A new Gronwall-Bellman inequality for discontinuous functions

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Abstract
This article concerns new integral inequalities of Gronwall-Bellman type for discontinuous functions that generalize results obtained for difference inequalities.

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1. Introduction
The numerous linear and nonlinear generalizations of Gronwall-Bellman-Bihari inequalities for continuous and discontinuous functions have been very important in investigations qualitative characteristics: existence, uniqueness, boundedness, stability solutions of systems of ordinary differential equations with different kinds of perturbations.

In this paper are established new linear integral inequalities for discontinuous functions (“integro-sum inequalities”). This term firstly used in [3], and later in [4], [5], [2].

Section 2 is devoted to some preliminary considerations; Section 3 concerns generalization of results in [1] for discontinuous functions.

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2. Preliminary considerations

Consider the integro-sum inequality of the form
\[
    u(t) \leq p(t) + q(t) \left[ \sum_{t_0 < t_i < t} \beta_i u(t_i - 0) + \int_{t_0}^{t} u(\tau)v(\tau)d\tau \right],
\] (1)
where:
- \(u(t)\) is a piecewise-continuous non-negative function with 1-st kind discontinuities in points \(t_i, t_1 < t_2 < \ldots, \lim_{i \to \infty} t_i = \infty\);
- \(p(t), q(t)\) are positive functions at \(t \geq t_0\);
- \(\beta_i > 0, \forall \ i = 1, 2, \ldots\).

Defined the function
\[
    r(t) = \sum_{t_0 < t_i < t} \beta_i u(t_i - 0) + \int_{t_0}^{t} u(\tau)v(\tau)d\tau,
\] (2)
the previous inequality turns
\[
    u(t) \leq p(t) + q(t)r(t).
\] (3)

In what follows it’s studied the behaviour of function \(r(t)\) to obtain, through (3), an estimate for the function \(u(t)\) of inequality (1).

First of all observe that if \(t \in [t_0, t_1]\), using (2) and (3), it can write
\[
    r(t) = \int_{t_0}^{t} u(\tau)v(\tau)d\tau
    \leq \int_{t_0}^{t} v(\tau)[p(\tau) + q(\tau)r(\tau)]d\tau
    = \int_{t_0}^{t} v(\tau)p(\tau)d\tau + \int_{t_0}^{t} v(\tau)q(\tau)r(\tau)d\tau.
\]
Denote
\[
    \psi(t) = \int_{t_0}^{t} v(\tau)p(\tau)d\tau;
\]
it’s obvious that \(\psi(t)\) is non-decreasing function. By use of classical Bellman result, we obtain such estimate:
\[
    r(t) \leq \left( \int_{t_0}^{t} v(\tau)p(\tau)d\tau \right) \exp \left( \int_{t_0}^{t} v(\tau)q(\tau)d\tau \right).
\] (4)
From this and (3) it’s concluded that
\[
    u(t) \leq p(t) + q(t) \left( \int_{t_0}^{t} v(\tau)p(\tau)d\tau \right) \exp \left( \int_{t_0}^{t} v(\tau)q(\tau)d\tau \right).
\] (5)
Now consider \( t \in [t_k, t_{k+1}] \) \((k \geq 1)\) and introduce the function
\[
\varphi(t) = \sum_{t_0 < t_i < t} \beta_i p(t_i) + \beta_i q(t_i) r(t_i - 0) \\
+ \int_{t_0}^{t} v(\tau) [p(\tau) + q(\tau) r(\tau)] d\tau.
\]

Using (3) it’s possible to compare functions \( r(t) \) and \( \varphi(t) \) and deduce that
\[
r(t) \leq \varphi(t).
\]

