Measuring process capability index $C_{pk}$ with fuzzy data

Chang-Chun Tsai

Department of Industrial and Information Management
National Cheng Kung University
1 Ta Hsueh Road, Tainan
Taiwan
R.O.C.

Cheng-Che Chen

Department of Industrial and Business
Far East University
49 Chung Hua Road, Hsin Shi
Taninan, Taiwan
R.O.C.

Abstract

Process capability index $C_{pk}$ is important in manufacturing industry. This paper extends its applications to calculate the process capability index $\tilde{C}_{pk}$ of fuzzy numbers. Unlike previous researches, the $\alpha$-cuts of fuzzy observations are first derived based on various values of $\alpha$. The membership function of fuzzy process capability index $\tilde{C}_{pk}$ is then constructed based on the $\alpha$-cuts of fuzzy observations. Two examples are used to illustrate how to interpret the fuzzy process capability index $\tilde{C}_{pk}$. When the quality characteristic cannot be precisely determined, the proposed method not only provides the most possible value and spread of fuzzy process capability index $\tilde{C}_{pk}$, but also can be easily applied to the fuzzy number with different types of membership functions. With crisp data, the proposed method reduces to the classical method of process capability index $C_{pk}$.

Keywords : Process, index, fuzzy number.

*E-mail: chengche@cc.fec.edu.tw

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1. Introduction

Process capability analysis has proven in practice to be a very valuable engineering decision tool. Managers are often faced with decision problems such as whether to reject or accept a batch of process output in production. Process capability indices $C_p$ and $C_{pk}$ have been used in the manufacturing industry to not only provide numerical measures on process potential and performance, but also quantify the relationship between the actual process performance and the specification limits. The two indices are defined as [9]

$$C_p = \frac{USL - LSL}{6\sigma} \quad (1)$$

$$C_{pk} = (1 - K) \times C_p \quad (2)$$

where $K = \frac{2|\mu - M|}{USL - LSL}$, USL is the upper specification limit, LSL is the lower specification limit, $\mu$ is the process mean, $\sigma$ is the process standard deviation, $M = \frac{USL + LSL}{2}$ is the midpoint of the specification interval. The index $C_p$ was used to measure the magnitude of the process variation. Assume the target value $T = M$, for simplicity, then $K$ is a capability index designed to quantify deviation from the specification target. The index $C_{pk}$ was used to measure the capability of accuracy and the capability of precision concerning the process. Sample data must be collected in order to calculate these indices. The estimators of $C_p$ and $C_{pk}$ can be calculated as the following, where $\bar{X} = \frac{1}{n} \sum_{i=1}^{n} X_i$ and $S = \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (X_i - \bar{X})^2}$ are estimators of $\mu$ and $\sigma$:

$$\hat{C}_p = \frac{USL - LSL}{6S} \quad (3)$$

$$\hat{C}_{pk} = (1 - \hat{K}) \times \hat{C}_p \quad (4)$$

where $\hat{K} = \frac{2|\bar{X} - M|}{USL - LSL}$. The index $C_p$ only measures potential capability as defined by the actual process spread and does not consider the mean of the process. In order to capture the actual process performance, this paper considers the index $C_{pk}$ because it involves the information of location and dispersion of observations simultaneously.
Within the context of the process capability indices, observations are crisp. However, in real world applications, the observations usually contain fuzziness owing imprecise measurements or described by linguistic variables, such as “about 7”, “somewhere between 6 and 8” and so on. Therefore, measuring the process capability indices is a larger challenge when observations are fuzzy. Yongting [2] defined a formula of process capability index to measure fuzzy quality. In his definition, the construction of membership function of $\hat{C}_{pk}$, which will affect the final decision, is not discussed. Besides, the fuzzy process capability index $\hat{C}_{pk}$ lies between 0 and 1, which is different from the conventional range of $[-\infty, \infty]$. The resulting fuzzy framework in his approach was not extension of the original crisp problem. Lee [4] conducted the membership function of mean and standard deviation of fuzzy numbers. The $C_{pk}$ index estimation presented by fuzzy number is constructed and its membership function is approximated. The characteristics of his approach is not only complicated concerning the construction of membership function of the index $\hat{C}_{pk}$, but also, as mentioned in his paper, the complexity of constructing the membership function of the type other than triangular is much more difficult.

