Mathematical model of conflict with non-annihilating multi-opponent

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Abstract

In this paper a conflict composition for non-annihilating multi-opponent for a finite collection of positions is introduced and consider the associated dynamical system. The existence of limiting distribution of the conflict interaction is investigated. By means of conflict interaction how the segregation phenomena emerge in the society is shown.

Keywords: Conflict interaction, stochastic vector, segregation.

1. Introduction

In some recent papers, [2], [3], V. Koshmanenko describes a conflict model, for non-annihilating two opponents, in a situation where none of the opponents have any strategic priority. The conflict interaction between the opponents only produces a certain redistribution of common area of interests. He develops this model as an alternative approach to a well known mathematical theory of population dynamics to describe the quantitative changes of conflicting spices (e.g., Lotka-Volterra equation).

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We observe that there are many multi-opponent situations, in our social phenomena, where they are making conflicts to each other. For example, there are multi race (e.g., Black, White, Chinese, Hispanic, etc), multi religion (e.g., Islam, Christian, Hindu, etc) and different political opinions exist in the society and because of their difference they have conflicts to each other. Therefore it is very important to construct a conflict model for multi-opponent situation to understand realistic conflict situations in the society.

This paper proposes such a conflict model for non-annihilating multi-opponent which is a natural extension of [2]. By means of conflict among races how segregation emerges in the society is shown through computer experiment.

This paper is structured as follows. In Section 2 we present our concepts and formal model, in Section 3 we present computer experimental results and in Section 4 we discuss our results.

2. Mathematical model of conflict with multi-opponent

In order to give a good understanding of our model to the reader, we firstly explain it for the case of three opponents denoted by $A, B, C$ and four positions. We denote by $\Omega = \{\omega_1, \omega_2, \omega_3, \omega_4\}$ the set of positions which $A, B$ and $C$ try to occupy. Hence $\omega_1, \omega_2, \omega_3$ and $\omega_4$ represents different positions in $\Omega$. By a social scientific interpretation, each $\omega_j$, $j = 1, 2, 3, 4$ represent an area of a big city $\Omega$. Let $\mu_0$, $\nu_0$ and $\gamma_0$ denote the probability measures on $\Omega$. We define the probability that the opponents $A, B$ and $C$ occupy the position $\omega_j$, $j = 1, 2, 3, 4$ as follows:

$A, B$ and $C$ independently occupy $\omega_j$, $j = 1, \ldots, 4$ with probabilities $\mu_0(\omega_j)$, $\nu_0(\omega_j)$ and $\gamma_0(\omega_j)$, respectively. As we are thinking about the probability measures and a priori the opponents are assumed to be non-annihilating, it holds that

$$\sum_{j=1}^{4} \mu_0(\omega_j) = 1, \quad \sum_{j=1}^{4} \nu_0(\omega_j) = 1, \quad \sum_{j=1}^{4} \gamma_0(\omega_j) = 1.$$\[1\]

Since $A, B$ and $C$ are incompatible, this generates a conflicting interaction and we express this mathematically in a form of conflict composition. Namely, we define the conflict composition in terms of the conditional probability to occupy, for example, $\omega_1$ by each of the opponents. Therefore for the opponent $A$ this conditional probability should be proportional to
the product,

\[ \mu_0(\{w_1\}) \times \nu_0(\{w_2\}, \{w_3\}, \{w_4\}) \times \gamma_0(\{w_2\}, \{w_3\}, \{w_4\}). \]

We note that this corresponds to the probability for \( A \) to occupy \( w_1 \) and the probability for \( B \) and \( C \) to be absent in that position \( w_1 \). Similarly for the opponents \( B \) and \( C \) we define the corresponding quantities. As a result, we obtain a redistribution of the conflicting areas. We can repeat the above described procedure for infinite number of times, which generates a trajectory of the conflicting dynamical system. The limiting distribution of the conflicting areas is investigated.

Now, we precisely formulate the above concept for multi-opponent and multi position:

For given \( n, m \in \mathbb{N} \), s.t., \( n \geq m \geq 2 \), let \( \Omega = \{w_1, w_2, \ldots, w_n\} \) be the finite set of positions that can be occupied by each of \( m \) opponents. We denote these \( m \) opponents by \( A_1, A_2, \ldots, A_m \).

We explain the intense competition among the opponents \( A_1, A_2, \ldots, A_m \) to occupy the positions \( w_1, w_2, \ldots, w_n \) in a form of a conflict composition among the stochastic vectors by introducing a stochastic dynamics for each coordinate of the vectors.

