Stochastic self-similar processes and random walk in nature

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Abstract

In this paper we consider the result which was obtained by the first author of the present work to describe scaling rules in nature, \( R(N) = \left( \frac{h}{m_{\infty}} \right) N^{\theta} \), in connection with the well-known Random Walk. By adopting this prospect, we present the result as a Brownian motion process developed as a limit of a Random Walk in the context of Mohamed El Naschie’s \( \epsilon(\infty) \) Cantorian space-time applied in cosmology.

Keywords: Random walk, Brownian motion, stochastic self-similar processes, cosmology.

1. Introduction

The observations show that Universe has structure with scaling rules, where the clustering properties of cosmological objects reveal a form of
hierarchy. In [1], [2], [3] the author presented the observed segregated Universe as the result of a fundamental self-similar law, which generalizes the Compton wavelength relation, \( R(N) = \left( \frac{h}{Mc} \right) N^{1+\phi} \), where \( R \) is the radius of the structures, \( h \) is the Planck constant, \( M \) is the total mass of the self-gravitating system, \( c \) the speed of light, \( N \) the number of the nucleons within the structures, and \( \phi \cong 1/2 \) [4]. As noted by Mohamed El Naschie, this expression agrees with the Golden mean and with the gross law of Fibonacci and Lucas [5] and [6].

In the context of El Nashie's \( \epsilon^{(\infty)} \) Cantorian space-time [7] starting from a universal scaling law, the author showed its agreement with the well-known Random Walk equation or Brownian motion relation used by Eddington for the first time [8], [9]. Consequently, he arrived at a self-similar Universe. In [4], [10], [12] and [13] the relevant consequences of a stochastic self-similar and fractal Universe were presented. It appears that Universe has a memory of its quantum origin as suggested by Sir Roger Penrose with respect to quasi-crystal [14]. Particularly, the model is related to Penrose tiling and thus to \( \epsilon^{(\infty)} \) theory (Cantorian space-time theory) as proposed by El Naschie [15], and [16] as well a Connes Noncommutative Geometry [17].

In order to understand the intricacies of Iovane’s result, \( R(N) = \left( \frac{h}{Mc} \right) N^{1+\phi} \), where \( m_n \) is the mass of a nucleon (proton or neutron), first of all we present the well-know stochastic process named Random Walk. In detail, in this paper we use a Brownian motion process and we develop this type of process as limit of a Random Walk. Let us consider this point of view, more specifically the relation \( R(N) = \left( \frac{h}{m_n c} \right) N^{\phi} \) will be seen as a Brownian motion process. Consequently, the results give us a mathematical formulation of the model presented in [1] in terms of a deep analysis with respect to the Brownian motion.

The paper is organized as follows: in Section 2, first we present the background of a Simple Random Walk by analyzing the case of: Symmetric Random Walk, Unrestricted, Absorbing Barriers; then we present some mathematical properties of a Brownian motion process. Section 3 is devoted to some properties of astrophysical scenario in \( \epsilon^{(\infty)} \) Cantorian space-time; in Section 4 we consider the application of the Random Walk process to the segregated Universe and the application to cosmology; finally the conclusions are drawn in Section 5.
2. Preliminaries: simple random walk

Thinking of our problem the simplest and more interesting stochastic process is called random walk, [18], [19], [20], [21], [22]. As well known, it represents the movement of a particle in the space, identifying its position to the time $n$. Such a position depends on the preceding position and on an independent random variable; formally, it is defined as the sum of a sequence $\{Y_i\}$ of independent and identically distributed random variables, for which the total path $X_n = \sum_{i=1}^{n} Y_i$. Alternatively, the random walk process consists of a sequence of discrete steps of fixed length.

The state space of the process $X_n$ will be discrete or continuous corresponding to the variables $Y_i$ (discrete or continuous).

Let us consider specifically the random walk in one-dimension.

Suppose that the particle moves on the $x$-axis in the following hypotheses:

1. the particle occupies the position $X_0 = 0$ to the $n = 0$ instant, with $n \in \mathbb{N}_0$,
2. the particle occupies the following $X_n$ position for all of the aforesaid moments:

$$X_n = X_0 + Y_1 + \ldots + Y_n,$$

where $\{Y_i\}$ is a sequence of independent and identically distributed random variables.

Alternatively, we may write (1) as:

$$X_n = X_{n-1} + Y_n \quad (n = 1, 2, \ldots).$$

(2)

Therefore, the relation (2) gives the equation of the particle motion; consequently, we obtain the following system of difference equations of the first order:

$$X_0 = 0$$
$$X_1 = Y_1$$
$$X_2 = Y_1 + Y_2$$
$$\vdots$$

By iterating the procedure we have from (2):

$$X_n = \sum_{i=1}^{n} Y_i.$$

(3)
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In the specific case in which the random variables $Y_i$ can only take the values 1, 0, −1 with distribution:

\[
P(Y_i = 1) = p, \quad P(Y_i = -1) = q, \quad P(Y_i = 0) = 1 - p - q,
\]

we name the process a simple random walk. Sometimes in literature, it is usual to find a simple random walk as one for which each step is either +1 or −1 with $p + q = 1$. However, we will assume that $p + q \leq 1$ with $1 - p - q$ as the probability of a zero step.

Besides, we denote with $\mu$ and $\sigma^2$ the mean value and variance of a step respectively:

1. $E[Y_i] = \mu = p - q$;
2. $\text{Var}[Y_i] = \sigma^2 = p + q - (p - q)^2 = 4pq$.

where, to calculate the variance we have used the relation $p + q = 1$.

