A two-echelon inventory model with fuzzy annual demand in a supply chain

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Abstract

It requires a new spirit of cooperation between the buyer and the vendor in supply chain environment today. A two-echelon inventory model with such consideration is based on the total cost optimization under a common stock strategy and business formula. However, the supposition of known annual demand in most related publications may not be realistic. This paper proposes the inclusion of fuzzy annual demand and then employs the signed distance, to find the estimation of the common total cost in the fuzzy sense, and derives the corresponding optimal buyer’s quantity consequently and the integer number of lots in which the items are delivered from the vendor to the purchaser. Numerical example is included to illustrate the procedures of the solution.

Keywords: Supply chain, inventory, signed distance.

1. Introduction

In the current supply chain management environment, JIT requires cooperation between the buyer and the vendor, and it has shown that forming a partnership between the buyer and the vendor is helpful in achieving tangible benefits for both parties Kelle et al.[1]. Many researchers have demonstrated that buyers and vendors can both obtain greater benefit through strategic collaboration with each other Stefan et al. [2].
Banerjee [3] proposed an economic lot size model by assuming that the vendor produces orders for a buyer on a lot-for-lot basis under deterministic conditions. Goyal [4] generalized Banerjee’s model [3] by relaxing the assumption of the lot-for-lot policy of the vendor and showed that joint economic lot size model where the vendor’s economic production quantity per cycle being an integer multiple of the buyer’s purchase quantity provides a lower or equal joint total relevant cost as compared to Banerjee’s model [3]. Lu [5] relaxed Goyal’s assumption of completing a batch before a shipment is started Goyal [4] and explored a model that allowed shipments to take place during production and the delivery quantity to the buyer is known.


Most of the related literature assumed that the average demand per year is fixed constant. However, it is usually difficult for the managers to set the demand as crisp values in reality.

This paper presents two-echelon inventory system with fuzzy annual demand. Building upon the work of Pan and Yang [7], this paper proposes the model incorporates the fuzziness of annual demand. For the model, Yao and Wu’s ranking method [16] for fuzzy number is employed to find the estimation of the joint total expected annual cost in the fuzzy sense, and the corresponding order quantity of the purchaser is derived accordingly.

2. The two-echelon inventory model

2.1 Notation and assumptions

To develop the proposed model, the following notation is used:

- \( D \): average demand per year;
- \( P \): production rate;
- \( Q \): order quantity of the buyer;
- \( A \): purchaser’s ordering cost per order;
- \( S \): vendor’s set-up cost per set-up;
- \( L \): length of lead time;
- \( C_V \): unit production cost paid by the vendor;
- \( C_p \): unit purchase cost paid by the purchaser;
- \( m \): an integer representing the number of lots in which the items are delivered from the vendor to the buyer;
- \( r \): annual inventory holding cost per dollar invested in stocks;
- \( k \): safety stock factor.

The assumptions made in this paper are as follows.

1. The product is manufactured with a finite production rate \( P \), and \( P > D \).
2. The demand \( X \) during lead time \( L \) follows a normal distribution with mean \( \mu L \) and standard deviation \( \sigma \sqrt{L} \).
3. The reorder point (ROP) equals the sum of the expected demand during lead time and safety stock.
4. Inventory is continuously reviewed.

2.2 A model with fuzzy annual demand

Because the inventory pattern for this model is shown in Figure 1. So the joint total expected annual cost \( JTEC(Q, m) \) for the problem under
study can be established as Pan and Yang [7]

\[
J_{TEC}(Q, m) = \frac{D}{Q} \left( A + \frac{S}{m} \right) + \frac{Q}{2} r \left[ \left( m \left( 1 - \frac{D}{P} \right) \right) - 1 + \frac{2D}{P} \right] C_V + C_P \right] + rC_Pk\sigma\sqrt{L}. \tag{1}
\]

\[ \text{quantity} \]

\[ \text{time} \]

\[ Q/P \]

\[ Q/D \]

\[ mQ/P \]

\[ mQ/D \]

\[ mQ \]

\[ \text{Accumulated Inventory for vendor} \]

\[ \text{Accumulated inventory for buyer} \]

\[ \text{Figure 1} \]

