_{\mathcal{V}_n}$ can be classified. For $\lambda_1 = 1$, $\lambda_2, \lambda_3 \in \mathbb{R}$, the fundamental matrix for 2PGQs with generalized Tetranacci numbers components is obtained. For $\lambda_1 = \lambda_2 = 1, \lambda_3 = -1$, then the fundamental matrix for split quaternions with generalized Tetranacci numbers components is obtained. Additionally, for $\lambda_1 = \lambda_2 = \lambda_3 = 1$, then the fundamental matrix for Hamilton quaternions with generalized Tetranacci numbers components is obtained.

Remark 3.11: For all $m, n \in \mathbb{Z}$, let $\mathcal{V}_n$ and $\mathcal{V}_m$ be n th and m th 3PGQ with generalized Tetranacci numbers components. Then, the following matrix computation is written:

$$\mathcal{M}_{\mathcal{V}_n}(\mathcal{V}_m^*)^t = \mathcal{M}_{\mathcal{V}_m}(\mathcal{V}_n^*)^t = (\mathcal{V}_n \mathcal{V}_m)^*,$$

where the superscript $*$ represents column matrix forms. Therefore, here

$$\mathcal{V}_n^* = (V_n \quad V_{n+1} \quad V_{n+2} \quad V_{n+3})^t, \mathcal{V}_m^* = (V_m \quad V_{m+1} \quad V_{m+2} \quad V_{m+3})^t,$$

and

$$(\mathcal{V}_n \mathcal{V}_m)^* = \begin{pmatrix} V_n V_m - \lambda_1 \lambda_2 V_{n+1} V_{m+1} - \lambda_1 \lambda_3 V_{n+2} V_{m+2} - \lambda_2 \lambda_3 V_{n+3} V_{m+3} \\ V_n V_{m+1} + V_{n+1} V_m + \lambda_3 V_{n+2} V_{m+3} - \lambda_3 V_{n+3} V_{m+2} \\ V_n V_{m+2} + V_{n+2} V_m - \lambda_2 V_{n+1} V_{m+3} + \lambda_2 V_{n+3} V_{m+1} \\ V_n V_{m+3} + V_{n+3} V_m + \lambda_1 V_{n+1} V_{m+2} - \lambda_1 V_{n+2} V_{m+1} \end{pmatrix}.$$

We obtain the following Definition 3.3, according to the study [69].

Definition 3.3: Let $\mathcal{V}_n$ be the n th 3PGQ with generalized Tetranacci numbers components for all $n \in \mathbb{Z}$. Then, the following mathematical expressions hold:

- The determinant of $\mathcal{M}_{\mathcal{V}_n}$: $\det(\mathcal{M}_{\mathcal{V}_n}) = N_{\mathcal{V}_n}^2$.
- The characteristic polynomial of $\mathcal{M}_{\mathcal{V}_n}$:

$$P_{\mathcal{M}_{\mathcal{V}_n}}(u) = (u^2 - 2uV_n + V_n^2 + \lambda_1\lambda_2V_{n+1}^2 + \lambda_1\lambda_3V_{n+2}^2 + \lambda_2\lambda_3V_{n+3}^2)^2.$$
- The characteristic equation of $\mathcal{M}_{\mathcal{V}_n}$:

$$\det(\mathcal{M}_{\mathcal{V}_n} - uI_4) = 0 \Leftrightarrow P_{\mathcal{M}_{\mathcal{V}_n}}(u) = (u^2 - 2uV_n + V_n^2 + \lambda_1\lambda_2V_{n+1}^2 + \lambda_1\lambda_3V_{n+2}^2 + \lambda_2\lambda_3V_{n+3}^2)^2 = 0.$$
- The eigenvalues of $\mathcal{M}_{\mathcal{V}_n}$:

$$\begin{aligned}
\mathfrak{T}_{1,2} &= V_n + \sqrt{-\lambda_1\lambda_2V_{n+1}^2 - \lambda_1\lambda_3V_{n+2}^2 - \lambda_2\lambda_3V_{n+3}^2}, \\
\mathfrak{T}_{3,4} &= V_n - \sqrt{-\lambda_1\lambda_2V_{n+1}^2 - \lambda_1\lambda_3V_{n+2}^2 - \lambda_2\lambda_3V_{n+3}^2}.
\end{aligned}$$