Consequently, we obtain for the entire process

1. $E[X_n] = n\mu = n(p - q)$;
2. $\text{Var}[X_n] = n\sigma^2 = 4npq$.

If the distribution of the steps −1 and 1 assumes value 1/2:

\[
P(Y_i = 1) = P(Y_i = -1) = 1/2, \quad (n = 1, 2, \ldots),
\]

(4)

it is well known that

\[
E[X_n] = 0, \quad \text{Var}[X_n] = 1.
\]

(5)

The $X_n$ process with probabilities (4) is called symmetric random walk (for more details see [19], p. 321).

2.1 Unrestricted

In the present section we briefly introduce the unrestricted random walk ([18], p. 25). We consider the equation (3) and suppose that the random walk starts at the origin and that the particle is free to move indefinitely in either directions.

The possible positions of the particle at some times $n$ are $k = 0, \pm 1, \ldots, \pm n$. To obtain the position of the particle we fix $N_1^+, N_2^-$ and $N_3^0$ non-negative integers that represent respectively positive steps, negative steps and zero steps. The integers have to fulfill the simultaneous equalities:

\[
N_1^+ + N_2^- + N_3^0 = n, \quad N_1^+ - N_2^- = k.
\]

(6)
Let $p$ be the probability of taking a step to the right, $q$ the probability of taking a step to the left, $(1 - p - q)$ the probability of zero steps.

Hence, the probability that $X_n = k$ is conditioned from initial condition $X_0 = 0$, where $P(X_0 = 0) = 1$, is given by:

$$
P(X_n = k | X_0 = 0) = \sum_{N_1^+!N_2^-!N_3^0!} \frac{n}{N_1^+!N_2^-!N_3^0!} p^{N_1^+} q^{N_2^-} (1 - p - q)^{N_3^0},$$

(7)

where the summation is over the values of $N_1^+, N_2^-$ and $N_3^0$, satisfying (6).

We notice that for all integers $j$ it follows:

$$
P(X_n = k | X_0 = 0) = P(X_n = k + j | X_0 = j).
$$

In general, the relation (7) gives the probability for any initial state $(j)$ too.

However, the summation (7) of multinomial probabilities introduces a lot of difficulty above all when $n$ takes great values. In order to find an approximation, for this inconvenient summation of a large number of multinomial probabilities, we introduce the central limit theorem. Using such theorem $X_n$ will be approximate by a normal distribution with mean equal to $n\mu$ and variance equal to $n\sigma^2$ (when $n$ is sufficiently large).

Hence, we have:

$$
\sum_{i} Y_i \approx N(n\mu; n\sigma^2).
$$

In general, for the central limit theorem the succession $Z_n$ of random variables which are defined as follows:

$$
Z_n = \frac{X_n - n\mu}{\sqrt{n}\sigma}
$$

convergence in law to $N(0, 1)$ random variable; $\forall \epsilon > 0, \exists n_0 \in Z, \forall n \geq n_0$ we have:

$$
\left| P(Z_n \leq x) - \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy \right| < \epsilon.
$$

Thus, with $n$ sufficiently large:

$$
P(Z_n \leq x) \approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy .
$$

Hence

$$
P\left( \frac{X_n - n\mu}{\sqrt{n}\sigma} \leq x \right) = P(X_n \leq n\mu + x\sqrt{n}\sigma)
$$

$$
\approx \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{x} e^{-y^2/2} dy ,
$$
where, for \( k = n\mu + x\sqrt{n}\sigma \) it follows:

\[
P(X_n \leq k) \simeq \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\frac{k-n\mu}{\sqrt{n}\sigma}} e^{-y^2/2} dy.
\]  

(8)

The equation (8) allows us to give immediately an answer to some interesting questions:

1. the determination of the probability \( P(X_n > a) \) that (for \( n \) sufficiently large) the particle occupies a position with an abscissa greater than a fixed real number \( a \), with \( a \in \mathbb{R} \), arbitrarily;

2. the determination of the probability \( P(-b < X_n < a) \) that (with \( n \) sufficiently large) the particle is found in an interval \((-b; a)\).

By using the relation (8) to calculate the probability distribution of the particle in the first case:

\[
P(X_n > a) \simeq \frac{1}{\sqrt{2\pi}} \int_{\frac{a - n\mu}{\sqrt{n}\sigma}}^{+\infty} e^{-y^2/2} dy,
\]

we have:

\[
\lim_{n \to \infty} P(X_n > a) = \begin{cases} 
1 & \text{for } \mu > 0 \\
1/2 & \text{for } \mu = 0 \\
0 & \text{for } \mu < 0. 
\end{cases}
\]

Thus, when \( n \to \infty \) we can say that, for:

- \( \mu > 0 \) (with \( p > q \)) the particle occupies the positions over a predetermined level (spatial threshold);

- \( \mu < 0 \) (with \( p < q \)) the particle occupies the positions under a predetermined level (spatial threshold);

- \( \mu = 0 \) (with \( p = q \)) the particle asymptotically occupies, with equal probability, the positions over and under a predetermined level (spatial threshold).

These considerations justify the term drift to which it is designated the parameter \( \mu = p - q \).

By using the relation (8), the distribution of the probability that the particle is found in the inclusive strip \((-b; a)\) (second case) is:

\[
P(-b < X_n < a) \simeq \frac{1}{\sqrt{2\pi}} \int_{\frac{-b - n\mu}{\sqrt{n}\sigma}}^{\frac{a - n\mu}{\sqrt{n}\sigma}} e^{-y^2/2} dy.
\]

When \( n \to \infty \) the distribution of probability \( P(-b < X_n < a) \to 0. \)
Hence, the study of the asymptotic behaviour of the random walk provides an application of central limit theorem.