Here, we suppose that the initial probabilities of opponent \( A_1, A_2, \ldots, A_m \) are associated with stochastic vectors, say \( P_1^{(0)}, P_2^{(0)}, \ldots, P_n^{(0)} \in \mathbb{R}_+^n \) such that \( \sum_{j=1}^{n} p_{ij}^{(0)} = 1 \), where \( p_{ij}^{(0)} \) represent the initial probability that \( i \)-th opponent occupy the \( j \)-th position.

Let us define the procedure by which the transition from a previous stage to the next is characterized. Precisely, giving the initial vectors \( P_1^{(0)}, P_2^{(0)}, \ldots, P_m^{(0)} \), the stochastic vectors in the next stage is defined by:

\[
\begin{pmatrix}
P_1^{(1)} \\
\vdots \\
P_m^{(1)}
\end{pmatrix} = 
\begin{pmatrix}
p_{11}^{(1)} & \cdots & p_{1n}^{(1)} \\
\vdots & \ddots & \vdots \\
p_{m1}^{(1)} & \cdots & p_{mn}^{(1)}
\end{pmatrix}
\]

where,

\[ p_{ij}^{(1)} := \frac{1}{z_i^{(0)}} p_{ij}^{(0)} \prod_{\substack{l=1 \atop l \neq j}}^{n} (1 - p_{lj}^{(0)}); \quad i, l = 1, \ldots, m; \quad j = 1, \ldots, n, \] (1)
Theorem 1. Suppose there are \( n \) opponents and \( n \) positions. For the initial stochastic vectors \( \mathbf{P}_1^{(0)} = (a_1^{(0)}, a_2^{(0)}, a_3^{(0)}, \ldots, a_{n-1}^{(0)}, a_n^{(0)}) \), \( \mathbf{P}_2^{(0)} = (a_n^{(0)}, a_1^{(0)}, a_2^{(0)}, \ldots, a_{n-2}^{(0)}, a_{n-1}^{(0)}) \), \( \ldots \), \( \mathbf{P}_n^{(0)} = (a_1^{(0)}, a_2^{(0)}, a_3^{(0)}, \ldots, a_{n-1}^{(0)}, a_n^{(0)}) \in \mathbb{R}^n_+ \) with
the coordinates $0 < a_1^{(0)} < 1, a_1^{(0)} + a_2^{(0)} + \ldots + a_n^{(0)} = 1, 0 \leq a_n^{(0)} < a_{n-1}^{(0)} < \ldots < a_1^{(0)} < 1$, there exists a limit
\[
\mathbb{P}_i^{(\infty)} = \lim_{N \to \infty} \mathbb{P}_i^{(N)}, \quad i = 1, 2, \ldots, n.
\]
where $\mathbb{P}_i^{(N)}$ gives the stochastic vector after $N$-th conflict.

If $\mathbb{P}_1^{(0)} \neq \mathbb{P}_2^{(0)} \neq \ldots \neq \mathbb{P}_n^{(0)}$ then the limiting elements are 0 or 1. If all of $\mathbb{P}_i^{(0)}$ are same i.e., every opponents has same probability to occupy each positions, then the limiting distribution is given by:
\[
\mathbb{P}_1^{(\infty)} = \mathbb{P}_2^{(\infty)} = \ldots = \mathbb{P}_n^{(\infty)} = \left( \frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n} \right)
\]

**Proof.** If $a_1^{(0)} = 1$ and $a_2^{(0)} = a_3^{(0)} = \ldots = a_n^{(0)} = 0$, then obviously $a_1^{(N)} = 1$ and $a_2^{(N)} = a_3^{(N)} = \ldots = a_n^{(N)} = 0$ for all $N \geq 1$.

Thus, we have to prove the case $0 < a_n^{(0)} < a_{n-1}^{(0)} < \ldots < a_1^{(0)} < 1$.