- Multiplication of the eigenvalues of $\mathcal{M}_{V_n}$:
 $\mathfrak{T}_{1,2}\mathfrak{T}_{3,4} = V_n^2 + \lambda_1\lambda_2V_{n+1}^2 + \lambda_1\lambda_3V_{n+2}^2 + \lambda_2\lambda_3V_{n+3}^2 = N_{V_n}$.
- The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{1,2}$ of $\mathcal{M}_{V_n}$:

$$\begin{pmatrix} \frac{\lambda_1 V_{n+2} \sqrt{-\lambda_1 \lambda_2 V_{n+1}^2 - \lambda_1 \lambda_3 V_{n+2}^2 - \lambda_2 \lambda_3 V_{n+3}^2} - \lambda_1 \lambda_2 V_{n+1} V_{n+3}}{\lambda_1 V_{n+2}^2 + \lambda_2 V_{n+3}^2} \\ \frac{V_{n+3} \sqrt{-\lambda_1 \lambda_2 V_{n+1}^2 - \lambda_1 \lambda_3 V_{n+2}^2 - \lambda_2 \lambda_3 V_{n+3}^2} + \lambda_1 V_{n+1} V_{n+2}}{\lambda_1 V_{n+2}^2 + \lambda_2 V_{n+3}^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \frac{\lambda_2 V_{n+3} \sqrt{-\lambda_1 \lambda_2 V_{n+1}^2 - \lambda_1 \lambda_3 V_{n+2}^2 - \lambda_2 \lambda_3 V_{n+3}^2} + \lambda_1 \lambda_2 V_{n+1} V_{n+2}}{\lambda_1 V_{n+2}^2 + \lambda_2 V_{n+3}^2} \\ - \frac{V_{n+2} \sqrt{-\lambda_1 \lambda_2 V_{n+1}^2 - \lambda_1 \lambda_3 V_{n+2}^2 - \lambda_2 \lambda_3 V_{n+3}^2} - \lambda_2 V_{n+1} V_{n+3}}{\lambda_1 V_{n+2}^2 + \lambda_2 V_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}.$$

- The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{3,4}$ of $\mathcal{M}_{V_n}$:

$$\begin{pmatrix} - \frac{\lambda_1 V_{n+2} \sqrt{-\lambda_1 \lambda_2 V_{n+1}^2 - \lambda_1 \lambda_3 V_{n+2}^2 - \lambda_2 \lambda_3 V_{n+3}^2} + \lambda_1 \lambda_2 V_{n+1} V_{n+3}}{\lambda_1 V_{n+2}^2 + \lambda_2 V_{n+3}^2} \\ - \frac{V_{n+3} \sqrt{-\lambda_1 \lambda_2 V_{n+1}^2 - \lambda_1 \lambda_3 V_{n+2}^2 - \lambda_2 \lambda_3 V_{n+3}^2} - \lambda_1 V_{n+1} V_{n+2}}{\lambda_1 V_{n+2}^2 + \lambda_2 V_{n+3}^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} - \frac{\lambda_2 V_{n+3} \sqrt{-\lambda_1 \lambda_2 V_{n+1}^2 - \lambda_1 \lambda_3 V_{n+2}^2 - \lambda_2 \lambda_3 V_{n+3}^2} - \lambda_1 \lambda_2 V_{n+1} V_{n+2}}{\lambda_1 V_{n+2}^2 + \lambda_2 V_{n+3}^2} \\ \frac{V_{n+2} \sqrt{-\lambda_1 \lambda_2 V_{n+1}^2 - \lambda_1 \lambda_3 V_{n+2}^2 - \lambda_2 \lambda_3 V_{n+3}^2} + \lambda_2 V_{n+1} V_{n+3}}{\lambda_1 V_{n+2}^2 + \lambda_2 V_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}.$$

In light of the above theorems and properties, we give the following Corollary 3.1. We scrutinize a member of this special number family; 3PGQ with fourth-order Pell-Lucas numbers components. The other special cases of this number family which are given in the Table 7 can be seen similarly.