Moreover if derive (6) with respect to \( t \) obtain
\[
\frac{d\varphi(t)}{dt} = v(t) p(t) + v(t) q(t) r(t) \\
\leq v(t) p(t) + v(t) q(t) \varphi(t)
\]
therefore, multiplying both members of this inequality by \( \exp \left( -\int_{t_0}^{t} v(\tau) q(\tau) d\tau \right) \) and integrating within \( t_0 \) and \( t \), it achieves to
\[
\varphi(t) \leq \left[ \psi_k + \int_{t_0}^{t} v(\tau) p(\tau) d\tau \right] \exp \left( \int_{t_0}^{t} v(\tau) q(\tau) d\tau \right),
\]
with
\[
\psi_k = \varphi(t_0) = \sum_{t_0 < t_i < t} \left[ \beta_i p(t_i) + \beta_i q(t_i) r(t_i - 0) \right].
\]

Since (8) is fulfilled whatever the interval \([t_k, t_{k+1}]\) may be and (9) depends on \( k \), for the aim of this article it’s enough to verify how it changes \( \psi_k \) with respect to \( k \).

3. Main result

By using the inductive method now it’s proved that in interval \([t_k, t_{k+1}]\) the initial condition \( \psi_k \) fulfills inequality
\[
\psi_k \leq \sum_{j=1}^{k} \left[ \beta_j p(t_j) + \beta_j q(t_j) \left( \int_{t_0}^{t_j} v(\tau) p(\tau) d\tau \right) \right] \\
\times \exp \left( \int_{t_0}^{t_j} v(\tau) q(\tau) d\tau \right) \\
\times \prod_{j=1}^{k} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau) q(\tau) d\tau \right) \right],
\]
\(\forall k \geq 1.\)
Consider \( t \in [t_1, t_2] \); from (9) and (4) it’s deduced immediately that

\[
\psi_1 \leq \beta_1 p(t_1) + \beta_1 q(t_1) \left( \int_{t_0}^{t_1} v(\tau)p(\tau)d\tau \right) \exp \left( \int_{t_0}^{t_1} v(\tau)q(\tau)d\tau \right)
\]

that is the inequality (10) for the case \( k = 1 \).

Then suppose that (10) holds in \([t_{k-1}, t_k]\) so that it’s possible to write

\[
\psi_{k-1} = \sum_{i=1}^{k-1} \left[ \beta_i p(t_i) + \beta_i q(t_i) \left( \int_{t_0}^{t_i} v(\tau)p(\tau)d\tau \right) \times \exp \left( \int_{t_0}^{t_i} v(\tau)q(\tau)d\tau \right) \right] \\
\times \prod_{j=1+i}^{k-1} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau)q(\tau)d\tau \right) \right]. \quad (11)
\]

Consider \( t \in [t_k, t_{k+1}] \) and evaluate \( \psi_k \) taking into account (9), (7) and (8):

\[
\psi_k = \psi_{k-1} + \beta_k p(t_k) + \beta_k q(t_k) \left( t_k - t_0 \right) \\
\leq \psi_{k-1} + \beta_k p(t_k) + \beta_k q(t_k) \left( \psi_{k-1} + \int_{t_0}^{t_k} v(\tau)p(\tau)d\tau \right) \\
\times \exp \left( \int_{t_0}^{t_k} v(\tau)q(\tau)d\tau \right) \\
= \beta_k p(t_k) + \left[ 1 + \beta_k q(t_k) \exp \left( \int_{t_0}^{t_k} v(\tau)q(\tau)d\tau \right) \right] \psi_{k-1} \\
\times \beta_k q(t_k) \left( \int_{t_0}^{t_k} v(\tau)p(\tau)d\tau \right) \exp \left( \int_{t_0}^{t_k} v(\tau)q(\tau)d\tau \right).
\]

Using the hypothesis of inductivity (11) the last inequality becomes

\[
\psi_k \leq \sum_{i=1}^{k-1} \left[ \beta_i p(t_i) + \beta_i q(t_i) \left( \int_{t_0}^{t_i} v(\tau)p(\tau)d\tau \right) \right] \\
\times \exp \left( \int_{t_0}^{t_i} v(\tau)q(\tau)d\tau \right) \\
\times \prod_{j=1+i}^{k-1} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau)q(\tau)d\tau \right) \right] \\
\times \left[ 1 + \beta_k q(t_k) \exp \left( \int_{t_0}^{t_k} v(\tau)q(\tau)d\tau \right) \right].
\]
Theorem 1. Let the non-negative piecewise-continuous function \( u(t) \) satisfy inequality (1) for \( t \geq t_0 \), where \( p(t) \), \( q(t) \) are non-negative functions at \( t \geq t_0 \), \( t_i \) the 1-st kind discontinuity points of the function \( u(t) \): \( t_1 < t_2 < \ldots \), \( \lim_{i \to \infty} t_i = \infty \), \( \beta_i > 0 \). Then is fulfilled the estimate