By (1) and (2) we get,
\[
\begin{align*}
\mathbb{P}_1^{(1)} &= (a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots, a_{n-1}^{(1)}, a_n^{(1)}), \\
\mathbb{P}_2^{(1)} &= (a_1^{(1)}, a_2^{(1)}, a_3^{(1)}, \ldots, a_{n-1}^{(1)}, a_n^{(1)}), \\
&\vdots \\
\mathbb{P}_n^{(1)} &= (a_2^{(1)}, a_3^{(1)}, a_4^{(1)}, \ldots, a_n^{(1)}, a_1^{(1)}).
\end{align*}
\]
Such that $a_1^{(1)} + a_2^{(1)} + \ldots + a_n^{(1)} = 1$, and where
\[
\begin{align*}
a_1^{(1)} &= \frac{1}{z^{(0)}}a_1^{(0)}(1 - a_2^{(0)})(1 - a_3^{(0)})\ldots(1 - a_n^{(0)}), \\
a_2^{(1)} &= \frac{1}{z^{(0)}}a_2^{(0)}(1 - a_1^{(0)})(1 - a_3^{(0)})\ldots(1 - a_n^{(0)}), \\
&\vdots \\
a_n^{(1)} &= \frac{1}{z^{(0)}}a_n^{(0)}(1 - a_1^{(0)})(1 - a_2^{(0)})\ldots(1 - a_{n-1}^{(0)}).
\end{align*}
\]
Here, $z_1^{(0)} = z_2^{(0)} = \ldots = z_m^{(0)} = z^{(0)}.$

Now, denoted $R_{11}^{(1)} = a_1^{(0)}/a_1^{(N)}$ and $R_{11}^{(N)} = a_1^{(N)}/a_1^{(N)}$ for all $N \geq 1$. We get
\[
R_{11}^{(1)} = \frac{a_1^{(1)}}{a_1^{(1)}} = \frac{1}{z^{(0)}}a_1^{(0)}(1 - a_2^{(0)})(1 - a_3^{(0)})\ldots(1 - a_n^{(0)}).
\]
Thus, we get $R_{11}^{(1)} > R_{11}^{(0)}$ since $k_{11}^{(0)} > 1$.

$$R_{12}^{(1)} = \frac{a_2^{(1)}}{a_1^{(1)}} = \frac{1}{z(0)} \frac{a_2^{(0)} (1 - a_1^{(0)}) (1 - a_2^{(0)}) \cdots (1 - a_n^{(0)})}{a_1^{(0)} (1 - a_2^{(0)}) (1 - a_3^{(0)}) \cdots (1 - a_n^{(0)})}$$

Thus, we get $R_{12}^{(1)} < R_{12}^{(0)}$ since $k_{12}^{(0)} < 1$.

$$R_{1n}^{(1)} = \frac{a_n^{(1)}}{a_{n-1}^{(1)}} = \frac{1}{z(0)} \frac{a_n^{(0)} (1 - a_1^{(0)}) (1 - a_2^{(0)}) \cdots (1 - a_{n-1}^{(0)})}{a_{n-1}^{(0)} (1 - a_n^{(0)}) (1 - a_{n-2}^{(0)}) \cdots (1 - a_1^{(0)})}$$

Thus we get $R_{1n}^{(1)} < R_{1n}^{(0)}$ since $k_{1n}^{(0)} < 1$.

Similarly, by computing

$$R_{21}^{(1)}, \ldots, R_{22}^{(1)}, \ldots, R_{nn}^{(1)}$$

we get $0 < a_n^{(1)} < a_{n-1}^{(1)} < \ldots < a_1^{(1)} < 1$.

Similarly, by induction we can get $0 < a_n^{(N-1)} < a_{n-1}^{(N-1)} < \ldots < a_1^{(N-1)} < 1$.

Now by induction, we get

$$R_{11}^{(N)} = \frac{a_1^{(N)}}{a_n^{(N)}} = \frac{a_1^{(N-1)}}{a_n^{(N-1)}} \cdot \frac{(1 - a_n^{(N-1)})}{(1 - a_1^{(N-1)})}$$

$$= R_{11}^{(N-1)} \cdot k_{11}^{(N-1)} = R_{11}^{(0)} \cdot k_{11}^{(0)} \cdot k_{11}^{(1)} \cdots k_{11}^{(N-1)}$$

(5)

where, $k_{11}^{(N)} = \frac{(1 - a_n^{(N-1)})}{(1 - a_1^{(N-1)})} > 1$ for all $N$ and since for each iteration $a_1^{(N)}$ is monotonically increasing and $a_n^{(N)}$ is monotonically decreasing, thus we get $K_{11}^{(N)}$ is monotonically increasing, i.e.,

$$1 < k_{11}^{(0)} < k_{11}^{(1)} < \ldots < k_{11}^{(N)} < \ldots$$

(6)
Thus, by (5) and (6) we get

$$K_{11}^{(N)} = \frac{a_1^{(N)}}{a_n^{(N)}} \to \infty; \quad \text{as } N \to \infty.$$  

This yields \(a_n^{(N)} \to 0\) since \(a_1^{(N)} < 1\). Similarly we get \(a_2^{(N)}, a_3^{(N)}, \ldots, a_{n-1}^{(N)} \to 0\).