Corollary 3.1: *Let us consider the n th 3PGQ with fourth-order Pell-Lucas numbers components denoted by $\mathcal{D}_n$ and defined as follows:*

$$\begin{aligned} \mathcal{D}_n &= \mathcal{D}_n + \mathcal{D}_{n+1}e_1 + \mathcal{D}_{n+2}e_2 + \mathcal{D}_{n+3}e_3 \text{ for all } n \geq 0, \\ \mathcal{D}_{-n} &= \mathcal{D}_{-n} + \mathcal{D}_{-n+1}e_1 + \mathcal{D}_{-n+2}e_2 + \mathcal{D}_{-n+3}e_3 \text{ for all } n \geq 1. \end{aligned}$$

Moreover, some values are written as follows:

$$\begin{cases} \mathcal{D}_0 = 4 + 2e_1 + 6e_2 + 17e_3, \\ \mathcal{D}_1 = 2 + 6e_1 + 17e_2 + 46e_3, \\ \mathcal{D}_2 = 6 + 17e_1 + 46e_2 + 117e_3, \\ \mathcal{D}_3 = 17 + 46e_1 + 117e_2 + 303e_3, \end{cases} \quad \text{and} \quad \begin{cases} \mathcal{D}_{-1} = -1 + 4e_1 + 2e_2 + 6e_3, \\ \mathcal{D}_{-2} = -1 - e_1 + 4e_2 + 2e_3, \\ \mathcal{D}_{-3} = -4 - e_1 - e_2 + 4e_3, \\ \mathcal{D}_{-4} = 11 - 4e_1 - e_2 - e_3. \end{cases}$$

Then, the followings are satisfied:

- The recurrence relations hold:

$$\mathcal{D}_n = 2\mathcal{D}_{n-1} + \mathcal{D}_{n-2} + \mathcal{D}_{n-3} + \mathcal{D}_{n-4} \text{ for all } n \geq 4,^4 \quad (3.14)$$

$$\mathcal{D}_{-n} = \mathcal{D}_{-(n-4)} - \mathcal{D}_{-(n-1)} - \mathcal{D}_{-(n-2)} - 2\mathcal{D}_{-(n-3)} \text{ for all } n \geq 1.$$

- For all $n \in \mathbb{Z}$, the Binet formula is as follows:

$$\mathcal{D}_n = x_1^n \underline{x}_1 + x_2^n \underline{x}_2 + x_3^n \underline{x}_3 + x_4^n \underline{x}_4,$$

where (see [52] for $x_k, k = 1, 2, 3, 4$)

$$\begin{cases} x_1 = 2.592052792, \\ x_2 = -0.6631378984, \\ x_3 = 0.03554255298 - 0.7619107877i, \\ x_4 = 0.03554255298 + 0.7619107877i. \end{cases}$$

- The generating functions are as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{D}_n x^n &= \frac{\mathcal{D}_0 + (\mathcal{D}_1 - 2\mathcal{D}_0)x + (\mathcal{D}_2 - 2\mathcal{D}_1 - \mathcal{D}_0)x^2 + (\mathcal{D}_3 - 2\mathcal{D}_2 - \mathcal{D}_1 - \mathcal{D}_0)x^3}{1 - 2x - x^2 - x^3 - x^4}, \\ \sum_{n=0}^{\infty} \mathcal{D}_{-n} x^n &= \frac{\mathcal{D}_0 + (\mathcal{D}_{-1} + \mathcal{D}_0)x + (\mathcal{D}_{-2} + \mathcal{D}_{-1} + \mathcal{D}_0)x^2 + (\mathcal{D}_{-3} + \mathcal{D}_{-2} + \mathcal{D}_{-1} + 2\mathcal{D}_0)x^3}{1 + x + x^2 + 2x^3 - x^4}. \end{aligned}$$