**Remark 2.1.** The random walk shows the utility of the central limit theorem when we wish a quantitative analysis of the behaviour for a system of the described type, abdicating to quantitative evaluations.

Besides, there are cases, such as the Brownian motion, in which it is natural and necessary to consider a sufficiently large \( n \). The Brownian motion is characterized by a time constant smaller than the measurements or the observations time, so that, for example, between two following measurements the test particle suffers an extremely high number of impacts, and so it is subjected to possible displacements.

### 2.2 Absorbing barriers

In this section, we briefly consider the equation (3) when the random walk starts at the origin and the particle moves in the presence of two absorbing barriers to the points \(-b\) and \(a\) (with \(a, b > 0\)) [18], [21]; for our purpose we do not consider the case of reflecting barriers.

The probability that the particle moves indefinitely between the two barriers is zero.

The particle moves in the strip \((-b; a)\) and its motion is stopped when it touches one of the two barriers. We calculate the probability that the particle is absorbed by barrier \(a\) (likewise \(-b\)) at exactly the time \(n\) under the hypothesis that the particle starts at the point \(X_0 = j\) \((-b \leq j \leq a)\):

\[
f^{(n)}_{j,a} = P(-b < X_i < a, i = 1, 2, \ldots, n - 1; X_n = a \mid X_0 = j) \quad (n = 1, 2, \ldots).
\]

While, for \(n = 0\) it follows the initial condition:

\[
f^{(0)}_{j,a} = \begin{cases} 
1 & \text{for } j = a \\
0 & \text{for } j \neq a.
\end{cases}
\]

The hypothesis makes sense when the process is time- and spatial-homogeneous because it is invariant for temporal and spatial translations.

We still notice that:

\[
\sum_{n=0}^{\infty} (f^{(n)}_{j,a} + f^{(n)}_{j,-b}) = 1.
\]

The probabilities for each step are the following:

- \(p\) for a forward step (that is +1) when the particle starts at the point \(X_0 = j + 1\);
• $q$ for a backward step (that is $-1$) when it stars at the point $X_0 = j - 1$;
• $1 - p - q$ for a zero step (that is 0) when it starts at the point $X_0 = j$.

Hence, we can write $f_{j,a}^{(n)}$ as:

$$f_{j,a}^{(n)} = p f_{j+1,a}^{(n-1)} + q f_{j-1,a}^{(n-1)} + (1 - p - q) f_{j,a}^{(n-1)}$$  \hspace{1cm} (9)

where, $j = -b + 1, \ldots, a - 1$ and $n = 0, 1, \ldots$.

Again, together with the initial condition we must write the boundary condition:

$$f_{a,a}^{(n)} = 0$$

and

$$f_{-b,a}^{(n)} = 0$$

with $n \in \mathbb{N}$.

Since, $f_{j,a}^{(n)}$ is a function of the two discrete variables $(n$ and $j)$ and we have a difference equation of the first order in $n$ and of the second order in $j$, we can use generating functions in order to calculate the solution of (9). By using the generating functions, we eliminate one of the variables and we obtain:

$$F_{j,a}(s) = \sum_{n=0}^{\infty} f_{j,a}^{(n)} s^n = F_j(s).$$ \hspace{1cm} (10)

Let us observe that when the barrier $b \to \infty$ we obtain the case of one absorbing barrier.

Let us suppose that the particle starts from the state $X_0 = j$ (or $X_0 = 0$ equivalently) and that an absorbing barrier is placed at the point $a > j$ (or $a > 0$), so that the particle is free to move among the states $x < a$ if and until it reaches the state $a$ which, once entered, holds the particle permanently.

Let $g_{j,a}^{(n)}$ the probability that the particle is absorbed by the barrier which is posed in $a$ at exactly the time $n$ under the hypothesis that the particle is free to move among the states $x < a$ and it starts at the state $X_0 = j$:

$$g_{j,a}^{(n)} = P \{ X_i < a, i = 1, 2, \ldots, n-1; X_n = a | X_0 = j \}.$$ \hspace{1cm} (11)

For $n = 0$ we have the initial conditions:

$$g_{j,a}^{(0)} = \begin{cases} 1 & \text{for } j = a \\ 0 & \text{for } j \neq a. \end{cases}$$
Moreover, we have the boundary conditions:

\[ g_{\mu}(n) = 0, \quad \text{if} \ n \neq 0. \]

It is possible to define the generating function as:

\[
G_{j,\mu}(z) = \sum_{n=0}^{\infty} g_{j,\mu}^{(n)} z^n.
\] (12)

This case, corresponding to a single barrier, will be used in the following with respect to a practical application in cosmology.

2.3 Brownian motion

In this section, we are going to show the main probabilistic tools generally used to formulate the Brownian motion. Since, it is not our aim to expose this continuous-time stochastic process thoroughly, we will emphasize the main results and formulae useful for our purpose.

We begin by considering the Brownian motion process \( \{ B_t \}, t \geq 0 \), sometimes called Wiener’s process. Typically, the term “Brownian motion” is used to describe a wide class of processes; \( \{ B_t \}_{t \geq 0} \) is only a particular case (i.e. motion of a free particle with negligible acceleration). In detail, the Brownian motion will be developed as a limit of a random walk.