Thus, the limiting vectors are

\[
P_1^{(1)} = (1, 0, 0, \ldots, 0),
\]

\[
P_2^{(1)} = (0, 1, 0, \ldots, 0),
\]

\[\vdots \]

\[
P_n^{(1)} = (0, 0, 0, \ldots, 1).
\]

In the case, \(P_1^{(0)} = P_2^{(0)} = \ldots = P_n^{(0)}\), suppose \(P_1^{(0)} = (p_{11}^{(1)}, p_{12}^{(1)}, \ldots, p_{1n}^{(1)})\), \(P_2^{(0)} = (p_{21}^{(1)}, p_{22}^{(1)}, \ldots, p_{2n}^{(1)})\), \ldots, \(P_n^{(0)} = (p_{n1}^{(1)}, p_{n2}^{(1)}, \ldots, p_{nn}^{(1)})\). It is easy to show that if \(0 < p_{1j}^{(1)} = p_{2j}^{(1)} = \ldots = p_{nj}^{(1)} < 1\) then \(0 < p_{1i}^{(N)} = p_{2i}^{(N)} = \ldots = p_{ni}^{(N)} < 1\) for all \(N\). Since by (3), \(P_1^{(N)} = P_2^{(N)} = \ldots = P_n^{(N)}\) if and only if,

\[
p_{1j}^{(N-1)} (1 - p_{2j}^{(N-1)}) \ldots (1 - p_{nj}^{(N-1)})
\]

\[
= p_{2j}^{(N-1)} (1 - p_{1j}^{(N-1)}) \ldots (1 - p_{nj}^{(N-1)}) = \ldots
\]

\[
= p_{nj}^{(N-1)} (1 - p_{1j}^{(N-1)}) \ldots (1 - p_{n-1,j}^{(N-1)}).
\]

i.e., if and only if, \(P_1^{(N-1)} = P_2^{(N-1)} = \ldots = P_n^{(N-1)}\).

Thus, if \(p_{1j} = p_{2j} = \ldots = p_{nj}\) then \(p_{1j}^{(N)} = p_{2j}^{(N)} = \ldots = p_{nj}^{(N)}\) for all \(N\). i.e., non-zero limit appear \(p_{1j}^{(\infty)} = p_{2j}^{(\infty)} = \ldots = p_{nj}^{(\infty)} \neq 0\). From

\[
\|P_1^{(\infty)}\| = \|P_2^{(\infty)}\| = \ldots = \|P_n^{(\infty)}\| = 1
\]

where,

\[
\|P^{(\infty)}\| := p_{11}^{(\infty)} + p_{22}^{(\infty)} + \ldots + p_{nn}^{(\infty)} = 1.
\]

We get \(p_{ij}^{(\infty)} = \frac{1}{n}\). Thus the limiting vectors are \(P_i^{(\infty)} = \left(\frac{1}{n}, \frac{1}{n}, \ldots, \frac{1}{n}\right)\). \(\square\)

Now, the following computer experimental results show how segregation appear due to conflict.
3. Results

1. \( M^{(0)} = B \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ 0.4 & 0.35 & 0.25 \\ 0.25 & 0.4 & 0.35 \\ 0.35 & 0.25 & 0.4 \end{pmatrix} \quad M^{(\infty)} = B \begin{pmatrix} \omega_1 & \omega_2 & \omega_3 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \)
2. $M^{(0)} = \begin{bmatrix} 0.2 & 0.5 & 0.3 \\ 0.2 & 0.5 & 0.3 \\ 0.2 & 0.5 & 0.3 \end{bmatrix}$

$M^{(\infty)} = \begin{bmatrix} \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & \omega_2 & \omega_3 \\ \omega_1 & \omega_2 & \omega_3 \end{bmatrix}$

$B$

$W$

$C$

$\forall i$
3. \[ M^{(0)} = \begin{bmatrix} 0.4 & 0.4 & 0.2 \\ 0.4 & 0.5 & 0.1 \\ 0.3 & 0.1 & 0.6 \end{bmatrix} \]

\[ M^{(\infty)} = \begin{bmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} \]
4. \( M^{(0)} = \begin{pmatrix} B & 0.25 & 0.24 & 0.24 & 0.27 \\ W & 0.23 & 0.26 & 0.25 & 0.26 \\ C & 0.24 & 0.24 & 0.26 & 0.26 \end{pmatrix} \)