\[
    u(t) \leq p(t) + \sum_{t_0 < t_i < t} \left[ \beta_i p(t_i) + \beta_i q(t_i) \left( \int_{t_0}^{t_i} v(\tau) p(\tau) \, d\tau \right) \exp \left( \int_{t_0}^{t_i} v(\tau) q(\tau) \, d\tau \right) \right]
\]

that is the thesis.

At last (3), (7), (8) and (10) imply the following estimate for the function \( u(t) \) in \( [t_k, t_{k+1}] \):

\[
    u(t) \leq p(t) + q(t) r(t) \\
    \leq p(t) + q(t) \left( \sum_{i=1}^{k} \beta_i p(t_i) + \beta_i q(t_i) \left( \int_{t_0}^{t_i} v(\tau) p(\tau) \, d\tau \right) \right. \\
    \left. \times \exp \left( \int_{t_0}^{t_i} v(\tau) q(\tau) \, d\tau \right) \right) \\
    \times \prod_{j=1}^{k+1} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau) q(\tau) \, d\tau \right) \right] \\
    + \int_{t_0}^{t} v(\tau) p(\tau) \, d\tau \exp \left( \int_{t_0}^{t} v(\tau) q(\tau) \, d\tau \right) . \quad (12)
\]

After all consider the inequality

\[
    \prod_{j=1}^{k+1} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau) q(\tau) \, d\tau \right) \right] \\
    \leq \prod_{j=1}^{k} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau) q(\tau) \, d\tau \right) \right].
\]

In this way it has been proved the following
A consequence of it is the

**Corollary 1.** Let in Theorem 1 the function \( p(t) \) be non-decreasing and \( q(t) \geq 1 \), for \( t \geq t_0 \). Then for function \( u(t) \) holds the estimate

\[
\begin{align*}
  u(t) &\leq p(t)q(t) \left( 1 + \int_{t_0}^{t} v(\tau)q(\tau)d\tau \right) \\
       &\times \prod_{t_0 < t_i < t} \left[ 1 + \beta_i q(t_i) \exp \left( \int_{t_0}^{t_i} v(\tau)q(\tau)d\tau \right) \right].
\end{align*}
\]

(14)

**Proof.** To prove the thesis observe that hypothesis leads to:

\[
q(t) \exp \left( \int_{t_0}^{t} v(\tau)q(\tau)d\tau \right) \geq 1,
\]

(15)

\[
\int_{t_i}^{t} v(\tau)d\tau \leq \int_{t_0}^{t} v(\tau)d\tau; \quad i = 1, 2, \ldots
\]

(16)

and furthermore if \( \tau \leq t_i \leq t \) then

\[
\frac{p(t_i)}{p(t)} \leq 1 \quad (i = 1, 2, \ldots), \quad \frac{1}{p(t)} \leq \frac{1}{p(\tau)}.
\]

From (5) and (7) follows

\[
u(t) \leq p(t) + q(t)r(t) \leq p(t) + q(t)\varphi(t)
\]

so, dividing both members of the inequality by \( p(t) \), it obtains:

\[
\frac{u(t)}{p(t)} \leq 1 + q(t)\frac{\varphi(t)}{p(t)}.
\]