In all this section, \( (\Omega, \mathcal{F}, P) \) is a fixed probability space with \( \Omega \) nonempty set, \( \mathcal{F} \) \( \sigma \)-algebra of subsets of \( \Omega \) and \( P \) probability measure on \( \mathcal{F} \). On this space we consider a stochastic process \( \{ B_t \}_{t \geq 0} \) with \( t \geq 0 \) and a real random variable \( B_t \).

**Definition 2.1.** The stochastic process \( \{ B_t \}_{t \geq 0} \) is said to be a Brownian motion if:

1. \( B(0) = 0 \) almost surely;
2. \( \{ B_t \}_{t \geq 0} \) is a process with independent increments;
3. \( B_t \) is a centered Gaussian random variable with \( \sigma^2 t \) variance\(^1\), \( \forall \ t \) and for some positive constant \( \sigma \);
4. \( \gamma : t \to B(t)_t \), with \( t \geq 0 \) is continuous almost surely.

A full development of the Brownian motion can be found in [18], [19], [20], [21], [22].

\(^1\)When \( \sigma = 1 \), the process is often called standard Brownian motion or standard Wiener process.
Thanks to the previous definitions, one obtains that for $s < t$, $B_t - B_s$ is of Gaussian type, centered with variance $t - s$, which says that Brownian motion is stationary.

Indeed, one has

$$B_t = B_s + (B_t - B_s).$$

Therefore, using characteristic functions and the independence of $B_s$ and $B_t - B_s$ one has

$$E[e^{ir(B_t - B_s)}] = e^{-r^2(t-s)/2}$$

because $s < t$, $B_t - B_s \overset{L}{=} N(0,t-s)$, where “L” means that we have a convergence in law.

One can also compute the covariance function of the Brownian motion: for all $(s,t) \in (\mathbb{R}^+)^2$,

$$E(B_sB_t) = s \wedge t$$

where the symbol $\wedge$ denotes the minimum between $s$ and $t$.

Indeed, suppose that $s < t$ then

$$E(B_sB_t) = E\{B_s[B_s + (B_t - B_s)]\}$$

$$= E(B_s^2) + E[B_s(B_t - B_s)]$$

$$= E(B_s^2)$$

$$= Var(B_s)$$

$$= s.$$

Since, a standard Brownian motion $\{B_t\}$ is normal with mean 0 and variance $t$, its density function is given by

$$f_t(x) = \frac{1}{\sqrt{2\pi t}} \exp[-x^2/2t].$$

The Brownian motion process can also be defined as the limit of the random walk. The random walk model is a first approximation of the theory of diffusion and Brownian motion. In the limit, the process will appear as a continuous motion (for example see [21], p. 323; [20], p. 356; [19], p. 323).

In order to prove the intuitive properties (see Definition 2.1) of this limiting process we consider the relation (5) and the central limit theorem (see symmetric random walk, relation 4). In detail, if we begin with a
simple random walk we obtain a Brownian motion process with drift coefficient \( \mu \) of the form:

\[
X_t = B_t + \mu t, \quad t \geq 0
\]  

(13)

where \( \{B_t\} \) is the standard Brownian motion [20].

**Definition 2.2.** The stochastic process \((X_t)_{t \geq 0}\) is said to be a *Brownian motion with drift coefficient* if:

1. \( X(0) = 0 \) almost surely;
2. \((X_t)_{t \geq 0}\) is a process with independent increments;
3. \( X_t \) is normally distributed with mean \( \mu t \) and variance \( \sigma^2 t \), with \( \mu \) and \( \sigma \) positive real constants.

### 3. Astrophysical scenario

In [10], [11] the authors presented a study on the dynamical systems on Cantorian space-time to explain some relevant stochastic and quantum processes, where the space acts a harmonic oscillating support, such as it often happens in Nature. The observations show a structure of Universe with scaling rules, where clustering properties, from cosmological to nuclear objects, reveals a form of hierarchy. As an example, it is possible to distinguish among globular cluster, galaxies, clusters and superclusters of galaxies through their spatial lengths [23], [24]. Table 1 recalls the dimensions and masses of the previous systems [25].

<table>
<thead>
<tr>
<th>System type</th>
<th>Length</th>
<th>Mass ((M_\odot))</th>
</tr>
</thead>
<tbody>
<tr>
<td>Globular clusters</td>
<td>( R_{GC} \sim 10 \text{pc} )</td>
<td>( M_{GC} \sim 10^{6-7} )</td>
</tr>
<tr>
<td>Galaxies</td>
<td>( R_G \sim 100 \text{kpc} )</td>
<td>( M_G \sim 10^{10-12} )</td>
</tr>
<tr>
<td>Cluster of galaxies</td>
<td>( R_{CG} \sim 1.5 h^{-1} \text{Mpc} )</td>
<td>( M_{CG} \sim 10^{15} h^{-1} )</td>
</tr>
<tr>
<td>Supercluster of galaxies</td>
<td>( R_{SCG} \sim 100 h^{-1} \text{Mpc} )</td>
<td>( M_{SCG} \sim 10^{15+17} h^{-1} )</td>
</tr>
</tbody>
</table>

In [1], [4], [12], [13] the first author of the present paper considered the compatibility of a Stochastic Self-Similar Fractal Universe by the observation and the consequences of the model. In detail, it was
demonstrated that the observed segregated Universe is the result of a fundamental self-similar law, which generalizes the Compton wavelength relation, \( R(N) = \left(\frac{\hbar}{m_n c}\right)N^{\phi} \). A typical interaction length can be defined as a quantity, which is proportional to the size of the system which contains the constituents [26]. In other words, consider a maximum length corresponding to its size for each a system. In 1985 Sakharov argued that quantum primordial fluctuations had to be related to cosmological evolution and to the dynamics of astrophysical systems [27]. Eddington and later on Weinberg wrote the relevant relationship between quantum quantities and the cosmological ones:

\[
\hbar \cong \frac{G}{2^{\frac{1}{2}}} m^{\frac{3}{2}} R^{\frac{1}{2}},
\]

where \( \hbar \) is the Plank constant, \( G \) is the gravitational constant, \( m \) is the mass of nucleon, and \( R \) is the radius of Universe.

