# Finding proper efficient solutions in fuzzy multiobjective fractional programming

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**Abstract**

A solution algorithm to fuzzy multiobjective fractional programming problems is suggested. These problems involve fuzzy parameters in the right-hand side of the constraints and the concept of $\alpha$-level set of the fuzzy numbers is given. Results of Geoffrion for efficient and properly efficient solutions of multiobjective programming problems are extended to fuzzy multiobjective fractional programming problems. An illustrative numerical example is presented to clarify the developed theory and the solution algorithm.

***Keywords :*** *Multiobjective fractional programming, fuzzy numbers, $\alpha$-level set, efficient solutions, proper efficiency.*

## 1. Introduction

Linear fractional objective functions occur frequently as measures of performance in a variety of circumstances such as when satisfying objectives under uncertainty [7, 9, 10, 11]. Real-valued linear objective function fractional programming was introduced into literature by Charnes and Cooper [1]. Results of Geoffrion for efficient and properly efficient solutions of multiobjective programming problems are extended in [13] to multiobjective fractional programming problems.

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A number of reported studies has been done by the author and others in the field of linear, nonlinear, integer fractional programming and multiobjective fractional programming, for example, [6, 7, 8, 9, 10, 11, 13].

In the previous work, Hughes [6] introduced a geometric approach for finding interior efficient solutions in bicriterion linear fractional programming problems. The general conditions for these problems to have interior efficient solutions have been derived. It has been shown that when these conditions are satisfied, the interior efficient solutions form a hyperplane separating the points where the individual objective functions are optimized. On the other hand, Saad [7] presented a solution procedure for solving linear fractional programs having fuzzy parameters in the right-hand side of the constraints. These parameters have been characterized by fuzzy numbers and the concept of $\alpha$-optimality has been introduced. Later, Saad and Hughes [8] considered bicriterion integer linear fractional programs with single-scalar parameter in the objective functions. For such programs, the stability of efficient solution in the decision space has been studied and the stability set of the first kind has been defined and characterized. Bicriterion integer nonlinear fractional programs (*BINOLFP*) involving fuzzy parameters in the objective functions have been studied by Saad and Abdelkader in [9]. Moreover, a solution algorithm has been described to solve the (*BINOLFP*). A solution algorithm has been proposed by Saad and Abd-Rabo [10] for solving integer linear fractional programs involving random parameters in the right-hand side of the constraints. The suggested procedure in [10] was based upon the chance-constrained programming technique [12] along with the branch-and bound method [14]. Saad and Sharif [11] developed a solution method to solve integer linear fractional programs with chance constraints and having statistically independent random parameters. The main features of their proposed method are based on the Charnes and Cooper transformation [1] with the use of the cutting-plane technique [14].

In the present paper we consider a fuzzy multiobjective fractional programming problem with each component of the objective function having a different denominator. In this setting, use of the Charnes and Cooper transformation [1] seems inhibitive. The problem of concern involves fuzzy parameters in the right-hand side of the constraints. A solution algorithm is described in finite number of steps to solve the model of fuzzy multiobjective fractional programming problem.

Throughout the paper we distinguish between $\leq$ and $\underset{\equiv}{\prec}$ or $\geq$ and $\underset{\equiv}{\succ}$.

This paper consists of the following main sections: Section 2 contains the mathematical formulation of fuzzy multiobjective fractional programming problem and surveys some basic definitions in the fuzzy theory. In addition, a nonfuzzy version of the formulated model is stated along with the concepts of efficient and proper efficient solutions. Results of Geoffrion for efficient and proper efficient solutions to multiobjective programming problems are extended to fuzzy multiobjective fractional programming problems. Section 3 outlines the solution algorithm in finite steps. Section 4 gives a numerical example and finally the paper is concluded in Section 5.