\text{The vendor’s inventory pattern}

Consider the problem with fuzzy annual demand by fuzzifying \( D \) to a triangular fuzzy number \( \tilde{D} \), where \( \tilde{D} = (D - \Delta_1, D, D - \Delta_2) \), \( 0 < \Delta_1 < D \), \( 0 < \Delta_2 \) and \( \Delta_1, \Delta_2 \) are both determined by decision-makers. In this case, the joint total expected annual cost is a fuzzy function and can be
expressed as
\[
\tilde{W}(Q, m) = \frac{D}{Q} \left( A + \frac{S}{m} \right) + \frac{Q}{2} \left[ \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) C_V + C_F \right] + rC_P k \sigma \sqrt{T}. \quad (2)
\]

**Definition 1.** From Kaufmann and Gupta (1991) [17], Zimmermann (1996) [18], Yao and Wu [16], for any \( a \) and \( 0 \in R \), define the signed distance from \( a \) to 0 as \( d_0(a, 0) = a \). If \( a > 0 \), \( a \) is on the right hand side of origin 0; and the distance from \( a \) to 0 is \( d_0(a, 0) = a \). If \( a > 0 \), \( a \) is on the left hand side of origin 0; and the distance from \( a \) to 0 is \( -d_0(a, 0) = -a \). This is the reason why \( d_0(a, 0) = a \) is called the signed distance from \( a \) to 0.

Let \( \Omega \) be the family of all fuzzy sets \( \tilde{A} \) defined on \( R \), the \( \alpha \)-cut of \( \tilde{A} \) is \( A(\alpha) = [A_L(\alpha), A_U(\alpha)] \), \( 0 \leq \alpha \leq 1 \), and both \( A_L(\alpha) \) and \( A_U(\alpha) \) are continuous functions on \( \alpha \in [0, 1] \). Then, for any \( \tilde{A} \in \Omega \), we have
\[
\tilde{A} = \bigcup_{0 \leq \alpha \leq 1} [A_L(\alpha)_a, A_U(\alpha)_a]. \quad (3)
\]

Besides, for every \( \alpha \in [0, 1] \), the \( \alpha \)-level fuzzy interval \( [A_L(\alpha)_\alpha, A_U(\alpha)_\alpha] \) has a one-to-one correspondence with the crisp interval \( [A_L(\alpha), A_U(\alpha)] \), that is, \( [A_L(\alpha)_\alpha, A_U(\alpha)_\alpha] \rightarrow [A_L(\alpha), A_U(\alpha)] \) is one-to-one mapping. From Definition 1, the signed distance of two end points, \( A_L(\alpha) \) and \( A_U(\alpha) \) to 0 are \( d_0(A_L(\alpha), 0) = A_L(\alpha) \) and \( d_0(A_U(\alpha), 0) = A_U(\alpha) \), respectively.

Hence, the signed distance of interval \( [A_L(\alpha), A_U(\alpha)] \) to 0 can be represented by their average, \( (A_L(\alpha) + A_U(\alpha))/2 \). Therefore, the signed distance of interval \( [A_L(\alpha), A_U(\alpha)] \) to 0 can be represented as
\[
d_0([A_L(\alpha), A_U(\alpha)], 0) = \frac{d_0(A_L(\alpha), 0) + d_0(A_U(\alpha), 0)}{2} = \frac{(A_L(\alpha) + A_U(\alpha))}{2}. \quad (4)
\]

Further, because of the 1-level fuzzy point \( \tilde{0}_1 \) is mapping to the real number 0, the signed distance of \( [A_L(\alpha)_\alpha, A_U(\alpha)_\alpha] \) to \( \tilde{0}_1 \) can be defined as
\[
d([A_L(\alpha)_\alpha, A_U(\alpha)_\alpha], \tilde{0}_1) = d_0([A_L(\alpha), A_U(\alpha)], 0) = \frac{(A_L(\alpha) + A_U(\alpha))}{2}. \quad (5)
\]