By following Eddington Weinberg is (E–W) approach, the first author and his team wrote a general relationship between the radius \( R \) of the self-equilibrated system and its number of nucleons. While the E–W relationship was only written for the radius of Universe, they presented a relationship which is scale invariant, so adoptable for all types of self-gravitating systems (and also for the entire universe):

\[
R(N) = \frac{\hbar}{m_n c} N^{\alpha} = \frac{\hbar}{m_n c} N^{\phi}
\]

with \( \alpha = \frac{3}{2} \), for \( M = M_G = 10^{10–12} M_\odot, m_n \) mass of the nucleons, \( \phi = \frac{1}{2} \) \( N = 10^{68} \) (this is approximately the number of nucleons in a galaxy), again they reproduce exactly \( R \approx 1–10 \) kpc. In general, the authors evaluate the number of nucleons in a self-gravitating system as

\[
N = \frac{M}{m_n},
\]

where \( N \) is the number of nucleons of mass \( m_n \) into self-gravitating system of total mass \( M \). Then, they obtain the relevant results recalled in Table 2. In the second column the number of evaluated nucleons is shown, while they find the expected radius of self-gravitating system in the last column.

\[\text{The value } \phi = \frac{1}{2} \text{ is what is found by the observation. If we assume a Cantorian } e^{(\infty)} \text{ space time, as suggested by Mohamed El Nashie the expectation value is } \phi = \frac{\sqrt{5} - 1}{2}, \text{ that is the golden Mean.}\]
Table 2
Evaluated Length for different self-gravitating systems

<table>
<thead>
<tr>
<th>System type</th>
<th>Number of nucleons</th>
<th>Evaluated length</th>
</tr>
</thead>
<tbody>
<tr>
<td>Global clusters</td>
<td>$N_G \sim 10^{63} \div 10^{64}$</td>
<td>$R_{GC} \sim 1 \div 10,\text{pc}$</td>
</tr>
<tr>
<td>Galaxies</td>
<td>$N_G \sim 10^{68}$</td>
<td>$R_G \sim 1 \div 10,\text{kpc}$</td>
</tr>
<tr>
<td>Cluster of galaxies</td>
<td>$N_{CG} \sim 10^{72}$</td>
<td>$R_{CG} \sim 1,\text{h}^{-1},\text{Mpc}$</td>
</tr>
<tr>
<td>Supercluster of galaxy</td>
<td>$N_{SCG} \sim 10^{73}$</td>
<td>$R_{SCG} \sim 10 \div 100,\text{h}^{-1},\text{Mpc}$</td>
</tr>
</tbody>
</table>

Moreover, the relation (14) can be written in terms of Plakian quantities:

$$R_p(N) = \frac{l_p}{m_p} \sqrt{\frac{\hbar c}{G}} N^{(1+\phi)}$$

and from eq. (15) we obtain:

$$R_p(N) \propto l_p N^{3/2}$$

where we have assumed $\phi = 1/2$.

If we consider the $R$ radius as a fixed quantities, equalizing the equations (14) and (15), we get:

$$\frac{l_p}{m_p} \sqrt{\frac{\hbar c}{G}} N^{3/2} = \frac{h}{M_c} N^{3/2}$$

and so

$$M = \frac{m_p}{l_p} \sqrt{\frac{G \hbar}{c^3}} .$$

The mass $M$ of the structure is written, through the relation (16), in terms of Plank’s length.

As reported in [28], and [1], the following theorems can be obtained.

**Theorem 1.** The structures of the Universe appear as if they were a classically self-similar random process at all astrophysical scales. The characteristic scale length has a self-similar expression

$$R(N) = \frac{h}{M_c} N^{1+\phi} = \frac{h}{m_n c} N^{\phi} ,$$

where the mass $M$ is the mass of the structure, $m_n$ is the mass of a nucleon, $N$ is the number of nucleons into the structure and $\phi$ is the Golden Mean value.

In terms of Plankian quantities the scale length can be recast in

$$R_p(N) = \frac{l_p}{m_p} \sqrt{\frac{\hbar c}{G}} N^{(1+\phi)} .$$
The previous expression reflects the quantum (stochastic) memory of the Universe at all scales, which appears as a hierarchy in the clustering properties.

**Theorem 2.** The mass and the extension of a body are connected with its quantum properties, through the relation

\[ E_{E,N}(N) = E_P N^{1+\phi}, \]

that links Planck's and Einstein's energies.

The quantum (stochastic) memory is reflected at all scales and it manifests itself through a clusterization principle of mass and extension of the body.

4. Application to dynamical system and cosmology

4.1 Random Walk process and the segregated Universe

In [26] the authors noted that all astrophysical scales have a particular length. For this reason, they obtained the exact lengths of the self-gravitation system just by using an interesting power law. An invariant scale relation, from the quantum lengths to the astrophysical ones, plays a fundamental role. As a macroscopic system, Universe shows a sort of quantum and relativistic memory of its primordial phase. The choice to start with \( \alpha = 1/2 \) is suggested by the Statistical Mechanics.