Intuitively, when observations are fuzzy, the process capability index $C_{pk}$ should be fuzzy as well. Therefore, derive a crisp $C_{pk}$ using fuzzy observations, contradicting to human intuition. In this paper, the fuzzy analytic method concerning process capability index $C_{pk}$ is presented and used to calculate $\hat{C}_{pk}$ for fuzzy observations. The index is more realistically calculated as a fuzzy number.

The rest of this paper is organized as follows. Firstly, the importance of process capability index $C_{pk}$ is introduced. Secondly, the construction of membership function of the fuzzy process capability index $\hat{C}_{pk}$ is developed. Finally, two numerical examples are presented for illustration and conclusions are drawn.

2. The basic concept of process capability index $C_{pk}$

Process capability index $C_{pk}$ can not only measure the robustness and consistency of the process, but also estimate the proportion of product within specification which a particular in-control process should be expected to produce. From Eq. (4), the relationship between $\hat{K}$, $\hat{C}_p$ and $\hat{C}_{pk}$ can be discussed as follows. If $\hat{K}$ is small and $\hat{C}_p$ is large then $\hat{C}_{pk}$
is large which guarantees that the process capability is excellent and the product specifications are met. If $\hat{K}$ is small and $\hat{C}_p$ is small which means that there is high consistency between process mean and midpoint of specification interval; however, there could be nonconformities because the process variance cannot satisfy the product quality standard (specification tolerance). If $\hat{K}$ is large and $\hat{C}_p$ is large which means that the process variance can satisfy the product quality standard (specification tolerance); however, process mean and midpoint of specification interval are inconsistency that would cause high defect rate. If $\hat{K}$ is large and $\hat{C}_p$ is small then $\hat{C}_{pk}$ tends to be too small which means that not only process mean and midpoint of specification interval are inconsistency but also the process variance cannot satisfy the product quality standard (specification tolerance). If $\hat{K} = 0$, then $\hat{C}_{pk} = \hat{C}_p$.

When process is in-control and is normally distributed, $\hat{C}_{pk}$ index is related to the process capability. Table 1 lists various values of $\hat{C}_{pk}$, the corresponding $Z$ values, and the fractions of nonconformity (defect rate) in parts per million (ppm). A process is called “super” if $\hat{C}_{pk} \geq 2.00$, corresponding defect rate is only 0.002 ppm, shows the process capability is super excellent. A process is called “excellent” if $1.67 \leq \hat{C}_{pk} \leq 2.00$, the corresponding defect rate is only 0.54 ppm which is relative low. A process is called “satisfactory” if $1.33 \leq \hat{C}_{pk} \leq 1.67$: this indicates that process capability is satisfactory; sufficient to inspect at start of operations; can consider speeding up process or otherwise increasing load. A process is called “capable” if $1.00 \leq \hat{C}_{pk} \leq 1.33$: this shows that danger of producing defects, needs watching. A process is called “inadequate” if $0.67 \leq \hat{C}_{pk} \leq 1.00$: this shows that the process is not adequate with respect to the production tolerances. A process is called “poor” if $\hat{C}_{pk} \leq 0.67$: this shows that process capability is poor and the production line should be stop produce immediately; need to consider changing procedures, changing equipment and changing tolerance. Table 2 summarizes the six quality levels and the corresponding $\hat{C}_{pk}$ values.

However, in reality, observations are usually fuzzy. Suppose that the observations $\tilde{X}_i$’s are approximately known and can be represented by fuzzy numbers. Based on extension principle $[3, 5, 8]$, the fuzzy process capability index $\hat{C}_{pk}$ is also fuzzy. Deriving the exact membership function of $\hat{C}_{pk}$ is critical because the spread of the membership function plays an
important role in expressing the characteristics of a fuzzy number. Besides, the membership function can also provide the most possible value as well as the smallest and largest values of the fuzzy process capability index $\hat{C}_{pk}$.