Thus, from (4) and (5), since the above function is continuous on \( 0 \leq \alpha \leq 1 \) for \( \tilde{A} \in \Omega \), we can use the following equation to define the signed distance of \( \tilde{A} \) to \( \tilde{0}_1 \) as follows.
Next, defuzzify $\tilde{W}(Q, m)$ by using the signed distance method. From Definition 1, the signed distance of $\tilde{W}$ to $\tilde{0}_1$ is given by

$$d(\tilde{W}, \tilde{0}_1) = \frac{d(\tilde{D}, \tilde{0}_1)}{Q} \left( A + \frac{S}{m} \right) + \frac{Q}{2} \left[ \left( m - \frac{d(\tilde{D}, \tilde{0}_1)}{p} \right) - 1 + \frac{2d(\tilde{D}, \tilde{0}_1)}{p} \right] C_V + C_P$$

$$+ r C_p k \sigma \sqrt{L}$$

$$= d(\tilde{D}, \tilde{0}_1) \left[ \frac{A}{Q} + \frac{S}{mQ} + \frac{(2 - m)rQC_V}{2p} \right]$$

$$+ \frac{rQ}{2} [(m - 1)C_V + C_p] + r C_p k \sigma \sqrt{L}$$

(6)

where $d(\tilde{D}, \tilde{0}_1)$, the signed distance of fuzzy number $\tilde{D}$ to $\tilde{0}_1$, by Appendix 1, is

$$d(\tilde{D}, \tilde{0}_1) = \frac{1}{4} [(D - \Delta_1) + 2D + (D + \Delta_2)] = D + \frac{1}{4} (\Delta_2 - \Delta_1).$$

(7)

Substituting the result of (6) into (7), we have

$$W(Q, m) \equiv d(\tilde{W}, \tilde{0}_1)$$

$$= \left[ D + \frac{(\Delta_2 - \Delta_1)}{4} \right] \left[ \frac{A}{Q} + \frac{S}{mQ} + \frac{(2 - m)rQC_V}{2p} \right]$$

$$+ \frac{rQ}{2} [(m - 1)C_V + C_p] + r C_p k \sigma \sqrt{L}$$

(8)

where $W(Q, m)$ is regarded as the estimate of the joint total expected annual cost in the fuzzy sense.

The objective of this problem is to determine the optimal order quantity of the purchaser $Q^*$ and the optimal integer number of lots in which the items are delivered from the vendor to the purchaser such that $W(Q, m)$ achieves its minimum value. Utilizing classical optimization, we take the first and second derivatives of $W(Q, m)$ with respect to $Q$, and obtain

$$\frac{\partial W(Q, m)}{\partial Q} = - \frac{(\tilde{D}, \tilde{0}_1)}{Q^2} \left( A + \frac{S}{m} \right)$$

$$+ \frac{r}{2} \left[ \left( m - \frac{d(\tilde{D}, \tilde{0}_1)}{p} \right) - 1 + \frac{2d(\tilde{D}, \tilde{0}_1)}{p} \right] C_V + C_P$$

(9)
and
\[
\frac{\partial^2 W(Q,m)}{\partial Q^2} = \frac{2d(\tilde{D},\tilde{D}_1)}{Q^3} \left( A + \frac{S}{m} \right). \tag{10}
\]

Since \( \frac{\partial^2 W(Q,m)}{\partial Q^2} > 0 \), i.e., \( W(Q,m) \) is convex in \( Q \), and hence the minimum value of \( W(Q,m) \) will occur at the point that satisfies \( \frac{\partial W(Q,m)}{\partial Q} = 0 \). Setting (9) equal to zero and solving for \( Q \), we obtain the optimal order quantity of the purchaser as:
\[
Q = \sqrt{\frac{2d(\tilde{D},\tilde{D}_1)(A + \frac{S}{m})}{r \left[ C_V \left( m \left( 1 - \frac{d(\tilde{D},\tilde{D}_1)}{p} \right) - 1 + \frac{2d(\tilde{D},\tilde{D}_1)}{p} \right) + C_P \right]}} \tag{11}
\]
where \( d(\tilde{D},\tilde{D}_1) = D + 1/4(\Delta_2 - \Delta_1) \).

The derivation of equation (11) is shown in Appendix 2.

In order to find the optimal \( m \), we can use equation (12) which is proved in Appendix 3.
\[
m^*(m^* - 1) \leq \frac{S \left[ C_P - \left( 1 - \frac{2d(\tilde{D},\tilde{D}_1)}{p} \right) C_V \right]}{AC_V \left( 1 - \frac{d(\tilde{D},\tilde{D}_1)}{p} \right)} \leq m^*(m^* + 1), \tag{12}
\]
where \( d(\tilde{D},\tilde{D}_1) = D + 1/4(\Delta_2 - \Delta_1) \).