Indeed eq. (14) is strictly equivalent to

\[ R(N) = l N^\alpha \]

where \( l = h/m_nc \). The relation (17) is the well-known Random Walk or Brownian motion developed as limit of a Random Walk, when \( \alpha = 1/2 \).

In this paragraph, we consider a segregated Universe as the result of an aggregation process, in which a test particle in its motion in the Fractal Cantorian \( e^{(\infty)} \) spacetime can be captured or not.

**The model.** Let us consider a test particle with mass \( m_n \), moving in a physical space \( S \) (like the entire Universe). Moreover, let us also consider \( S \) composed by some substructures \( S_i \) (like a galaxy, a cluster of galaxy, and so on) on the \( N \)-axis (see Figures 1-2).

Due to the interaction between the systems and the test particle, the probability that the particle is captured by a component \( S_i \) can be expressed through a Random Walk \( X_n \) process. In detail, we can make the hypothesis that \( X_n \) is the aggregation process of a fixed structure \( S_i \).
Figure 1
A sketch of a long bang in the universe’s expansion

Figure 2
A simulated result for the long bang in the universe’s expansion
In this case, if we introduce a sequence of independent and identically distributed random variables \( \{Y_i\} \), which are the single steps of aggregation, that is the single possibilities that a test particle is captured or not, then \( X_N = \sum_{i=1}^{N} Y_i \) is the aggregation process, thanks to which a structure \( S_i \) after a fixed time (or a fixed number of steps) reaches the mass \( M_i \) and so the radius \( R_i \).

We consider the simple random walk process (see Paragraph 2) since in this case, the mass-step \( Y_i \) can only take the value 1 or -1 with the distribution:

\[
P(Y_i = 1) = p = \phi, \quad P(Y_i = -1) = q = 1 - \phi,
\]

where in the symmetric case \( \phi = 1/2 \); in the main context of our application \( \phi \cong 1/2 \). Specifically when \( \phi = \frac{\sqrt{5}-1}{2} \), that is the Golden Mean, we have a Cantorian \( \varepsilon(\infty) \) Universe, according to Mohamed El Naschie is theory.

It is assumed that each mass-step is either +1 (with probability \( \phi \)) or -1 (with probability \( 1 - \phi \)).

In our context the steps assume the following meaning:

- corresponding to \( Y_i = +1 \) the test particle is captured by the system \( S_i \), and \( m_{S_i} \) grows;
- corresponding to \( Y_i = -1 \) the test particle is not captured by \( S_i \).

If the particle is not captured by \( S_i \), it can be captured by an \( S_j \), that is at the same scale (for example a galaxy scale) or at a different scale (for example globular cluster, cluster of galaxy and so on).

From the point of view of the structure \( S_i \) the particle can be captured (corresponding to \( Y_i = +1 \)) or not captured (corresponding to \( Y_i = -1 \)). Consequently, the case with \( Y_i = 0 \) becomes irrelevant.

Anyway, in a more realistic model, where we consider a spacetime structure which is not homogeneous and isotropic and where the aggregation process follows a hierarchical approach, we will analyze if the case \( Y_i = 0 \) makes sense.

**Remark 4.1.** On the other hand, either there is a +1 mass-step linked to the energy of bounding or there is a -1 less-mass-step linked to the dissipative energy from the structure.

**Remark 4.2.** We notice that in our model, for all integer \( j \), it remains the equality between the probability that the particle starts from zero
and the probability that the particle starts from $k + j$:

$$P(X_N = k \mid X_0 = 0) = P(X_N = k + j \mid X_0 = j),$$

where $k = 0, \pm 1, \pm 2, \ldots$ and $N = 1, 2, \ldots$

While from a mathematical point of view this is not so interesting, from a physical point of view this is relevant due to the fact that we must start from a system with a mass $m_n$ corresponding to a system composed by a single mass and not a zero mass system.

Let us denote $\mu$ and $\sigma^2$ the mean and variance of a mass-step.

Then $\mu = p - q$ and $\sigma^2 = p + q - (p - q)^2$ and hence in our case, the mean and variance are respectively:

$$E(Y_i) = \mu = p - q = (2\phi - 1) = \phi^3,$$

$$\text{Var}(Y_i) = \sigma^2 = p + q - (p - q)^2 = 4pq = 4\phi(1 - \phi) = 4/\langle \dim_{\mathcal{H}}e^{(\infty)} \rangle$$

with

$$\frac{1}{\phi(1 - \phi)} = \frac{1}{\phi\phi^2} = \frac{1}{\phi^3} = 4 + \phi^3 = \langle \dim_{\mathcal{H}}e^{(\infty)} \rangle$$

the Hausdorff’s dimension.

We notice that $E(Y_i)$ is the mean of the number of mass-steps to obtain the structure of radius $R_i$; consequently the mean and variance of our simple random walk for the structure $S_i$ are respectively:

$$E(X_N) = \mu N = N(2\phi - 1) = N\phi^3$$

$$\text{Var}(X_N) = \sigma^2 N = 4Npq = 4N\phi(1 - \phi) = 4N/\langle \dim_{\mathcal{H}}e^{(\infty)} \rangle.$$ 

In the case of $\phi = \sqrt{5} - 1$, it follows $p > q$; consequently, the probability that a new mass will be captured by the structure $S_i$ is greater than the particle is not captured.