### Table 1

<table>
<thead>
<tr>
<th>$\hat{C}_{pk}$</th>
<th>0.67</th>
<th>1.00</th>
<th>1.33</th>
<th>1.67</th>
<th>2</th>
</tr>
</thead>
<tbody>
<tr>
<td>$Z$</td>
<td>2</td>
<td>3</td>
<td>4</td>
<td>5</td>
<td>6</td>
</tr>
<tr>
<td>ppm</td>
<td>45500</td>
<td>2700</td>
<td>66</td>
<td>0.54</td>
<td>0.002</td>
</tr>
</tbody>
</table>

### Table 2

<table>
<thead>
<tr>
<th>Quality levels</th>
<th>$\hat{C}_{pk}$ values</th>
</tr>
</thead>
<tbody>
<tr>
<td>Super</td>
<td>$\hat{C}_{pk} \geq 2.00$</td>
</tr>
<tr>
<td>Excellent</td>
<td>$1.67 \leq \hat{C}_{pk} \leq 2.00$</td>
</tr>
<tr>
<td>Satisfactory</td>
<td>$1.33 \leq \hat{C}_{pk} \leq 1.67$</td>
</tr>
<tr>
<td>Capable</td>
<td>$1.00 \leq \hat{C}_{pk} \leq 1.33$</td>
</tr>
<tr>
<td>Inadequate</td>
<td>$0.67 \leq \hat{C}_{pk} \leq 1.00$</td>
</tr>
<tr>
<td>Poor</td>
<td>$\hat{C}_{pk} \leq 0.67$</td>
</tr>
</tbody>
</table>

3. **Membership function of fuzzy process capability index $\hat{C}_{pk}$**

At the present time, the $\hat{C}_{pk}$ index is used more than any other index for measuring process capability. Eq. (4) shows that $\hat{C}_{pk}$ index is the multiplication of the accuracy capability and the precision capability of a process. Besides, these two capacities are calculated from the sample mean and sample standard deviation. Nevertheless, $\hat{C}_{pk}$ index applied only the case where the observations are crisp. This is not suitable for the case when the observations are fuzzy. In order to overcome this problem, fuzzy set theory [6] is applied to improve the fuzzy situation.

When the observations are measured imprecisely, then the observations can be treated as fuzzy numbers. Without loss of generality, assume all observations are fuzzy numbers, since crisp values can be treated as degenerated fuzzy numbers. For simplicity, assume that all
the observations are triangular fuzzy numbers defined as
\[ \tilde{X}_i = [X_{il}, X_{im}, X_{iu}] \quad \text{for} \quad i = 1, 2, \ldots, n. \] (5)

The sample mean \( \bar{X} \) and sample standard deviation \( S \) are substituted by natural fuzzy statistics \( \tilde{\bar{X}} \) and \( \tilde{S} \) respectively, which are defined as
\[ \tilde{\bar{X}} \approx \frac{1}{n} \sum_{i=1}^{n} \tilde{X}_i \quad \text{and} \quad \tilde{S} \approx \sqrt{\frac{1}{n-1} \sum_{i=1}^{n} (\tilde{X}_i - \tilde{\bar{X}})^2}. \] Eq. (4) for calculating the index \( \tilde{C}_{pk} \) becomes
\[ \tilde{C}_{pk} \approx (1 - \tilde{K}) \times \tilde{C}_p \] (6)

where \( \tilde{K} \approx \frac{2|\tilde{X} - M|}{\text{USL} - \text{LSL}} \) and \( \tilde{C}_p \approx \frac{\text{USL} - \text{LSL}}{6\tilde{S}} \). Eq. (6) is also a fuzzy number based on extension principle \([3, 5, 8]\). From Eq. (6), the function relationship between the index \( \tilde{C}_{pk} \) and the fuzzy observations \( \tilde{X}_i \)'s is nonlinear. Deducing the exact membership function \( \mu_{\tilde{C}_{pk}} \) is nearly impossible since Eq. (6) includes quadratic terms of the fuzzy observations. The approximate membership function \( \mu_{\tilde{C}_{pk}} \) of the index \( \tilde{C}_{pk} \) can be obtained by Kao and Liu \([1]\).