- The exponential generating functions are as follows:

$$\begin{aligned} \sum_{n=0}^{\infty} \mathcal{D}_n \frac{y^n}{n!} &= \underline{x}_1 e^{x_1 y} + \underline{x}_2 e^{x_2 y} + \underline{x}_3 e^{x_3 y} + \underline{x}_4 e^{x_4 y}, \\ \sum_{n=0}^{\infty} \mathcal{D}_{-n} \frac{y^n}{n!} &= \underline{x}_1 e^{\frac{y}{x_1}} + \underline{x}_2 e^{\frac{y}{x_2}} + \underline{x}_3 e^{\frac{y}{x_3}} + \underline{x}_4 e^{\frac{y}{x_4}}. \end{aligned}$$

- The Poisson generating functions are written as follows:

$$\begin{aligned} e^{-y} \sum_{n=0}^{\infty} \mathcal{D}_n \frac{y^n}{n!} &= \underline{x}_1 e^{(x_1-1)y} + \underline{x}_2 e^{(x_2-1)y} + \underline{x}_3 e^{(x_3-1)y} + \underline{x}_4 e^{(x_4-1)y}, \\ e^{-y} \sum_{n=0}^{\infty} \mathcal{D}_{-n} \frac{y^n}{n!} &= \underline{x}_1 e^{y(\frac{1}{x_1}-1)} + \underline{x}_2 e^{y(\frac{1}{x_2}-1)} + \underline{x}_3 e^{y(\frac{1}{x_3}-1)} + \underline{x}_4 e^{y(\frac{1}{x_4}-1)}. \end{aligned}$$

- The summation formulae are satisfied:

$$\begin{aligned} - \sum_{n=0}^m \mathcal{D}_n &= \frac{1}{4} \left(\mathcal{D}_{m+4} - \mathcal{D}_{m+3} - 2\mathcal{D}_{m+2} - 3\mathcal{D}_{m+1} - \mathcal{D}_3 + \mathcal{D}_2 \right) \\ &\quad + 2\mathcal{D}_1 + 3\mathcal{D}_0, \\ - \sum_{n=0}^m \mathcal{D}_{2n} &= \frac{1}{8} \left(-\mathcal{D}_{2m+2} + 5\mathcal{D}_{2m+1} + 2\mathcal{D}_{2m} + 3\mathcal{D}_{2m-1} - 3\mathcal{D}_3 \right) \\ &\quad + 7\mathcal{D}_2 - 2\mathcal{D}_1 + 9\mathcal{D}_0, \\ - \sum_{n=0}^m \mathcal{D}_{2n+1} &= \frac{1}{8} \left(3\mathcal{D}_{2m+2} + \mathcal{D}_{2m+1} + 2\mathcal{D}_{2m} - \mathcal{D}_{2m-1} + \mathcal{D}_3 - 5\mathcal{D}_2 \right) \\ &\quad + 6\mathcal{D}_1 - 3\mathcal{D}_0, \\ - \sum_{n=1}^m \mathcal{D}_{-n} &= \frac{1}{4} \left(-\mathcal{D}_{-m+3} + \mathcal{D}_{-m+2} + 2\mathcal{D}_{-m+1} + 3\mathcal{D}_{-m} \right) \\ &\quad + \mathcal{D}_3 - \mathcal{D}_2 - 2\mathcal{D}_1 - 3\mathcal{D}_0 \end{aligned}$$

⁴ (3.14) is valid for all $n \in \mathbb{Z}$.

$$\begin{aligned}
- \quad \sum_{n=1}^m \mathcal{D}_{-2n} &= \frac{1}{8} \left(\mathcal{D}_{-2m+2} - 5\mathcal{D}_{-2m+1} + 6\mathcal{D}_{-2m} - 3\mathcal{D}_{-2m-1} + 3\mathcal{D}_3 \right), \\
- \quad \sum_{n=1}^m \mathcal{D}_{-2n+1} &= \frac{1}{8} \left(-3\mathcal{D}_{-2m+2} + 7\mathcal{D}_{-2m+1} - 2\mathcal{D}_{-2m} + \mathcal{D}_{-2m-1} \right).
\end{aligned}$$