Vice-versa is for $p > q$.

In the case of $p = q$ we have the same probabilities that the particle is accepted or refused.

In a different vision for a fixed structure $S_i$ we could have the above three cases at different time, corresponding to different eras in the life of $S_i$. By adopting this point of view $\phi = \sqrt{5} - 1$ could be linked to the present, in which we carry out measurements.

This analysis is not conclusive; obviously this is just a toy model to start the investigation of cosmology in $\mathcal{E}^{(\infty)}$ Cantorian spacetime by using a stochastic approach based on Random Walk.
Indeed, in future analysis we will consider some other parameters such as:

1. the homogeneity and the isotropic of the spacetime;
2. the hierarchy between structures at different scale for accepting or not new mass in terms of new particles;
3. the presence of hidden variables for describing internal rules for a structure $S_i$ to accept or do not accept a particle.

This hidden variables could represent the maximum capability to accept new mass with respect to the spatial dimension of the structure, or a changed number as it often it happens in particle physics, when we consider the colour, the flower and so on.

By using the result of the central limit theorem (see Paragraph 2.2) $X_N$ will be approximated by a normal distribution with mean $N\mu$ and variance $N\sigma^2$ (when $N$ is sufficiently large):

$$\sum_i Y_i \approx N(N\phi^3; 4N\phi(1-\phi)).$$

In general, for the central limit theorem the succession $Z_N$ of random variable:

$$Z_N = \frac{X_N - N\mu}{\sqrt{N} \sigma}$$

converges in law to $N(0,1)$ random variable.

In detail, the particle will be within a distance of order $N^\phi$ from its starting point after $N = M/m_p$ mass-steps. From the previous considerations and by reading the earlier papers of first author, it clearly appears that the relation $R(N) = (\hbar/m_n c)N^\phi$, is a Brownian motion process developed as a limit of the Random Walk (see Paragraph 2.3.) Indeed, by using the central limit theorem, the Random Walk $R(N) = (\hbar/m_n c)N^\phi$ will appear as a Brownian motion with drift coefficient $\phi^3$ and from the relation (13) it follows:

$$(X_N)_N \overset{L}{\to} \tilde{B}_N = B_N + \phi^3 N$$

(20)

where $L$ is a convergence in Law and $B_N$ is the standard Brownian motion process.

Hence, we notice that the process $R(N) = (\hbar/m_n c)N^\phi$ is a Brownian motion with drift coefficient $\phi^3$.

It is interesting to note that each scale (with radius $R_i$) of Universe
has a Gaussian distribution with mean $\mu_i = N\Phi^3$ and variance $\sigma_i^2 = 4N/(\dim H e^{(\infty)})$:

$$f(x) = \frac{1}{\sqrt{2\pi\sigma_i^2}} \exp \left[ -\frac{(x - \mu_i)^2}{2\sigma_i^2} \right] , \quad x \in \mathbb{R}.$$  

In our model, we can see the different scales of Universe and so the segregation of Universe, as sequence of fundamental lengths. This lengths has a gaussian distribution with mean $\mu_i$ and variance $\sigma_i^2$.

Moreover, we are assuming that there is no overlapping among different scales.

In other words, each scale has a Gaussian distribution, with the mean value that is $R_i$ and with a dispersion that is given by the standard deviation.

For example, this implies that in the case of a system $S_i$ on galaxy scale the best value will be of the order of $R_i = 10^{10}$, corresponding to $\mu_i = 10^{68}$.

By taking into account the results in [29] we can easily understand that the present Universe can be obtained as a Brownian motion at some different scales. Equivalently, we can consider a fundamental length scale that has Gaussian distribution and that generates the segregated Universe thanks to translation and processes in scale.

4.2 Application to cosmology

From the previous analysis, the state of a system with scale length $R_i$ appears to be more probable than the other with length $R_{i+\alpha}$, $\alpha \in \mathbb{R}$, $\alpha \neq j$, where $j$ is another scale with high probability. The choice of fixed length (see Table 1 and Table 2) instead of others is more stable, since it is more probable.

In order to prove the stability of the structures, we introduce the specific case of an absorbing barrier (see Section 2.2). Let us suppose that the structure starts in the state $X_0 = j$ and that an absorbing barrier is placed at the point $a > j$ [18]. We suppose that the different scales of Universe are the absorbing barriers $a_i$ and $e^{(\infty)}$ Cantorian space gives the minimal distance between two points among which it is possible to allocate an unitary mass $(m_n)$, in connection with the Plank’s length.

In other words, we have a process thanks to which a test system grows its mass step by step of quantities $m_n$. This goes on until the upper limit, fixed by the barriers value $a_i$, is reached. This means that if a particle
arrives at the structure when this one has a mass smoother than the value linked to \( a_i \), then the particle with mass \( m_n \) is absorbed; otherwise the mass \( m_n \) is refused.

In order to have the barriers \( a_i \), as dimensionless number, we define:

\[
a_i = \frac{R_i}{R_p(1)} = \left( \frac{m_{nc}}{m_p} N_i^\phi \right)^{1/\sqrt{\frac{G}{hc}}} = \frac{m_p}{m_{nc}^2} \sqrt{\frac{Gh}{c^3}} N_i^\phi.
\]

Since, the barriers \( a_i \) are the absorbing barriers, the equation (12) gives the probability that the absorption occurs at \( a_i \) at the time \( N_i \). Here we measure the time evolution in connection with the aggregation of the matter. In other words, \( N_i \) indicates both the number of components into a structure and a possible time scale, that is the number of steps to obtain the mass \( M_i \) corresponding to the length \( R_i \).

