On the oscillation of second order differential equations

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Abstract

Some sufficient conditions are established for the oscillation of second order linear differential equations with variable coefficients which obey certain conditions. An example has been given to illustrate the results.

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1. Introduction

The study of oscillation of second order differential equations is of great interest. Many criteria have been found which involve the behavior of the integral of a combination of the coefficients. This approach has been motivated by some authors (for example see [1, 4, 5, 8, 9, 10]).

The purpose of this paper is to establish oscillation criterion for second order linear differential equations with variable coefficients of the form

\[ y''(x) + B(x)y'(x) + C(x)y(x) = 0. \] (1.1)

Throughout the paper we shall focus our attention only to certain restrictions on the coefficients functions namely, \( B(x) \), and \( C(x) \) are continuously differentiable function on the interval \([\alpha, \infty)\), where \( \alpha \) non negative real number.

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Definition 1. A solution \( y(x) \) of the differential (1.1) is said to be nontrivial if \( y(x) \neq 0 \) for at least an \( x \in [\alpha, \infty) \).

Definition 2. The differential equation (1.1) is said to be oscillatory if a nontrivial solution \( y(x) \) of (1.1) has arbitrarily large zeros on \([T, \infty)\) for all \( T > \alpha \), otherwise it said to be “nonoscillatory”.

2. Oscillation criteria

We prove the following theorems

Theorem 1. If \( B(x) < 0 \) on \([\alpha, \infty)\) such that

\[
\lim_{x \to \infty} \int_{\beta}^{x} \left( C(s) - \left( \frac{B(s)}{2} \right)^2 \right) ds = \infty.
\]

Then any solution of the differential equation (1.1) is oscillatory on \([\alpha, \infty)\).

Proof. Suppose that the differential equation (1.1) is non-oscillatory. Then there exist a non-trivial solution \( y(x) \) of the differential equation that has no zero on the interval \([\beta, \infty)\) for \( \beta > \alpha \).

The differential equation (1.1) can be transformed to a system of linear first order differential equations of the form

\[
\begin{align*}
u'(x) &= v(x) \\
v'(x) &= -C(x)u(x) - B(x)v(x)
\end{align*}
\]

where \( u(x) = y(x) \).

Let \( w(x) \) be a function defined by \( w(x) = -u^{-1}(x)v(x) \), for \( x \in [\beta, \infty) \). Then \( w(x) \) is well defined and satisfies the Riccati equation

\[
w'(x) = w^2(x) - B(x)w(x) + c(x)
\]

on \([\beta, \infty)\). Integrating both sides of this equation from \( \beta \) to \( x \) we get

\[
w(x) = w(\beta) + \int_{\beta}^{x} \left( (w(s))^2 - B(s)w(s) + C(s) \right) ds
\]

\[
= w(\beta) + \int_{\beta}^{x} \left[ \left( w(s) - \frac{B(s)}{2} \right)^2 + \left( C(s) - \frac{(B(s))^2}{4} \right) \right] ds.
\]

Since \( B(x) < 0 \), the hypothesis (2.1) implies that there exists \( \gamma > \beta \) such that

\[
w(x) \geq \int_{\gamma}^{x} (w(s))^2 ds
\]

on \([\gamma, \infty)\).
Define
\[ Q(x) = \int_{\gamma}^{x} (w(s))^2 \, ds, \quad \text{for } x \geq \gamma. \]  
(2.3)

Then,
\[ w(x) \geq Q(x) \geq 0. \]

Now differentiating (2.3) with respect to \( x \) we get
\[ Q'(x) = w^2(x) \quad \text{for } x \geq \gamma. \]

Hence
\[ Q'(x) \geq Q^2(x). \]

therefore
\[ 1 \leq \frac{Q'(x)}{Q^2(x)}, \]

this inequality holds for \( s \geq \gamma \).

Integrating both sides of this inequality from \( \gamma \) to \( x \), we will have
\[ \int_{\gamma}^{x} ds \leq \frac{1}{Q(\gamma)} - \frac{1}{Q(x)}. \]

Therefore since \( Q(x) > 0 \), we conclude
\[ \lim_{x \to \infty} \int_{\gamma}^{x} ds < \frac{1}{Q(\gamma)} \]

which is not true, then the differential equation (1.1) is oscillatory. This completes the proof. \( \square \)

**Theorem 2.** If there exists a non-vanishing function \( f(x) \in C^1[\alpha, \infty) \), \( \alpha \) is nonnegative real number such that
\[ \lim_{x \to \infty} \left[ -\frac{1}{4} \int_{\alpha}^{x} \left( f(s)(B(s))^2 - 2f'(s)B(s) - f'(s)(f(s))^2 - 4f(s)G(s) \right) \, ds + \frac{1}{2}f'(x) \right] = \infty, \]  
(2.4)

\[ \lim_{x \to \infty} \int_{\beta}^{x} f^{-1}(s) \, ds = \infty \quad \text{for } \beta > \alpha. \]  
(2.5)

Then any solution of the differential equation (1.1) is oscillatory.