Based on Kao and Liu \([1]\), the \( \alpha \)-cuts of the fuzzy observation \( \tilde{X}_i \) are presented as
\[ (X_i)_\alpha = [(X_i)_L^L, (X_i)_R^U] = [\min_{x_i} \{ x_i \in X \mid \mu_{\tilde{X}_i}(x_i) \geq \alpha \}, \max_{x_i} \{ x_i \in X \mid \mu_{\tilde{X}_i}(x_i) \geq \alpha \}] \] (7)

where \( X \) is the crisp universal set on which \( \tilde{X}_i \) is defined and \( \mu_{\tilde{X}_i} \) is the membership function of fuzzy observation \( \tilde{X}_i \). To find the membership function \( \mu_{\tilde{C}_{pk}} \), it suffices to find the lower and upper bounds of the \( \alpha \)-cuts of \( \tilde{C}_{pk} \), which can be calculated via the following functions:
\[ (\tilde{C}_{pk})^L_\alpha = \min_{(X_i)_L^L \leq X_i \leq (X_i)_R^U} [(1 - \tilde{K}) \times \tilde{C}_p] \] (8a)
\[ (\tilde{C}_{pk})^U_\alpha = \min_{(X_i)_L^L \leq X_i \leq (X_i)_R^U} [(1 - \tilde{K}) \times \tilde{C}_p] \] (8b)

then \( (\tilde{C}_{pk})_\alpha = [(\tilde{C}_{pk})^L_\alpha, (\tilde{C}_{pk})^U_\alpha] \) is the \( \alpha \)-cuts of \( \tilde{C}_{pk} \).

\( (\tilde{C}_{pk})_\alpha \) is a pair of nonlinear function with bounded constraints. The nonlinear programming package, LINGO \([7]\), can be used to calculate \( (\tilde{C}_{pk})_\alpha \). The objective function in Eqs. (8a) and (8b) is the classical formula
for the process capability index $\tilde{C}_{pk}$. Consequently, the properties of the index for crisp values also apply here. When all observations are degenerated to crisp values, Eqs. (8a) and (8b) become identical and boil down to Eq. (4) of the classical model. For two possibility levels, $\alpha_1$ and $\alpha_2$ such that $0 \leq \alpha_1 \leq \alpha_2$, the feasible region defined by $\alpha_2$ in Eqs. (8a) and (8b), is smaller than that of $\alpha_1$. Consequently, the $\alpha$-cut of $\tilde{C}_{pk}$ at $\alpha_2$ is contained in that of $\alpha_1$. Obviously, the membership function, $\mu_{\tilde{C}_{pk}}$, is convex. From various values of $\alpha$, the membership function, $\mu_{\tilde{C}_{pk}}$, is constructed as

$$
\mu_{\tilde{C}_{pk}}(\tilde{C}_{pk}) = \begin{cases} 
L(\hat{\tilde{C}}_{pk}), & (\hat{\tilde{C}}_{pk})_0^L \leq \tilde{C}_{pk} \leq (\hat{\tilde{C}}_{pk})_1^L \\
1, & (\hat{\tilde{C}}_{pk})_1^L \leq \tilde{C}_{pk} \leq (\hat{\tilde{C}}_{pk})_1^U \\
R(\hat{\tilde{C}}_{pk}), & (\hat{\tilde{C}}_{pk})_1^U \leq \tilde{C}_{pk} \leq (\hat{\tilde{C}}_{pk})_0^U 
\end{cases}
$$

(9)

where $L(\hat{\tilde{C}}_{pk})$ and $R(\hat{\tilde{C}}_{pk})$ are the left and right shape functions of $\mu_{\tilde{C}_{pk}}$, respectively. Figure 1 shows that membership function of the fuzzy process capability index $\tilde{C}_{pk}$.