Let us consider the random variable \( T(a_i \mid j) \):

\[
T(a_i \mid j) = \inf \{ N_i : X_{N_i} = a_i \mid X_0 = j \} \quad (a_i > j)
\]

that denotes the time to absorption at \( a_i \) or equivalently the first passage time to state \( a_i \) from \( j \).

We can write:

\[
T(a_i \mid j) = T_{j+i} + T_{j+i+1} + \ldots + T_{a_i}
\]

where \( T_{j+i} \) \((i = 1, 2, \ldots, a_i - j)\) are independent and identically distributed random variables.

By taking into account that \( T(a_i \mid 0) \) is the sum of independent random variables with finite variance and by using the central limit theorem, in terms of the mean \( \mu = p - q \) and variance \( \sigma^2 = p + q - (p - q)^2 \) of a single mass-addition-step, we have

\[
E[T(a_i \mid 0)] = \frac{a_i}{\mu} = \frac{a_i}{(2\phi - 1)},
\]

\[
\text{Var}[T(a_i \mid 0)] = a_i \frac{\sigma^2}{\mu^3} = a_i \frac{4}{(\dim \text{cel}(\infty))}.\]

By using the number of absorbed particles at a \( T(a_i \mid j) \), we can write (see [18], p. 35):

\[
G_{j,a_i}(z) = \sum_{N_i=1}^{\infty} z^{N_i} P[T(a_i \mid j) = N_i]
\]
for $z = 1$ we have
\[ G_{j,a_i}(1) = P[T(a_i | j) < \infty]. \]
Thus
\[ G_{j,a_i}(1) = P[T(a_i | j) < \infty] = \begin{cases} \left[ \frac{p}{q} \right]^{a_i - j} & (p < q) \\ 1 & (p \geq q) \end{cases} \tag{21} \]

**Example 4.1.** The first passage time $T(1 | 0)$ denotes the number of absorbed particles (components) on the first length scale $R_1$ structure from state $j$ (free-state) to state 1 (position of structure); it has a generating function equally to $G_1(z)$.
Corresponding to $T(a_i | j)$, we have the mass of the fixed structure equal to $M(a_i | j) = m_n T(a_i | j)$. Since, $T(a_i | j)$ denotes the duration of the Random Walk, (21) is the probability that the absorption occurs for each fixed barriers $a_i$.

Having posed
\[ a_i = \frac{m_p}{m_n} \sqrt[3]{\frac{Gh}{c^3}} N_i^\phi = \alpha N_i^\phi \]
where $\alpha = \frac{m_p}{m_n} \sqrt[3]{\frac{Gh}{c^3}}$ and by taking into account the expression for the mean value $E_i(X_N)$, that we have introduced in the previous section, that is
\[ E_i = (2\phi - 1)N_i \]
and passing to the logarithms, we obtain
\[ E_i = \left\{ 2 \left[ \frac{\log a_i - \log \alpha}{\log N_i} \right] - 1 \right\} N_i. \]
By simple calculations is gotten
\[ a_i = \alpha N_i^{\frac{1}{\phi} \left( \frac{E_i}{N_i} + 1 \right)} \]
and since
\[ R_i = a_i R_p(1) \]
it follows that
\[ R_i = \frac{h}{m_n c} N_i^{\frac{1}{\phi} \left( \frac{E_i}{N_i} + 1 \right)}. \tag{22} \]
In the specific case \( N_i = 1 \) the relation (22) becomes

\[
R_1 = \frac{\hbar}{m_n c}.
\]

The relation (22) represents the scaling law in terms of the means \( E_i \) in connection with the Random Walk process. Thanks to this approach, when the structure reaches the maximum permitted mass \( M_{i, \text{max}} \) that is compatible with the fractal structure with the fixed \( R_i \), a free particle (test-particle) meets the barrier of that structure with probability equal to 1. This means that for:

\[
M = M_i = m_n T(a_i | j)
\]

the structure with length scale equal to \( R_i \) does not accept any other component (particle) but reflects it. Consequently, this free particle of mass \( m_n \) will tend to find a new acceptor structure with the same scale length \( R_i \) or with \( R_j \).

In this way, it is possible to create a segregated Universe at different length scales.

5. Conclusions

In this paper we have considered the link between the scale invariant law, \( R(N) = (\hbar/m_n c)N^{\phi} \), introduced by the first author, and the Random Walk. The first law represents the observed segregated Universe as the result of a fundamental self-similar law, which generalizes the Compton wavelength relation, while the second is a well-known stochastic process. We have used a Brownian process in order to prove that this type of process, developed as a limit of a Random Walk, has the same form of the relation \( R(N) = (\hbar/m_n c)N^{\phi} \). Moreover, we have noticed that \( R(N) = (\hbar/m_n c)N^{\phi} \) is a Brownian motion with drift coefficient \( \mu \) of the form: \( \tilde{B}_N = B_N + \phi^\alpha N \).

In the last section of this work, we have studied the state of the system with different length scales. In this way, we have seen that the system with a length \( R_i \) is more probable than other with \( R_{i+\alpha} (\alpha \in \mathbb{R}) \) with a different radius. To be more precise, choosing a fixed length the masses take values, which are more stable. In order to prove this, we have considered the random walk with an absorbing barrier; here the different scales of Universe are considered in connection with some absorbing barriers and the \( \epsilon^{(\infty)} \) Cantorian space-time is directly linked with the minimal distance between two massive points and Plank’s length.
References


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