- The following matrix properties are maintained:

$$\begin{aligned}
- \quad \begin{pmatrix} \mathcal{D}_{n+3} \\ \mathcal{D}_{n+2} \\ \mathcal{D}_{n+1} \\ \mathcal{D}_n \end{pmatrix} &= \begin{pmatrix} 2 & 1 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}^n \begin{pmatrix} \mathcal{D}_3 \\ \mathcal{D}_2 \\ \mathcal{D}_1 \\ \mathcal{D}_0 \end{pmatrix}, \\
- \quad \begin{pmatrix} \mathcal{D}_{-n} \\ \mathcal{D}_{-n-1} \\ \mathcal{D}_{-n-2} \\ \mathcal{D}_{-n-3} \end{pmatrix} &= \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & -2 & -1 & -1 \end{pmatrix} \begin{pmatrix} \mathcal{D}_0 \\ \mathcal{D}_{-1} \\ \mathcal{D}_{-2} \\ \mathcal{D}_{-3} \end{pmatrix}.
\end{aligned}$$

- For all $n \in \mathbb{N}$, the following equalities hold.

$$\begin{aligned}
- \quad \mathcal{D}_n &= \begin{vmatrix} \mathcal{D}_0 & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mathcal{D}_1 & 0 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mathcal{D}_2 & 0 & 0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mathcal{D}_3 & 0 & 0 & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & 1 & 1 & 2 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 1 & 1 & 2 & -1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 2 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 2 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & 1 & 1 & 2 \end{vmatrix}_{(n+1) \times (n+1)} \\
- \quad \mathcal{D}_{-(n+1)} &= \begin{vmatrix} \mathcal{D}_{-1} & -1 & 0 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mathcal{D}_{-2} & 0 & -1 & 0 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mathcal{D}_{-3} & 0 & 0 & -1 & 0 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ \mathcal{D}_{-4} & 0 & 0 & 0 & -1 & 0 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 1 & -2 & -1 & -1 & -1 & 0 & \dots & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & -2 & -1 & -1 & -1 & \dots & 0 & 0 & 0 & 0 \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots & \vdots & \vdots & \vdots \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -1 & -1 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & -2 & -1 & -1 & -1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \ddots & 1 & -2 & -1 & -1 \end{vmatrix}_{(n+1) \times (n+1)}
\end{aligned}$$

- Under the following conditions $N_{\mathcal{D}_n} > 0$ and $\lambda_1 \lambda_2 \mathcal{D}_{n+1}^2 + \lambda_1 \lambda_3 \mathcal{D}_{n+2}^2 + \lambda_2 \lambda_3 \mathcal{D}_{n+3}^2 \neq 0$, polar representation of $\mathcal{D}_n$ is expressed as follows:

$$\mathcal{D}_n = \sqrt{N_{\mathcal{D}_n}} (\cos \theta + \widehat{\mathcal{D}}_n \sin \theta), \quad (3.15)$$

where $\widehat{\mathcal{D}}_n$ is called generalized unit vector of 3PGQs with fourth-order Pell-Lucas numbers components and determined as follows:

$$\widehat{\mathcal{D}}_n = \frac{1}{\sqrt{\lambda_1 \lambda_2 \mathcal{D}_{n+1}^2 + \lambda_1 \lambda_3 \mathcal{D}_{n+2}^2 + \lambda_2 \lambda_3 \mathcal{D}_{n+3}^2}} (\mathcal{D}_{n+1}, \mathcal{D}_{n+2}, \mathcal{D}_{n+3})$$

and also

$$\cos\theta = \frac{D_n}{\sqrt{N_{\mathcal{D}_n}}}, \sin\theta = \sqrt{\frac{\lambda_1\lambda_2D_{n+1}^2 + \lambda_1\lambda_3D_{n+2}^2 + \lambda_2\lambda_3D_{n+3}^2}{N_{\mathcal{D}_n}}}.$$