Thus, we can use the following procedure to find the optimal values of \( Q \) and \( m \).

**Step 1.** Obtain \( \Delta_1 \) and \( \Delta_2 \) from the decision-maker.

**Step 2.** Compute the optimal integer number of lots in which the items are delivered from the vendor to the purchaser by equation (12).

**Step 3.** Compute the optimal order quantity of the purchaser by equation (11).

**Step 4.** The \( W(Q^*,m^*) \) is the optimal joint total expected annual cost.

**Remark 1.** If \( \Delta_1 = \Delta_2 = \Delta \), then (7) reduces to \( d(\tilde{D},\tilde{D}_1) = D \); thus the estimate of the joint total expected annual cost in fuzzy sense (6) is identical to the crisp case. Hence, the crisp average demand per year model is a special case of the fuzzy model presented here. Besides, for the optimal order quantity of the purchaser (11), when \( \Delta_1 = \Delta_2 = \Delta \), it...
reduces to

\[ Q = \sqrt{\frac{2D \left( A + \frac{S}{m} \right)}{r \left[ C_V \left( m \left( 1 - \frac{D}{P} \right) - 1 + \frac{2D}{P} \right) + C_p \right]}} \] (13)

and the derivation of equation (12) reduces to

\[ m^* (m^* - 1) \leq \left[ \frac{S \left[ C_p - \left( 1 - \frac{D}{P} \right) C_V \right]}{AC_V \left( 1 - \frac{D}{P} \right)} \right] \leq m^* (m^* + 1). \] (14)

3. Numerical examples

To illustrate the results of the proposed models, consider an inventory system with data: annual demand \( D = 1500 \) unit/year, production rate \( P = 3000 \) unit/year, purchaser’s ordering cost per order \( A = $25/\text{order} \), vendor’s set-up cost \( S = $400/\text{set-up} \), lead time \( L = 8 \) weeks, purchase cost \( C_p = $25/\text{unit} \), production cost \( C_V = $20/\text{unit} \), annual inventory holding cost per dollar invested in stock \( r = 0.2 \), safety stock factor \( k = 2.33 \), and standard deviation \( \sigma = 7 \) unit/week.

For the model proposed in Section 2.2, solve for the optimal order quantity of purchaser and find the optimal joint total expected annual cost \( W(Q^*, m^*) \) in the fuzzy sense for various given sets of \((\Delta_1, \Delta_2)\). Note that in practical situations, \( \Delta_1 \) and \( \Delta_2 \) are determined by the decision-makers due to the uncertainty of the problem. The results are summarized in Table 1.

Furthermore, Table 1 lists the results of the fuzzy case results with those of the crisp one. The optimal order quantity of purchaser, \( Q^*_C \), and the corresponding joint total expected annual cost \( W(Q^*_C) \) of the crisp case can be derived easily from Pan and Yang [7] using the classical optimization technique. Consequently, we have \( Q^*_C = 127.1868 \) units and \( W(Q^*_C) = $2,392.83 \). Then, the relative variation between fuzzy case and crisp one for the optimal order quantity of purchaser and the optimal joint total expected annual cost can be measured by \( V_Q = (Q^* - Q^*_C)/Q^*_C \times 100\% \) and \( V_W = (W(Q^*, m^*) - W(Q^*_C))/Q^*_C \times 100\% \), respectively, as reported in the last two columns of Table 1.
Table 1