## 2. Problem formulation and the solution concept

The problem to be considered in this paper is the following fuzzy multiobjective fractional programming problem:

$$(FMOFP) \quad : \text{Max} \left[\frac{f(x)}{g(x)}\right] = \left[\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \ldots, \frac{f_p(x)}{g_p(x)}\right],$$
$$\text{subject to} \quad x \in X(A, \widetilde{b}) = \{x \in R^n / Ax \leq \widetilde{b}, x \geq 0\},$$

where $A$ is an $(m \times n)$-matrix, $\widetilde{b}$ is an $m$-vector of fuzzy parameters and we suppose that they are given by fuzzy numbers, estimated from the information provided by the decision maker.

A fuzzy number is defined differently by different authors. The most frequently used definition is the following one.

**Definition 1 ([4]).** It is appropriate to recall that a real fuzzy number $\widetilde{a}$ is a continuous fuzzy subset from the real line $R$ whose membership function $\mu_{\widetilde{a}}(a)$ is defined by:

(1) A continuous mapping from $R$ to the closed interval $[0, 1]$,
(2) $\mu_{\widetilde{a}}(a) = 0$ for all $a \in (-\infty, a_1]$,
(3) strictly increasing on $[a_1, a_2]$,
(4) $\mu_{\widetilde{a}}(a) = 1$ for all $a \in [a_2, a_3]$,
(5) strictly decreasing on $[a_3, a_4]$,
(6) $\mu_{\widetilde{a}}(a) = 0$ for all $a \in [a_4, +\infty)$.

Here, the vector of fuzzy parameters $\widetilde{b}$ involved in problem ($FMOFP$) is a vector of fuzzy numbers whose membership function is $\mu_{\widetilde{b}}(b)$.

In what follows, we give the definition of the $\alpha$-level set or $\alpha$-cut of the fuzzy vector $\widetilde{b} = [\widetilde{b}_1, \widetilde{b}_2, \dots, \widetilde{b}_m]$.

**Definition 2.** The $\alpha$-level set of the vector of fuzzy parameters $\widetilde{b}$ in problem (*FMOFP*) is defined as the ordinary set $L_\alpha(\widetilde{b})$ for which the degree of its membership function exceeds the level set $\alpha \in [0,1]$, where:

$$L_\alpha(\widetilde{b}) = \{b \in R^m / \mu_{\widetilde{b}}(b) \geq \alpha\}.$$

For a certain degree $\alpha$, the (*FMOFP*) can be understood as the following nonfuzzy $\alpha$-multiobjective fractional programming problem ($\alpha$-*MOFP*):

$$(\alpha\text{-}MOFP) \; : \; \text{Max} \left[\frac{f(x)}{g(x)}\right] = \left[\frac{f_1(x)}{g_1(x)}, \frac{f_2(x)}{g_2(x)}, \dots, \frac{f_p(x)}{g_p(x)}\right],$$
$$\text{subject to} \quad x \in X(A,b) = \{x \in R^n / Ax \leq b, \; x \geq 0, b \in L_\alpha(\widetilde{b})\}.$$

It should be emphasized here in the ($\alpha$-*MOFP*) above that the vector of parameters $b$ is treated as a vector of decision variables rather than constants.

Problem ($\alpha$-*MOFP*) can be reformulated in the following form:

$$(\text{P}) \; : \; \text{Max} \, F(x) = (F_1(x), F_2(x), \dots, F_p(x),$$
$$\text{subject to} \quad x \in X(A,b),$$

where $F_i(x) = f_i(x) - \theta_i^* g_i(x)$ and $\theta_i^* \underset{\equiv}{\succ} 0$, $(i = 1,2,\dots,p)$ are fixed parameters.

Based on Definition 2 of the $\alpha$-level set of the vector of fuzzy numbers $\widetilde{b}$, we introduce the concept of $\alpha$-efficient solution of problem (P) above as follows:

**Definition 3.** A point $x^* \in X(A,b)$ is said to be an $\alpha$-efficient solution of problem (P) if and only if there exists no other $x \in X(A,b)$, $b \in L_\alpha(\widetilde{b})$ such that $F_i(x^*) \leq F_i(x)$; $(i = 1,2,\dots,p)$ with strictly inequality holding for at least one $i$, where the corresponding values of parameters $b_r^*$, $(r = 1,2,\dots,m)$ are called the $\alpha$-level optimal parameters.