- The matrix representation of $\mathcal{D}_n$ is expressed as follows:

$$\mathcal{M}_{\mathcal{D}_n} = \begin{pmatrix} D_n & -\lambda_1\lambda_2D_{n+1} & -\lambda_1\lambda_3D_{n+2} & -\lambda_2\lambda_3D_{n+3} \\ D_{n+1} & D_n & -\lambda_3D_{n+3} & \lambda_3D_{n+2} \\ D_{n+2} & \lambda_2D_{n+3} & D_n & -\lambda_2D_{n+1} \\ D_{n+3} & -\lambda_1D_{n+2} & \lambda_1D_{n+1} & D_n \end{pmatrix},$$

where the matrix $\mathcal{M}_{\mathcal{D}_n}$ named the fundamental matrix for 3PGQs with fourth-order Pell-Lucas numbers components.

- For all $n \in \mathbb{Z}$, the following properties hold:
 - The determinant of $\mathcal{M}_{\mathcal{D}_n}$: $\det(\mathcal{M}_{\mathcal{D}_n}) = N_{\mathcal{D}_n}^2$.
 - The characteristic polynomial of $\mathcal{M}_{\mathcal{D}_n}$:
 $P_{\mathcal{M}_{\mathcal{D}_n}}(u) = (u^2 - 2uD_n + D_n^2 + \lambda_1\lambda_2D_{n+1}^2 + \lambda_1\lambda_3D_{n+2}^2 + \lambda_2\lambda_3D_{n+3}^2)^2$.
 - The characteristic equation of $\mathcal{M}_{\mathcal{D}_n}$:
 $\det(\mathcal{M}_{\mathcal{D}_n} - uI_4) = 0 \Leftrightarrow P_{\mathcal{M}_{\mathcal{D}_n}}(u) = (u^2 - 2uD_n + D_n^2 + \lambda_1\lambda_2D_{n+1}^2 + \lambda_1\lambda_3D_{n+2}^2 + \lambda_2\lambda_3D_{n+3}^2)^2 = 0$.
 - The eigenvalues of $\mathcal{M}_{\mathcal{D}_n}$:

$$\begin{cases} \mathfrak{T}_{1,2} = D_n + \sqrt{-\lambda_1\lambda_2D_{n+1}^2 - \lambda_1\lambda_3D_{n+2}^2 - \lambda_2\lambda_3D_{n+3}^2}, \\ \mathfrak{T}_{3,4} = D_n - \sqrt{-\lambda_1\lambda_2D_{n+1}^2 - \lambda_1\lambda_3D_{n+2}^2 - \lambda_2\lambda_3D_{n+3}^2}. \end{cases}$$
 - Multiplication of the eigenvalues of $\mathcal{M}_{\mathcal{D}_n}$:
 $\mathfrak{T}_{1,2}\mathfrak{T}_{3,4} = D_n^2 + \lambda_1\lambda_2D_{n+1}^2 + \lambda_1\lambda_3D_{n+2}^2 + \lambda_2\lambda_3D_{n+3}^2 = N_{\mathcal{D}_n}$.
 - The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{1,2}$ of $\mathcal{M}_{\mathcal{D}_n}$:

$$\begin{pmatrix} \lambda_1D_{n+2}\sqrt{\frac{-\lambda_1\lambda_2D_{n+1}^2 - \lambda_1\lambda_3D_{n+2}^2 - \lambda_2\lambda_3D_{n+3}^2 - \lambda_1\lambda_2D_{n+1}D_{n+3}}{\lambda_1D_{n+2}^2 + \lambda_2D_{n+3}^2}} \\ \frac{D_{n+3}\sqrt{-\lambda_1\lambda_2D_{n+1}^2 - \lambda_1\lambda_3D_{n+2}^2 - \lambda_2\lambda_3D_{n+3}^2 + \lambda_1D_{n+1}D_{n+2}}}{\lambda_1D_{n+2}^2 + \lambda_2D_{n+3}^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} \lambda_2D_{n+3}\sqrt{\frac{-\lambda_1\lambda_2D_{n+1}^2 - \lambda_1\lambda_3D_{n+2}^2 - \lambda_2\lambda_3D_{n+3}^2 + \lambda_1\lambda_2D_{n+1}D_{n+2}}{\lambda_1D_{n+2}^2 + \lambda_2D_{n+3}^2}} \\ -\frac{D_{n+2}\sqrt{-\lambda_1\lambda_2D_{n+1}^2 - \lambda_1\lambda_3D_{n+2}^2 - \lambda_2\lambda_3D_{n+3}^2 - \lambda_2D_{n+1}D_{n+3}}}{\lambda_1D_{n+2}^2 + \lambda_2D_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}$$

- The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{3,4}$ of $\mathcal{M}_{\mathcal{D}_n}$:

$$\begin{pmatrix} -\frac{\lambda_1 D_{n+2} \sqrt{-\lambda_1 \lambda_2 D_{n+1}^2 - \lambda_1 \lambda_3 D_{n+2}^2 - \lambda_2 \lambda_3 D_{n+3}^2 + \lambda_1 \lambda_2 D_{n+1} D_{n+3}}}{\lambda_1 D_{n+2}^2 + \lambda_2 D_{n+3}^2} \\ -\frac{D_{n+3} \sqrt{-\lambda_1 \lambda_2 D_{n+1}^2 - \lambda_1 \lambda_3 D_{n+2}^2 - \lambda_2 \lambda_3 D_{n+3}^2 - \lambda_1 D_{n+1} D_{n+2}}}{\lambda_1 D_{n+2}^2 + \lambda_2 D_{n+3}^2} \\ 1 \\ 0 \end{pmatrix}$$

and

$$\begin{pmatrix} -\frac{\lambda_2 D_{n+3} \sqrt{-\lambda_1 \lambda_2 D_{n+1}^2 - \lambda_1 \lambda_3 D_{n+2}^2 - \lambda_2 \lambda_3 D_{n+3}^2 - \lambda_1 \lambda_2 D_{n+1} D_{n+2}}}{\lambda_1 D_{n+2}^2 + \lambda_2 D_{n+3}^2} \\ \frac{D_{n+2} \sqrt{-\lambda_1 \lambda_2 D_{n+1}^2 - \lambda_1 \lambda_3 D_{n+2}^2 - \lambda_2 \lambda_3 D_{n+3}^2 + \lambda_2 D_{n+1} D_{n+3}}}{\lambda_1 D_{n+2}^2 + \lambda_2 D_{n+3}^2} \\ 0 \\ 1 \end{pmatrix}$$

Example 3.12: Let $\mathcal{D}_{-7}$ be the -7 th 3PGQ with fourth-order Pell-Lucas numbers components.

- With the help of the equation (3.15), polar representation of $\mathcal{D}_{-7}$ is as follows:

$$\mathcal{D}_{-7} = \sqrt{484 + 4\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 121\lambda_2\lambda_3} (\cos\theta + \widehat{\mathcal{D}}_{-7}\sin\theta),$$

where

$$\widehat{\mathcal{D}}_{-7} = \frac{(2, -6, 11)}{\sqrt{4\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 121\lambda_2\lambda_3}}$$

is 3-parameter generalized unit vector of $\mathcal{D}_{-7}$ and

$$\cos\theta = \frac{-22}{\sqrt{484 + 4\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 121\lambda_2\lambda_3}}, \sin\theta = \frac{\sqrt{4\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 121\lambda_2\lambda_3}}{\sqrt{484 + 4\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 121\lambda_2\lambda_3}}.$$