<table>
<thead>
<tr>
<th>$\Delta_1$</th>
<th>$\Delta_2$</th>
<th>$\bar{\delta}$</th>
<th>$m^*$</th>
<th>$Q^*$</th>
<th>$W(Q^<em>, m^</em>)$</th>
<th>$V_Q(%)$</th>
<th>$V_W(%)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>50</td>
<td>100</td>
<td>(1450, 1500, 1600)</td>
<td>6</td>
<td>127.9668</td>
<td>$2,397.60$</td>
<td>0.61</td>
<td>0.20</td>
</tr>
<tr>
<td>100</td>
<td>200</td>
<td>(1400, 1500, 1700)</td>
<td>6</td>
<td>128.7482</td>
<td>$2,402.37$</td>
<td>1.23</td>
<td>0.40</td>
</tr>
<tr>
<td>150</td>
<td>300</td>
<td>(1350, 1500, 1800)</td>
<td>6</td>
<td>129.531</td>
<td>$2,407.14$</td>
<td>1.84</td>
<td>0.60</td>
</tr>
<tr>
<td>200</td>
<td>400</td>
<td>(1300, 1500, 1900)</td>
<td>7</td>
<td>116.7972</td>
<td>$2,418.79$</td>
<td>−8.17</td>
<td>1.08</td>
</tr>
<tr>
<td>250</td>
<td>500</td>
<td>(1250, 1500, 2000)</td>
<td>7</td>
<td>117.5298</td>
<td>$2,421.57$</td>
<td>−7.59</td>
<td>1.20</td>
</tr>
<tr>
<td>500</td>
<td>500</td>
<td>(1000, 1500, 2000)</td>
<td>6</td>
<td>127.1868</td>
<td>$2,392.83$</td>
<td>0.00</td>
<td>0.00</td>
</tr>
<tr>
<td>500</td>
<td>250</td>
<td>(1000, 1500, 1750)</td>
<td>6</td>
<td>123.3058</td>
<td>$2,368.99$</td>
<td>−3.05</td>
<td>−1.00</td>
</tr>
<tr>
<td>400</td>
<td>200</td>
<td>(1100, 1500, 1700)</td>
<td>6</td>
<td>124.0796</td>
<td>$2,373.76$</td>
<td>−2.44</td>
<td>−0.80</td>
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<tr>
<td>300</td>
<td>150</td>
<td>(1200, 1500, 1650)</td>
<td>6</td>
<td>124.8546</td>
<td>$2,378.52$</td>
<td>−1.83</td>
<td>−0.60</td>
</tr>
<tr>
<td>200</td>
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<td>(1300, 1500, 1600)</td>
<td>6</td>
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<td>126.4081</td>
<td>$2,388.06$</td>
<td>−0.61</td>
<td>−0.20</td>
</tr>
</tbody>
</table>

From Table 1, we observe that

(1) when $\Delta_1 < \Delta_2$, as $(\Delta_2 - \Delta_1)$ increase up to 0 from 150, both $V_Q$ and $V_W$ increase. Thus, for the case that $(\Delta_2 - \Delta_1) > 150$, we have $Q^* < Q^*_C$ and $W(Q^*, m^*) > W(Q^*_C)$, which result in $V_Q < 0$ and $V_W > 0$. As the value $(\Delta_2 - \Delta_1)$ increase, $V_Q$ increases but $V_W$ decreases.

(2) when $\Delta_1 > \Delta_2$, then we have $d(\bar{\delta}, \tilde{\delta}_1) < D$. In this case, $Q^* < Q^*_C$ and $W(Q^*, m^*) < W(Q^*_C)$, which result in $V_Q < 0$ and $V_W < 0$. Further, as the value $(\Delta_2 - \Delta_1)$ decreases, both $V_Q$ and $V_W$ decrease, which means the smaller the difference between $\Delta_1$ and $\Delta_2$ the smaller the variation of the solutions between fuzzy case and crisp case.

(3) when $\Delta_1 = \Delta_2 = 500$, $d(\bar{\delta}, \tilde{\delta}_1) = D = 1500$. In this case, the solutions of the fuzzy case are identical to those of the crisp case, and hence $V_Q = 0$ and $V_W = 0$. This is consistent with what is shown in Remark 1.

From the example, although we can’t ascertain which of the solution is better, the major advantage of the fuzzy model is that the uncertainty of the real situation is captured well than the crisp model. In addition,
the decision-makers can use the solution which derived from the fuzzy model to perform sensitivity analysis, and to examine the effects of uncertainties.

4. Conclusions

Uncertainties of annual demand are inherented in real supply chain inventory systems. However, in practice, there may be a lack of historical data to estimate the annual demand. In this situation, using a crisp value is not appropriate. The inventory model of Pan and Yang [7] is worthwhile to reconsider and we provide an alternative approach.