Now, consider $\lambda$ is a $p$-dimensional strictly positive fixed vector, then problem (P) can be rewritten again in a problem of scalar single-objective function $(\mathrm{P}_\lambda)$ in the following form:

$$(\mathrm{P}_\lambda) \ : \ \mathrm{Max} \sum_{i=1}^{p} \lambda_i f_i(x)\,,$$
$$\text{subject to} \quad x \in X(A,b).$$

Let $X(A,b)$ denote the set of feasible solutions of problem ($\alpha$-*MOFP*) or (P) or $(\mathrm{P}_\lambda)$. We assume that $f(x) \underset{\equiv}{\succ} 0$, $g(x) \succ 0$, for all $x \in X(A,b)$. We further assume that $f, -g$ are concave functions and $X(A,b)$ is a convex set. It follows that $F$ is concave, (see[13]).

Problem $(\mathrm{P}_\lambda)$ can be solved at $\lambda_i = \lambda_i^* \succ 0$ and $\sum_{i=1}^{p} \lambda_i^* = 1$ with the corresponding fixed parameters $\theta_i = \theta_i^*$, $(i = 1, 2, \ldots, p)$ using any available nonlinear programming package, for example, GINO or SUPERGINO, to find the $\alpha$-optimal solution $x^*$ together with the optimal parameters, $b_r^*$, $(r = 1, 2, \ldots, m)$.

**Definition 4 (Geoffrion [5]).** Consider the multiobjective programming problem

$$\mathrm{Max}\, \phi(x) = (\phi_1(x), \phi_2(x), \ldots, \phi_k(x)),$$
$$\text{subject to} \quad x \in S \subseteq R^n\,.$$

We say that $x^0 \in S$ is efficient if and only if there exists no $x^0 \in S$ such that $\phi(x^0) \leq \phi(x)$.

**Definition 5 (Geoffrion [5]).** For the multiobjective programming problem in Definition 4, we say that an efficient solution $x^0$ is properly efficient if and only if for each $i$ and $x \in S$, there exists a positive real number $M$ and a $j$ such that

$$\phi_j(x^0) - \phi_j(x) \succ 0 \quad \text{and} \quad \phi_i(x) - \phi_i(x^0) \underset{\equiv}{\prec} M((\phi_j(x^0) - \phi_j(x)),$$

whenever $\phi_i(x) - \phi_i(x^0) \succ 0$.

Before we go further, it should be noted that for multiobjective linear fractional programming, when the emphasis is on finding efficient solutions, there is no general method for finding all the efficient solutions but Choo and Atkins [2] have developed an algorithm, using row parameters, for solving the bicriterion linear fractional programming problem (*BLFP*).

Choo [3] has also shown that if $x^0$ is an efficient solution to (*BLFP*) then $x^0$ is properly efficient [5].

The nonnegativity of $\theta_i^*$ is needed to establish Part (b) of Theorem 1 below.

**Theorem 1.**

(a) *If* $x^*$ *is an* $\alpha$*-optimal solution of* $(P_\lambda)$, *then* $x^*$ *is properly an* $\alpha$*-efficient for* $(\mathrm{P})$.

(b) *If* $f$ *and* $-g$ *are concave and* $x^*$ *is properly an* $\alpha$*-efficient for* $(\mathrm{P})$, *then it is an* $\alpha$*-optimal for* $(P_\lambda)$.

*We omit the proof of Theorem* 1 *and the reader is referred to* [5].

The following theorem does not require any convexity assumptions, (see [13]).

**Theorem 2.** *The point* $x^* \in X(A,b)$ *is an* $\alpha$*-efficient solution of* ($\alpha$*-MOFP*) *if and only if it is an* $\alpha$*-efficient of* $(\mathrm{P})$ *with* $F(x^*) = \overline{0}$.