![Figure 1](image_url)

**Figure 1**
Membership function of $\tilde{C}_{pk}$

4. **Numerical examples**

Two examples are provided in this section to illustrate the performance of the proposed approach for fuzzy data with two different types of fuzzy numbers.
Example 1. Suppose that we collected some fuzzy sample data from the factory. The production specification for a particular product are the following: USL = 6.4, LSL = 5.5, M = 5.95. The quality level was defined as “at least capable” (Ĉpk ≥ 1.00). A total of 30 fuzzy observations were collected which are shown in Table 3. Considering the space of the printed pages, Table 4 only lists the first five α-cuts of fuzzy observations X_i’s at 11 various α-values: α = 0, 0.1, . . . , 1.0. At each of these α-values, the lower and upper bounds of the α-cuts of Ĉpk can be solved from Eqs. (8a) and (8b). LINGO [7] is used to solve the nonlinear functions. Table 5 lists the lower and upper bounds of α-cuts of Ĉpk at 11 various α-values: α = 0, 0.1, . . . , 1.0. The membership function, μĈpk, is constructed as

\[
\mu_{Ĉpk}(Ĉpk) = \begin{cases} 
L(Ĉpk), & 0.3932 \leq Ĉpk \leq 0.6942 \\
R(Ĉpk), & 0.6942 \leq Ĉpk \leq 1.2323.
\end{cases}
\]

The L(Ĉpk) and R(Ĉpk) are depicted by ((Ĉpk)_L, α) and ((Ĉpk)_U, α), at the various α-values, respectively. The associated membership function is depicted in Figure 2.

![Figure 2](image_url)

**Figure 2**
Membership function of Ĉpk for Example 1

The fuzzy process capability index Ĉpk in this example has several characteristics to be noted. First, the support of Ĉpk is from 0.3932 to 1.2323. This range shows that, although the fuzzy process capability index Ĉpk is fuzzy, it is impossible for its value to fall below 0.3932 or exceed...
### Table 3
Collected 30 fuzzy observations in Example 1

| \( \tilde{X}_1 \) = [5.85 6.15 6.35] | \( \tilde{X}_{11} \) = [5.86 6.04 6.25] | \( \tilde{X}_{21} \) = [5.50 5.81 5.99] |
| \( \tilde{X}_2 \) = [5.79 5.90 5.98] | \( \tilde{X}_{12} \) = [6.13 6.23 6.33] | \( \tilde{X}_{22} \) = [5.60 5.92 6.05] |
| \( \tilde{X}_3 \) = [5.71 5.83 5.99] | \( \tilde{X}_{13} \) = [5.95 6.05 6.19] | \( \tilde{X}_{23} \) = [5.50 5.75 5.95] |
| \( \tilde{X}_4 \) = [6.05 6.18 6.32] | \( \tilde{X}_{14} \) = [5.60 5.65 5.70] | \( \tilde{X}_{24} \) = [5.84 6.03 6.15] |
| \( \tilde{X}_5 \) = [5.89 6.06 6.23] | \( \tilde{X}_{15} \) = [5.65 5.74 5.84] | \( \tilde{X}_{25} \) = [6.05 6.30 6.50] |
| \( \tilde{X}_6 \) = [6.01 6.10 6.25] | \( \tilde{X}_{16} \) = [5.70 5.77 5.83] | \( \tilde{X}_{26} \) = [6.25 6.35 6.45] |
| \( \tilde{X}_7 \) = [6.15 6.20 6.30] | \( \tilde{X}_{17} \) = [6.23 6.32 6.40] | \( \tilde{X}_{27} \) = [5.65 5.86 6.05] |
| \( \tilde{X}_8 \) = [5.64 5.81 6.05] | \( \tilde{X}_{18} \) = [5.60 5.70 5.80] | \( \tilde{X}_{28} \) = [5.70 5.87 5.95] |
| \( \tilde{X}_9 \) = [5.80 5.90 5.98] | \( \tilde{X}_{19} \) = [5.85 5.95 6.05] | \( \tilde{X}_{29} \) = [5.75 5.95 6.15] |
| \( \tilde{X}_{10} \) = [6.01 6.12 6.24] | \( \tilde{X}_{20} \) = [5.90 6.00 6.10] | \( \tilde{X}_{30} \) = [6.10 6.23 6.46] |
Table 4
The first five $\alpha$-cuts of $\tilde{X}_i$’s at 11 various $\alpha$-values in Example 1