**Proof.** Suppose that there exists a nontrivial solution \( y(x) \) of the differential equation (1.1) with no zeros on \( [\alpha, \infty) \). Now for \( x \geq \alpha \) define
\[ w(x) = -f(x)v(x)u^{-1}(x), \]
where $f(x)$ is non-vanishing function belongs to $C^1[\alpha, \infty)$. Then, on the
interval $[\alpha, \infty)$ $w(x)$ satisfies the Ricatti equation

$$w'(x) = \frac{1}{f(x)} [w^2(x) + f'(x)w(x) - B(x)f(x)w(x)] + f(x)C(x).$$

Now for $x \in [\alpha, \infty)$ defining

$$G(x) = w(x) + \frac{1}{2} f'(x)$$

we have

$$w'(x) = \frac{1}{f(x)} \left[ \left( G(x) - \frac{1}{2} f'(x) \right)^2 + f'(x) \left( G(x) - \frac{1}{2} f'(x) \right) - B(x)f(x) \left( G(x) - \frac{1}{2} f'(x) \right) \right] + f(x)C(x)$$

$$w'(x) = \frac{1}{f(x)} \left[ (G(x))^2 - f(x)B(x)G(x) \right.\left. - \frac{1}{4} (f'(x))^2 + B(x)f(x)f'(x) \right] + f(x)C(x)$$

$$= f^{-1}(x) \left( G(x) - \frac{B(x)f(x)}{2} \right)^2 - \frac{1}{4} \left( B(x) \right)^2 f(x)$$

$$+ (f'(x))^2 f(x) - 2B(x)f'(x) - 4f(x)C(x)].$$

Integrating both sides of the above equation from $\alpha$ to $x$ we get

$$w(x) = w(\alpha) + \int_{\alpha}^{x} f^{-1}(s) \left[ G(s) - \frac{B(s)f(s)}{2} \right]^2 ds$$

$$- \frac{1}{4} \int_{\alpha}^{x} \left( B(s) \right)^2 f(s) + (f'(s))^2 f(s) - 2B(s)f'(s) - 4f(s)C(s) ds.$$

By using equation (2.6) we get

$$G(x) = w(\alpha) + \int_{\alpha}^{x} f^{-1}(s) \left[ G(s) - \frac{B(s)f(s)}{2} \right]^2 ds$$

$$- \frac{1}{4} \int_{\alpha}^{x} \left( B(s) \right)^2 f(s) + (f'(s))^2 f(s) - 2B(s)f'(s) - 4f(s)C(s) ds + \frac{1}{2} f'(x).$$

The hypothesis (2.4) implies there exists $\beta > \alpha$ such that

$$G(x) \geq \int_{\beta}^{x} f^{-1}(s) \left[ G(s) - \frac{B(s)f(s)}{2} \right]^2 ds,$$

holds for $x > \beta$. 
Define a function $Q(x)$ for $x > \beta$ by

$$Q(x) = \int_\beta^x f^{-1}(s) \left[ G(s) - \frac{B(s) f(s)}{2} \right]^2 ds,$$  

(2.7)

then we have $G(x) \geq Q(x) > 0$.

Differentiating (2.7) we get

$$Q'(x) = f^{-1}(x) \left[ G(x) - \frac{B(x) f(x)}{2} \right]^2$$

$$\geq f^{-1}(x) \left[ Q(x) - \frac{B(x) f(x)}{2} \right]^2$$

$$\geq f^{-1}(x) Q^2(x).$$

Therefore

$$f^{-1}(x) \leq \frac{Q'(x)}{Q^2(x)}.$$

Integrating both sides of this inequality with respect to $x$ (with $x$ replaced by $s$) from $\beta$ to $x$, for $x > \beta$ we get

$$\int_\beta^x f^{-1}(s) ds < \frac{1}{Q(\beta)} - \frac{1}{Q(x)},$$

since $Q(x) > 0$ therefore

$$\lim_{x \to \infty} \int_\beta^x f^{-1}(s) ds < \frac{1}{Q(\beta)},$$

which contradicts the hypothesis of the theorem. Hence differential equation (1.1) is oscillatory.

\[\square\]

3. **Example**

Following is an illustrative example showing the applicability of both theorems.

Consider the second order differential equation

$$y''(x) - \frac{2}{x}y'(x) + \left( 1 + \frac{2}{x^2} \right) y(x) = 0, \text{ for } x > 0.$$ 

For this differential equation we have

$$B(x) = \frac{-2}{x} < 0, \text{ and } C(x) = \left( 1 + \frac{2}{x^2} \right).$$

To show the applicability of theorem (1), it is clear that the hypothesis
is satisfied as follows
\[
\lim_{x \to \infty} \int_\alpha^x \left( C(s) - \left( \frac{B(s)}{2} \right)^2 \right) ds \\
= \lim_{x \to \infty} \int_\alpha^x \left( 1 + \frac{2}{s^2} - \frac{1}{4} \frac{4}{s^2} \right) dx \\
= \lim_{x \to \infty} \int_\alpha^x \left( 1 + \frac{1}{s^2} \right) ds \lim_{x \to \infty} \left[ s - \frac{1}{s} \right]_\alpha^x \\
= \infty.
\]

Therefore the theorem implies that the differential equation is oscillatory.

This fact is directly verified by noting that the solution of the differential equation is given by
\[
y(s) = x(k_1 \cos x + k_2 \sin x).
\]

To show the applicability of Theorem 2. Choose \( f(x) = x \). It is clear that the hypothesis (2.4) is satisfied as follows
\[
\lim_{x \to \infty} \left\{ -\frac{1}{4} \left[ \int_\alpha^x (f(x)B(s))^2 - 2f'(x)B(s) - f'(s)f^2(s) \\
- 4f(s)C(s) \right] ds + \frac{1}{2} f'(s) \right\} \\
= \lim_{x \to \infty} \left\{ -\frac{1}{4} \left[ \int_\alpha^x \frac{4}{s} + \frac{4}{s} - s^2 - 4s - \frac{8}{s} \right] ds + \frac{1}{2} \right\} \\
= \lim_{x \to \infty} \left\{ \frac{s^3}{12} + \frac{s^3}{2} + \frac{1}{2} \right\} = \infty.
\]

Hence theorem (2) is applicable.

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References


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