***Proof.*** Suppose $x^* \in X(A,b)$ is an $\alpha$-efficient solution of ($\alpha$-*MOFP*). Then by Definition 4, there is no $x \in X(A,b)$ such that

$$\frac{f_i(x^*)}{g_i(x^*)} \leq \frac{f_i(x)}{g_i(x)}, \quad \forall\ i = 1,\dots,p\,.$$

Letting $\theta_i^* = \dfrac{f_i(x^*)}{g_i(x^*)}$ for $i = 1,\dots,p$, we see from the above inequality that there does not exist an $x \in X(A,b)$ such that

$$0 \leq f_i(x) - \theta_i^* g_i(x) = F_i(x), \quad \forall\ i = 1,\dots,p\,.$$

Since $0 = f_i(x^*) - \theta_i^* g_i(x^*) = F_i(x^*)$, $i = 1,\dots,p$, we see that there exists no $x$ in $X(A,b)$ such that $F_i(x^*) \leq F_i(x)$ for $i = 1,\dots,p$. Therefore, $x^*$ is an $\alpha$-efficient of $(\mathrm{P})$ with $F(x^*) = \overline{0}$.

Conversely, suppose $x^*$ is an $\alpha$-efficient solution of $(\mathrm{P})$ with $F(x^*) = 0 = f(x^*) - \theta^* g(x^*)$. That means, by Definition 4, there exists no $x \in X(A,b)$ such that

$$0 = F_i(x^*) \leq F_i(x) = f_i(x) - \theta_i^* g_i(x), \quad \forall\ i = 1,\dots,p\,.$$

That is, there exists no $x \in X(A,b)$ such that

$$\frac{f_i(x^*)}{g_i(x^*)} = \theta_i^* \leq \frac{f_i(x)}{g_i(x)}, \quad \forall\ i = 1,\dots,p\,.$$

Hence, $x^*$ is an $\alpha$-efficient solution of ($\alpha$-*MOFP*). □

For the development that follows, we assume that there exist real numbers $k \succ 0$, $K \succ 0$ such that $k \prec g_j(x) \prec K$ for all $i$. Applying Definition 5 of the proper efficiency to problem ($\alpha$-*MOFP*), we note that an $\alpha$-efficient solution $x^*$ of problem ($\alpha$-*MOFP*) is properly $\alpha$-efficient if there exists a real number $\bar{M} \succ 0$ such that for each $i$, we have

$$f_i(x)/g_i(x) - f_i(x^*)/g_i(x^*) \underset{\equiv}{\prec} M[f_j(x^*)/g_j(x^*) - f_j(x)/g_j(x)]$$

for some $j$ such that $f_j(x)/g_j(x) \prec f_j(x^*)/g_j(x^*)$ whenever $x \in X(A,b)$ and $f_i(x)/g_i(x) \succ f_i(x^*)/g_i(x^*)$. Or, rewriting these inequalities slightly differently, we say an $\alpha$-efficient solution $x^*$ of ($\alpha$-*MOFP*) is properly $\alpha$-efficient if there exists a real number $\overline{M} \succ 0$ such that for each $i$, we have

$$\begin{aligned}&[f_i(x)g_i(x^*) - f_i(x^*)g_i(x)]/g_i(x^*)\\ &\quad \leq \overline{M}[f_j(x^*)g_j(x) - f_j(x)g_j(x^*)]/g_j(x^*),\end{aligned} \tag{1}$$

where $\overline{M} = MK/k$ for some $j$ such that

$$f_j(x)g_j(x^*)g_j(x) \prec 0 \tag{2}$$

whenever $x \in X(A,b)$ and

$$f_i(x)g_i(x^*) - f_i(x^*)g_i(x) \succ 0\,. \tag{3}$$

To link proper $\alpha$-efficiency of problem ($\alpha$-*MOFP*) and (P), we prove the following theorem.