<table>
<thead>
<tr>
<th></th>
<th>$\alpha$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(X_1)^L_\alpha$</td>
<td>5.85</td>
<td>5.88</td>
<td>5.91</td>
<td>5.94</td>
<td>5.97</td>
<td>6.00</td>
<td>6.03</td>
<td>6.06</td>
<td>6.09</td>
<td>6.12</td>
<td>6.15</td>
<td></td>
</tr>
<tr>
<td>$(X_2)^L_\alpha$</td>
<td>5.79</td>
<td>5.80</td>
<td>5.81</td>
<td>5.82</td>
<td>5.83</td>
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<td>5.86</td>
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<td>5.90</td>
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</tr>
<tr>
<td>$(X_2)^U_\alpha$</td>
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<td>5.97</td>
<td>5.96</td>
<td>5.96</td>
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<tr>
<td>$(X_4)^L_\alpha$</td>
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<td>6.15</td>
<td>6.13</td>
<td>6.11</td>
<td>6.09</td>
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</table>
### Table 5

$\alpha$-cuts of $\hat{C}_{pk}$ at 11 various $\alpha$-values in Example 1

<table>
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<tr>
<th>$\alpha$</th>
<th>0</th>
<th>0.1</th>
<th>0.2</th>
<th>0.3</th>
<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\hat{C}<em>{pk})^L</em>\alpha$</td>
<td>0.3932</td>
<td>0.4025</td>
<td>0.4310</td>
<td>0.4592</td>
<td>0.4888</td>
<td>0.5090</td>
<td>0.5411</td>
<td>0.5734</td>
<td>0.6100</td>
<td>0.6506</td>
<td>0.6942</td>
</tr>
<tr>
<td>$(\hat{C}<em>{pk})^U</em>\alpha$</td>
<td>1.2323</td>
<td>1.1688</td>
<td>1.1085</td>
<td>1.0515</td>
<td>0.9960</td>
<td>0.9413</td>
<td>0.8876</td>
<td>0.8355</td>
<td>0.7858</td>
<td>0.7387</td>
<td>0.6942</td>
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</tbody>
</table>

### Table 6

$\alpha$-cuts of $\hat{C}_{pk}$ at 11 various $\alpha$-values in Example 2

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<th>0.1</th>
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<th>0.4</th>
<th>0.5</th>
<th>0.6</th>
<th>0.7</th>
<th>0.8</th>
<th>0.9</th>
<th>1.0</th>
</tr>
</thead>
<tbody>
<tr>
<td>$(\hat{C}<em>{pk})^L</em>\alpha$</td>
<td>0.1863</td>
<td>0.1975</td>
<td>0.2096</td>
<td>0.2229</td>
<td>0.2378</td>
<td>0.2545</td>
<td>0.2739</td>
<td>0.2971</td>
<td>0.3255</td>
<td>0.3605</td>
<td>0.4734</td>
</tr>
<tr>
<td>$(\hat{C}<em>{pk})^U</em>\alpha$</td>
<td>1.0541</td>
<td>1.0136</td>
<td>0.9715</td>
<td>0.9281</td>
<td>0.8834</td>
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<td>0.7902</td>
<td>0.7410</td>
<td>0.6884</td>
<td>0.6233</td>
<td>0.4734</td>
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</tbody>
</table>
1.2323. Second, the $\alpha$-cut at $\alpha = 1$, which contains only one value 0.6942, is the most possible value for the process capability index. Finally, despite the fact that the fuzzy observations are triangular, the membership function of $\tilde{C}_{pk}$ is no longer triangular. The information shows it is difficult for this factory to have quality level reach “at least capable” ($\hat{C}_{pk} \geq 1.00$) since the possibility is only 39.3% to have $\hat{C}_{pk}$ reach to one.