\( M^{(\infty)} = \begin{pmatrix} B & 0.68 & 0 & 0 & 0.32 \\ W & 0 & 1 & 0 & 0 \\ C & 0 & 0 & 1 & 0 \end{pmatrix} \)
5. $M^{(0)} = \begin{bmatrix}
\omega_1 & 0.2 & 0.18 & 0.21 & 0.17 & 0.24 \\
\omega_2 & 0.19 & 0.2 & 0.18 & 0.21 & 0.22 \\
\omega_3 & 0.18 & 0.19 & 0.22 & 0.22 & 0.21 \\
\omega_4 & 0.18 & 0.19 & 0.22 & 0.22 & 0.21 \\
\omega_5 & 0.18 & 0.19 & 0.22 & 0.22 & 0.21 \\
\end{bmatrix}$

$M^{(\infty)} = \begin{bmatrix}
\omega_1 & 0.25 & 0 & 0 & 0 & 0.75 \\
\omega_2 & 0 & 0.34 & 0.66 & 0 & 0 \\
\omega_3 & 0 & 0 & 1 & 0 & 0 \\
\omega_4 & 0 & 0 & 0 & 1 & 0 \\
\omega_5 & 0 & 0 & 0 & 0 & 1 \\
\end{bmatrix}$
4. Discussion

In our simulation results $M^{(0)}$ is the initial matrix where row vectors represent the distribution of each races, i.e. white, black and Chinese denoted by $W$, $B$ and $C$, respectively. $\omega_1, \omega_2, \ldots$, represents the districts of a city. Here all three races moving to occupy these districts, thus the conflict appear. Here $M^{(\infty)}$ gives the convergent or equilibrium matrix.

In result 1, we observe that black people are the largest in district $\omega_1$. After each conflict interaction they increase gradually. At stage 9 it reaches equilibrium. In district $\omega_2$ and $\omega_3$ black people decrease gradually. At stage 11th it reaches equilibrium to 0. By this way black, white and Chinese are segregated into $\omega_1$, $\omega_2$ and $\omega_3$, respectively.

In result 2, all three races have same distribution in each district. Here we observe that after little fluctuation their distributions reach equilibrium and they are uniform distribution. Here they do not segregate.

In result 3, in district $\omega_1$ the distribution of black and white people are same but in district $\omega_2$ white people are the largest thus they are segregated in district $\omega_2$ and black people are segregated to $\omega_1$. In district $\omega_3$ Chinese people are the largest and they are segregated to $\omega_3$.

In result 4, there are three races and four districts. Here in district $\omega_1$ and $\omega_4$ black segregated to $\omega_1$ and $\omega_4$. In district $\omega_2$ and $\omega_3$ white and Chinese people are the largest, respectively. After 21st conflict interaction they reach equilibrium and they segregated in district $\omega_2$ and $\omega_3$, respectively.

In result 5 there are three races and five districts. Here in district $\omega_1$ and $\omega_5$ black people are the largest. After 22nd conflict interaction they reach equilibrium and black are segregated to $\omega_1$ and $\omega_5$. In district $\omega_2$ and $\omega_4$ white people are the largest. After 22nd conflict interaction they reach equilibrium and white people are segregated to $\omega_2$ and $\omega_4$. In district $\omega_3$ Chinese people are the largest. After 22nd conflict interaction they reach equilibrium and Chinese are segregated to $\omega_3$.

The equilibrium distribution depends on initial distribution of each races. If there is a small change in the initial distribution the result would differ. Thus because of conflict among races they are segregated into different districts.

Usually people get separated in many ways. There is segregation by race, religion and opinion. Age and taste are also the motive of
segregation. Some segregation results from their language. It is observed in
the society that because of those differences there is a conflict among
them and these kinds of conflict interaction produce segregation in our
society. We have dealt with the segregation through conflict in [5].

Thomas Schelling proposed segregation model, [4], where he demon-
strated that how a city with different races, initially highly diversified,
might tip into a highly segregated city through individuals’ move-
ments. That is, he demonstrated how macro-behavior appears through
micro-motives i.e. individuals’ choices.

Our framework differs from Schelling’s segregation model in several
respects. Unlike Schelling’s model we do not suppose the individuals’
choices. Our model is semi-macroscopic model i.e. here we suppose
group’s choice.

There are some previous works, [1], on conflict theory from game
theoretical point of view. Our framework also differs from traditional
game theory. In our model limiting distribution depends on initial
distribution and here we use stochastic dynamics which produce conflict
among groups. Since the dynamic process is buffeted by random pertur-
bations that arise from a variety of factors, such as exogenous shocks or
unpredictability in people’s behavior. These shocks play a role similar
to that of mutations in biology by constantly testing the viability of the
current regime.

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