This paper proposes a fuzzy model for two-echelon inventory problem. For the fuzzy model, a method of defuzzification, namely the signed distance, is employed to find the estimation of total profit per unit time in the fuzzy sense, and then the corresponding optimal $m$ and $Q$ are derived to minimize the total cost. In addition, it is shown that in some cases, the proposed fuzzy model can be reduced to a crisp problem and the optimal order quantity of purchaser in the fuzzy sense can be reduced to that of the classical two-echelon inventory model. Although we are not sure the solution obtained from a fuzzy model is better than that of the crisp one, the advantage of the fuzzy approach is that it keeps the uncertainties which always fits real situations better than the crisp approach does. With this proposed fuzzy model, the real world inventory problem can be properly solved.

Appendix 1. Derivation of equation (7)

For a fuzzy set $\tilde{A} \in \Omega$ and $\alpha \in [0,1]$, the $\alpha$-cut of the fuzzy set $\tilde{A}$ is $A(\alpha) = \{ x \in \Omega | \mu_{\tilde{A}}(x) \geq \alpha \} = [A_L(\alpha), A_U(\alpha)]$, where $A_L(\alpha) = a + (b - a)\alpha$ and $A_U(\alpha) = c - (c - b)\alpha$. From Definition 1, we can obtain the following equation. The signed distance of $\tilde{A}$ to $\tilde{0}_1$ is defined as

$$d(\tilde{A}, \tilde{0}_1) = \frac{1}{2} \int_{0}^{1} (A_L(\alpha) + A_U(\alpha)) d\alpha.$$

So this equation is

$$a(\tilde{A}, \tilde{0}_1) = \frac{1}{4} \int_{0}^{1} [A_L(\alpha) + A_R(\alpha)] d\alpha = \frac{1}{4} (2b + a + c).$$
Appendix 2. Derivation of equation (11)

From equation (9), it follows that:

\[ \frac{d(\tilde{D}, \tilde{0}_1)}{Q^2} \left( A + \frac{S}{m} \right) = \frac{r}{2} \left[ \left( m \left( 1 - \frac{d(\tilde{D}, \tilde{0}_1)}{p} \right) - 1 + \frac{2d(\tilde{D}, \tilde{0}_1)}{p} \right) C_V + C_P \right]. \]

Solving for \( Q \), we have

\[ Q^2 = \frac{2d(\tilde{D}, \tilde{0}_1) \left( A + \frac{S}{m} \right)}{r \left[ \left( m \left( 1 - \frac{d(\tilde{D}, \tilde{0}_1)}{p} \right) - 1 + \frac{2d(\tilde{D}, \tilde{0}_1)}{p} \right) C_V + C_P \right]}, \]

\[ Q = \left[ \frac{2d(\tilde{D}, \tilde{0}_1) \left( A + \frac{S}{m} \right)}{r \left[ C_V \left( m \left( 1 - \frac{d(\tilde{D}, \tilde{0}_1)}{p} \right) - 1 + \frac{2d(\tilde{D}, \tilde{0}_1)}{p} \right) + C_P \right]} \right]^{1/2}. \]

Appendix 3. Derivation of equation (12)

For a particular value of \( m \), the joint total expected annual cost in fuzzy sense is described by:

\[ W(m) = \left[ 2rd(\tilde{D}, \tilde{0}_1) \left( A + \frac{S}{m} \right) \left( C_V \left( m \left( 1 - \frac{d(\tilde{D}, \tilde{0}_1)}{p} \right) - 1 + \frac{2d(\tilde{D}, \tilde{0}_1)}{p} \right) + C_P \right) \right]^{1/2} + rC_P k \sigma \sqrt{L}. \]

We can ignore the terms that are independent of \( m \) and take the square of \( W(m) \). Then, minimizing \( W(m) \) is equivalent to minimizing \( (W(m))^2 \)

\[ = 2rd(\tilde{D}, \tilde{0}_1) \left[ A \left( C_P - \left( 1 - \frac{2d(\tilde{D}, \tilde{0}_1)}{p} \right) C_V \right) + SC_V \left( 1 - \frac{d(\tilde{D}, \tilde{0}_1)}{p} \right) \right] + mAC_V \left( 1 - \frac{d(\tilde{D}, \tilde{0}_1)}{p} \right) + S \left( C_P - \left( 1 - \frac{2d(\tilde{D}, \tilde{0}_1)}{p} \right) C_V \right]. \]