**Theorem 3.** *The point $x^* \in X(A,b)$ is a properly $\alpha$-efficient solution of ($\alpha$-MOFP) if and only if it is a properly $\alpha$-efficient solution of (P) with $F(x^*) = \overline{0}$.*

***Proof.*** Suppose $x^*$ is a properly $\alpha$-efficient solution of ($\alpha$-*MOFP*). Then by Theorem 2, we know it is an $\alpha$-efficient solution of (P) with $F(x^*) = \overline{0}$. Now $x^*$ is a properly $\alpha$-efficient solution of (P) if there exists a positive real number $M$ such that for each $i$,

$$F_i(x) - F_i(x^*) \underset{\equiv}{\prec} M(F_j(x^*) - F_j(x)) \tag{4}$$

for some $j$ such that

$$F_j(x) - F_j(x^*) \prec 0\,, \tag{5}$$

whenever $x \in X(A, b)$ and

$$F_i(x) - F_j(x^*) \succ 0. \tag{6}$$

Or [in view of the fact that $F_i(x^*) = 0$ for all $i$ and $F_i(x) = f_i(x) - \theta_i^* g_j(x)$ with $\theta_i^* = f_i(x^*)/g_i(x^*)$ for $i = 1, \dots, p$], the result holds if and only if there exists an $M \succ 0$ such for each $i$,

$$[f_i(x)g_i(x^*) - f_i(x^*)g_i(x)]/g_i(x^*) \\ \underset{\equiv}{\prec} M[f_j(x^*)g_j(x) - f_j(x)g_j(x^*)]/g_j(x^*) \tag{7}$$

for some $j$ such that

$$f_j(x)g_j(x^*) - f_j(x^*)g_j(x) \prec 0, \tag{8}$$

whenever $x \in X(A, b)$ and

$$f_i(x)g_i(x^*) - f_i(x^*)g_i(x) \succ 0. \tag{9}$$

Relations (7)-(9) hold by (1)-(3) with $M = \overline{M}$. Conversely, suppose $x^*$ is a properly $\alpha$-efficient solution of $(\mathrm{P})$ with $F(x^*) = \overline{0}$. Then by Definition 5, relations (4)-(6) hold for some $M$ and each $i$ and $x \in X(A, b)$. From this it follows that (7)-(9) hold which are (1)-(3) with $\overline{M} = M$.

## 3. Solution algorithm

In this section, a solution algorithm to solve fuzzy multiobjective fractional programming problem (*FMOFP*) is described in a finite number of steps. The suggested algorithm can be summarized in the following manner:

*Step* 0. Start with an initial level set $\alpha = \alpha^* = 0$.

*Step* 1. Determine points $(b_1, b_2, b_3, b_4)$ for the vector of the fuzzy parameters $\widetilde{b}$ in problem (*FMOFP*) to elicit a membership function $\mu_{\widetilde{b}}(b)$ satisfying assumptions (1)-(6) in Definition 1.

*Step* 2. Convert problem (*FMOFP*) into its nonfuzzy version ($\alpha$*-MOFP*).

*Step* 3. Rewrite problem ($\alpha$*-MOFP*) in the form of problem $(\mathrm{P}_\lambda)$ of single-objective function.

*Step* 4. Choose $\lambda_i = \lambda_i^* \succ 0$ and $\sum_{i=1}^{p} \lambda_i^* = 1$ with fixed values of $\theta_i = \theta_i^*$, $(i = 1, 2, \dots, p)$ and use the GINO or SUPERGINO software package to find the $\alpha$-optimal solution $x^*$ of problem $(\mathrm{P}_\lambda)$.

*Step* 5. Set $\alpha = (\alpha^* + \text{step}) \in [0,1]$ and go to *Step* 1.

*Step* 6. Repeat again the above procedure until the interval $[0,1]$ is fully exhausted. Then, stop.

## 4. An illustrative examble

In what follows we provide a numerical example to clarify the developed theory and the solution algorithm suggested in this paper.

Let

$$f_1(x) = 1 - x_1^2\,, \quad g_1(x) = 1 + 2x_2^2$$
$$f_2(x) = 2\,, \qquad g_2(x) = 2 - x_2\,.$$

So

$$F_1(x) = \frac{f_1(x)}{g_1(x)} = \frac{1 - x_1^2}{1 + 2x_2^2}\,, \qquad F_2(x) = \frac{f_2(x)}{g_2(x)} = \frac{2}{2 - x_2}\,.$$