Now consider the term \( \frac{\varphi(t)}{p(t)} \); by virtue of (8) and of considerations made at the begin of this proof results

\[
\frac{\varphi(t)}{p(t)} \leq \left[ \frac{\psi_k}{p(t)} + \int_{t_0}^{t} v(\tau)d\tau \right] \exp \left( \int_{t_0}^{t} v(\tau)q(\tau)d\tau \right)
\]

therefore

\[
\frac{u(t)}{p(t)} \leq 1 + q(t) \left[ \frac{\psi_k}{p(t)} + \int_{t_0}^{t} v(\tau)d\tau \right] \exp \left( \int_{t_0}^{t} v(\tau)q(\tau)d\tau \right)
\]

\[
\leq q(t) \exp \left( \int_{t_0}^{t} v(\tau)q(\tau)d\tau \right) \left[ 1 + \frac{\psi_k}{p(t)} + \int_{t_0}^{t} v(\tau)d\tau \right]
\]

(17)

(the last inequality comes out from starting considerations).
On the other hand (10) provides what follows:

\[
1 + \frac{\psi_k}{p(t)} + \int_{t_0}^{t} v(\tau) d\tau \\
\leq \left( 1 + \int_{t_0}^{t} v(\tau) d\tau \right) + \sum_{i=1}^{k} \beta_i \left[ 1 + q(t_i) \left( \int_{t_0}^{t_i} v(\tau) d\tau \right) \right] \\
\times \exp \left( \int_{t_0}^{t_i} v(\tau) q(\tau) d\tau \right) \prod_{j=1+i}^{k} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau) q(\tau) d\tau \right) \right]
\]

and using (15) and (16) it’s deduced

\[
1 + \frac{\psi_k}{p(t)} + \int_{t_0}^{t} v(\tau) d\tau \\
\leq \left( 1 + \int_{t_0}^{t} v(\tau) d\tau \right) + \sum_{i=1}^{k} \beta_i q(t_i) \exp \left( \int_{t_0}^{t_i} v(\tau) q(\tau) d\tau \right) \\
\times \left( 1 + \int_{t_0}^{t} v(\tau) d\tau \right) \prod_{j=1+i}^{k} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau) q(\tau) d\tau \right) \right] \\
= \left( 1 + \int_{t_0}^{t} v(\tau) d\tau \right) \left\{ 1 + \sum_{i=1}^{k} \beta_i q(t_i) \exp \left( \int_{t_0}^{t_i} v(\tau) q(\tau) d\tau \right) \right\} \prod_{j=1+i}^{k} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau) q(\tau) d\tau \right) \right].
\]

To finish observe that, by using the inductive method, it’s easy to prove (analogously in problem 1.8.10, p. 35, [1])

\[
1 + \sum_{i=1}^{k} \beta_i q(t_i) \exp \left( \int_{t_0}^{t_i} v(\tau) q(\tau) d\tau \right) \prod_{j=1+i}^{k} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau) q(\tau) d\tau \right) \right] \\
= \prod_{i=1}^{k} \left[ 1 + \beta_i q(t_i) \exp \left( \int_{t_0}^{t_i} v(\tau) q(\tau) d\tau \right) \right]
\]

thus, substituting in previous result, it obtains

\[
1 + \frac{\psi_k}{p(t)} + \int_{t_0}^{t} v(\tau) d\tau \\
\leq \left( 1 + \int_{t_0}^{t} v(\tau) d\tau \right) \prod_{j=1+i}^{k} \left[ 1 + \beta_j q(t_j) \exp \left( \int_{t_0}^{t_j} v(\tau) q(\tau) d\tau \right) \right].
\]

The (14) follows immediately from this and (17). \(\square\)

The results of this paper generalize Theorem 4.1.1 and Corollary 4.1.3 in [1], as it can verify immediately for \(v(t) \equiv 0\).
As final remark observe that the corollary holds even if function $u(t)$ instead of inequality (1) verifies one of the following:

$$u(t) \leq p(t) + q(t) \sum_{t_0 < t_i < t} \beta_i u(t_i - 0) + \int_{t_0}^{t} u(\tau) v(\tau) d\tau$$

or

$$u(t) \leq p(t) + \sum_{t_0 < t_i < t} \beta_i u(t_i - 0) + q(t) \int_{t_0}^{t} u(\tau) v(\tau) d\tau.$$

References


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