Once again, while ignoring the terms that are independent of \( m \), the minimization of the problem can be reduced to the minimization of

\[ Z(m) = mAC_V \left( 1 - \frac{d(\tilde{D}, \tilde{0}_1)}{p} \right) + S \left( C_P - \left( 1 - \frac{2d(\tilde{D}, \tilde{0}_1)}{p} \right) C_V \right]. \]
The optimal value of \( m = m^* \) is obtained when \( Z(m^*) \leq Z(m^* - 1) \) and \( Z(m^*) \leq Z(m^* + 1) \). On substituting relevant values inequality, we get:

\[
mC_V A \left(1 - \frac{d(\tilde{D}, \tilde{O}_1)}{p}\right) + \frac{s}{m} \left(C_P - \left(1 - \frac{2d(\tilde{D}, \tilde{O}_1)}{p}\right)C_V\right) \\
\leq (m - 1)AC_V \left(1 - \frac{d(\tilde{D}, \tilde{O}_1)}{p}\right) + \frac{s}{m - 1} \left(C_P - \left(1 - \frac{2d(\tilde{D}, \tilde{O}_1)}{p}\right)C_V\right)
\]

and

\[
mC_V A \left(1 - \frac{d(\tilde{D}, \tilde{O}_1)}{p}\right) + \frac{s}{m} \left(C_P - \left(1 - \frac{2d(\tilde{D}, \tilde{O}_1)}{p}\right)C_V\right) \\
\leq (m + 1)AC_V \left(1 - \frac{d(\tilde{D}, \tilde{O}_1)}{p}\right) + \frac{s}{m + 1} \left(C_P - \left(1 - \frac{2d(\tilde{D}, \tilde{O}_1)}{p}\right)C_V\right).
\]

Then,

\[
AC_V \left(1 - \frac{d(\tilde{D}, \tilde{O}_1)}{p}\right) \leq \left(\frac{1}{m - 1} - \frac{1}{m}\right)S \left(C_P - \left(1 - \frac{2d(\tilde{D}, \tilde{O}_1)}{p}\right)C_V\right)
\]

and

\[
AC_V \left(1 - \frac{d(\tilde{D}, \tilde{O}_1)}{p}\right) \leq \left(\frac{1}{m} - \frac{1}{m + 1}\right)S \left(C_P - \left(1 - \frac{2d(\tilde{D}, \tilde{O}_1)}{p}\right)C_V\right).
\]

Accordingly, it follows that

\[
AC_V \left(1 - \frac{d(\tilde{D}, \tilde{O}_1)}{p}\right) \leq \frac{1}{m(m - 1)}S \left(C_P - \left(1 - \frac{2d(\tilde{D}, \tilde{O}_1)}{p}\right)C_V\right)
\]

and

\[
AC_V \left(1 - \frac{d(\tilde{D}, \tilde{O}_1)}{p}\right) \leq \frac{1}{m(m + 1)}S \left(C_P - \left(1 - \frac{2d(\tilde{D}, \tilde{O}_1)}{p}\right)C_V\right).
\]

Equivalently,

\[
m^*(m^* - 1) \leq \frac{S \left[C_P - \left(1 - \frac{2d(\tilde{D}, \tilde{O}_1)}{p}\right)C_V\right]}{AC_V \left(1 - \frac{d(\tilde{D}, \tilde{O}_1)}{p}\right)}
\]

and

\[
m^*(m^* + 1) \geq \frac{S \left[C_P - \left(1 - \frac{2d(\tilde{D}, \tilde{O}_1)}{p}\right)C_V\right]}{AC_V \left(1 - \frac{d(\tilde{D}, \tilde{O}_1)}{p}\right)}.
\]
Finally, it is concluded that
\[
m^* (m^* - 1) \leq \frac{S \left[ C_P - \left( 1 - \frac{2d(D, \tilde{D}_1)}{p} \right) C_V \right]}{AC_V \left( 1 - \frac{d(D, \tilde{D}_1)}{p} \right)} \leq m^* (m^* + 1).
\]

References


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