Consider the following fuzzy bicriterion fractional programming problem (*FBFP*):

$$\begin{aligned} &\text{Max } F(x) = (F_1(x), F_2(x))\,, \\ &\text{subject to } \quad x_1^2 + x_2^2 \le \widetilde{b}\,, \\ &\qquad\qquad\quad x_1, x_2 \ge 0\,, \end{aligned}$$

where $\widetilde{b}$ is a fuzzy parameter and is characterized by the following fuzzy numbers:

$$\widetilde{b} = (0, 1, 3, 5)\,.$$

Assume that the membership function of these fuzzy numbers in the following form:

$$\mu_{\widetilde{b}}(b) = \begin{cases} 0, & b \le b_1, \\ 1 - \left(\dfrac{b - b_2}{b_1 - b_2}\right)^2, & b_1 \le b \le b_2\,, \\ 1, & b_2 \le b \le b_3\,, \\ 1 - \left(\dfrac{b - b_3}{b_4 - b_3}\right)^2, & b_3 \le b \le b_4\,, \\ 0, & b \ge b_4\,. \end{cases}$$

Let $\alpha = 0.19$, for example, then we get:

$$0.1 \le b \le 4.8\,.$$

Choosing $b = 1$, the nonfuzzy $\alpha$-bicriterion fractional programming problem ($\alpha$-*BFP*) becomes:

$$\text{Max } F(x) = (F_1(x), F_2(x)),$$
$$\text{subject to } x_1^2 + x_2^2 \le 1,$$
$$x_1, x_2 \ge 0.$$

The point $x^* = (0,0)$ is an $\alpha$-efficient solution of problem ($\alpha$-*BFP*) since, for each feasible $x$, we have:

$$F_1(x) - F_1(x^*) = \frac{1 - x_1^2}{1 + 2x_2^2} - 1 = -\frac{x_1^2 + 2x_2^2}{1 + 2x_2^2} \underset{\equiv}{\prec} 0$$

and

$$F_2(x) - F_2(x^*) = \frac{2}{2 - x_2} - 1 = -\frac{x_2}{2 - x_2} \underset{\equiv}{\succ} 0,$$

and there is no other feasible point for which

$$F(x) = (F_1(x), F_2(x)) \ge (1,1).$$

Now considering the case $i = 2$, $j = 1$ in the definition of a properly efficient solution, it can be seen that $x^* = (0,0)$ is also a properly $\alpha$-efficient solution.

When

$$F_2(x) \succ F_2(x^*) \quad \text{we have} \quad \frac{x_2}{2 - x_2} \succ 0; \quad \text{that is, } x_2 \succ 0.$$

Then

$$F_1(x^*) - F_1(x) = \frac{x_1^2 + 2x_2^2}{1_2 x_2^2} \succ 0 \text{ and } F_2(x) - F_2(x^*) \succ 0.$$

Putting

$$M = \frac{x_2(1 + 2x_2^2)}{(2 - x_2)(x_1^2 + 2x_2^2)} \succ 0.$$

We have

$$F_2(x) - F_2(x^*) \le M(F_1(x^*) - F_1(x).$$

So the point $x^* = (0,0)$ is a properly $\alpha$-efficient solution for problem ($\alpha$-*BFP*) with the corresponding $\alpha$-level set equals 0.19.

## 5. Conclusion

In the present paper a solution algorithm to solve fuzzy multiobjective fractional programming problem (*FMOFP*) has been proposed and described in finite steps. Moreover, results of Geoffrion for efficient and properly efficient solutions of multiobjective problems have been extended to (*FMOFP*).

Many aspects and general questions remain to be studied and explored in the area of fuzzy multiobjective fractional programming problems. This paper is an attempt to establish underlying results which hopefully will help others to answer some of these questions. There are however several open points for research in the area of (*FMOFP*), in our opinion, to be studied in future. Some of these problems are:

(i) An algorithm is required for solving stochastic multiobjective fractional programming problems.

(ii) An algorithm is needed for solving large-scale stochastic and fuzzy multiobjective fractional programming problems.

(iii) Stability of the efficient and properly efficient solutions should be investigated to fuzzy multiobjective fractional programming problems for different values of $\alpha$-level sets.

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