**Example 2.** An example, that includes six fuzzy data points, is taken to show that the method can be applied to other types of fuzzy observations. Suppose these data are modeled by fuzzy numbers with following membership functions:

\[
\mu_{X_1}(x) = \begin{cases} 
1 - (x - 4)^2, & 3 \leq x \leq 5 \\
0, & \text{otherwise},
\end{cases}
\]

\[
\mu_{X_2}(x) = \begin{cases} 
1 - (x - 5)^2, & 4 \leq x \leq 6 \\
0, & \text{otherwise},
\end{cases}
\]

\[
\mu_{X_3}(x) = \begin{cases} 
1 - \frac{1}{2.25}(x - 5.5)^2, & 4 \leq x \leq 7 \\
0, & \text{otherwise},
\end{cases}
\]

\[
\mu_{X_4}(x) = \begin{cases} 
1 - 4(x - 3.5)^2, & 3 \leq x \leq 4 \\
0, & \text{otherwise},
\end{cases}
\]

\[
\mu_{X_5}(x) = \begin{cases} 
1 - (x - 6)^2, & 5 \leq x \leq 7 \\
0, & \text{otherwise},
\end{cases}
\]

\[
\mu_{X_6}(x) = \begin{cases} 
1 - (x - 7)^2, & 6 \leq x \leq 8 \\
0, & \text{otherwise}.
\end{cases}
\]

Suppose the production specification of interest is USL = 7, LSL = 3, and $M = 5$. In this example, 11 distinct $\alpha$-values: $\alpha = 0, 0.1, \ldots, 1.0$ are chosen. At each of those possibility levels, the $\alpha$-cuts of the fuzzy process capability index $\tilde{C}_{pk}$ are solved from Eqs. (8a) and (8b). LINGO [7] is again used to solve the associated nonlinear functions. Table 6 specifies the 11 $\alpha$-cuts of $\tilde{C}_{pk}$.

Figure 3 shows the membership function of $\tilde{C}_{pk}$ constructed from the 11 $\alpha$-cuts. When the fuzzy numbers are semi-circular, the membership
function of $\tilde{C}_{pk}$ is near semi-circular. The fuzzy process capability index $\tilde{C}_{pk}$ is bounded by 0.1863 and 1.0541. The most possible value of $\tilde{C}_{pk}$ is 0.4734. Though most of these six fuzzy data are conformity to the specifications, the $\tilde{C}_{pk}$ value is pretty low which indicates the process performance is not good.

![Membership function of $\tilde{C}_{pk}$ for Example 2](image)

**Figure 3**
Membership function of $\tilde{C}_{pk}$ for Example 2

5. Conclusions

In this paper, we proposes an analytic method to derive fuzzy measures according to the classical definition of process capability index $C_{pk}$. With crisp data, the proposed method reduces to the classical formula for calculating the process capability index $C_{pk}$. To derive exact membership function is important because the membership function can display the characteristics of a fuzzy number and affect the decision-making. The proposed method can construct the membership function of $\tilde{C}_{pk}$ not only for triangular fuzzy numbers, but also for other types of fuzzy numbers. Two examples show how to interpret the fuzzy measures for process capability index $\tilde{C}_{pk}$. In practice, many observations are fuzzy. In such a situation, the process capability index expressed in fuzzy numbers is more informative than in crisp numbers. When quality characteristics can not be precisely determined, the proposed method in this paper is appropriate for decision making.
References


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