- The following matrix can be constructed as follows:

$$\mathcal{M}_{\mathcal{D}_{-7}} = \begin{pmatrix} -22 & -2\lambda_1\lambda_2 & 6\lambda_1\lambda_3 & -11\lambda_2\lambda_3 \\ 2 & -22 & -11\lambda_3 & -6\lambda_3 \\ -6 & 11\lambda_2 & -22 & -2\lambda_2 \\ 11 & 6\lambda_1 & 2\lambda_1 & -22 \end{pmatrix}.$$

- The determinant of $\mathcal{M}_{\mathcal{D}_{-7}}$ is expressed as:

$$\det(\mathcal{M}_{\mathcal{D}_{-7}}) = (484 + 4\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 121\lambda_2\lambda_3)^2 = (N_{\mathcal{D}_{-7}})^2.$$

- The characteristic polynomial of $\mathcal{M}_{\mathcal{D}_{-7}}$ is expressed:

$$P_{\mathcal{D}_{-7}}(u) = (u^2 + 44u + 484 + 4\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 121\lambda_2\lambda_3)^2.$$

- The eigenvalues of $\mathcal{M}_{\mathcal{D}_{-7}}$ are written:

$$\begin{cases} \mathfrak{T}_{1,2} = -22 + \sqrt{-4\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 121\lambda_2\lambda_3}, \\ \mathfrak{T}_{3,4} = -22 - \sqrt{-4\lambda_1\lambda_2 - 36\lambda_1\lambda_3 - 121\lambda_2\lambda_3}. \end{cases}$$

- The multiplication of the eigenvalues of $\mathcal{M}_{\mathcal{D}_{-7}}$ is as follows:
 $\mathfrak{T}_{1,2}\mathfrak{T}_{3,4} = 484 + 4\lambda_1\lambda_2 + 36\lambda_1\lambda_3 + 121\lambda_2\lambda_3.$
- The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{1,2}$ of $\mathcal{M}_{\mathcal{D}_{-7}}$ are written as follows:

$$\begin{pmatrix} \frac{\lambda_1(-22\lambda_2-6\sqrt{-4\lambda_1\lambda_2-36\lambda_1\lambda_3-121\lambda_2\lambda_3})}{36\lambda_1+121\lambda_2} & \frac{-12\lambda_1+11\sqrt{-4\lambda_1\lambda_2-36\lambda_1\lambda_3-121\lambda_2\lambda_3}}{36\lambda_1+121\lambda_2} & 1 & 0 \end{pmatrix}^t$$

and

$$\begin{pmatrix} \frac{\lambda_2(-12\lambda_1+11\sqrt{-4\lambda_1\lambda_2-36\lambda_1\lambda_3-121\lambda_2\lambda_3})}{36\lambda_1+121\lambda_2} & -\frac{-22\lambda_2-6\sqrt{-4\lambda_1\lambda_2-36\lambda_1\lambda_3-121\lambda_2\lambda_3}}{36\lambda_1+121\lambda_2} & 0 & 1 \end{pmatrix}^t.$$

- The eigenvectors corresponding to the eigenvalue $\mathfrak{T}_{3,4}$ of $\mathcal{M}_{\mathcal{D}_{-7}}$ are written as follows:

$$\begin{pmatrix} -\frac{\lambda_1(22\lambda_2-6\sqrt{-4\lambda_1\lambda_2-36\lambda_1\lambda_3-121\lambda_2\lambda_3})}{36\lambda_1+121\lambda_2} & -\frac{12\lambda_1+11\sqrt{-4\lambda_1\lambda_2-36\lambda_1\lambda_3-121\lambda_2\lambda_3}}{36\lambda_1+121\lambda_2} & 1 & 0 \end{pmatrix}^t$$

and

$$\begin{pmatrix} \frac{\lambda_2(-12\lambda_1-11\sqrt{-4\lambda_1\lambda_2-36\lambda_1\lambda_3-121\lambda_2\lambda_3})}{36\lambda_1+121\lambda_2} & \frac{22\lambda_2-6\sqrt{-4\lambda_1\lambda_2-36\lambda_1\lambda_3-121\lambda_2\lambda_3}}{36\lambda_1+121\lambda_2} & 0 & 1 \end{pmatrix}^t.$$

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