The sum of a linear and a linear fractional transportation problem with restricted and enhanced flow

Archana Khurana *
Department of Mathematics
University of Delhi
Delhi 110 007
India

S. R. Arora †
Department of Mathematics
Hans Raj College
University of Delhi
Delhi 110 007
India

Abstract
In this paper, a transportation problem with an objective function as the sum of a linear and linear fractional function is considered. We present an algorithm to solve the above transportation problem. In addition, we consider two special cases of the problem, where the transportation flow is either restricted or enhanced. The first case deals with when one wishes to keep reserve stocks at the sources for emergencies, thereby restricting the total flow to a known specified level. The second case deals with when extra demand in the market compels some of the factories to increase their production, thereby enhancing the total flow to meet the customer demands fully. For the above special cases, we formulate a related linear plus linear fractional transportation problem. We present a numerical example to illustrate the proposed algorithm for the different cases. We show that the solution obtained is a local minimum occurring at an extreme point of the convex set of feasible solutions.

Keywords: Transportation problem, linear plus linear fractional, enhanced flow, restricted flow, local optimal solution and corner feasible solution.

*E-mail: archana2106@rediffmail.com
†E-mail: srarora@yahoo.com

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1. Introduction

Consider the following linear plus linear fractional optimization problem:

\[
\text{(P1)} \quad \sup Q(x) = a^t x + \frac{b^t x}{c^t x} \tag{1}
\]

subject to \( Ax = d \) \tag{2}

\( x \geq 0 \)

where \( A \) is an \( m \times n \) matrix, \( x, a, b, c \) are \( n \times 1 \) vectors, \( d \) is an \( m \times 1 \) vector, and \( t \) is the transpose of a vector. Further it is assumed that \( a \neq 0 \), \( b \neq 0 \); \( b \) and \( c \) are linearly independent vectors, \( c^t x > 0 \) and the set \( S = \{ x/Ax = b, x \geq 0 \} \subset \mathbb{R}^n \) is non-empty and bounded.

The linear plus linear fractional optimization problems do exist especially when a compromise between absolute and relative terms is to be maximized (Teterav (1969)). For example, the above problem arises when one wishes to maximize the linear combination of income and profitability.

Schaible (1977) investigated the problem in terms of quasi-convexity and quasi-concavity. He posed two assertions.

(a) A local maximum is a global maximum.

(b) A local maximum is attained at an extreme point of \( S \).

Assertion (a) is essentially equivalent to \( Q(x) \) being quasi-concave on \( S \), and (b) is essentially equivalent to \( Q(x) \) being quasi-convex on \( S \). He remarked that for a limited class of problems of type (P1), local optima have one of the properties (a) or (b). Further, (a) is true if \( a^t x \leq 0 \) on \( S \), and (b) is true if \( a^t x \geq 0 \) on \( S \).

Misra and Das (1981) discussed three dimensional linear plus linear fractional transportation problem. In this paper, we attempt to consider linear plus linear fractional transportation problem with the special cases of restricted and enhanced flows. We form a related transportation problem in both the cases and show that the solution obtained is a local minimum and occurs at an extreme point of the convex set of feasible solutions. In what follows, Section 1 presents an algorithm to solve a linear plus linear fractional transportation problem. Section 2 deals with the case of restricted flow and Section 3 with that of enhanced flow. Finally, Section 4 presents a numerical example.
2. Problem formulation

Consider the following transportation problem

\[(P2) \quad \text{Minimize} \quad Z = \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ij} x_{ij} + \sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij} x_{ij} \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} x_{ij} \quad (3)\]

subject to \(\sum_{j=1}^{n} x_{ij} = a_i, \quad i = 1, 2, \ldots, m\) \( (4)\)

\(\sum_{i=1}^{m} x_{ij} = b_j, \quad j = 1, 2, \ldots, n\) \( (5)\)

\(x_{ij} \geq 0, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n\) \( (6)\)

\(\sum_{i=1}^{m} a_i = \sum_{j=1}^{n} b_j\) \( (7)\)

\(a_i > 0, \quad b_j > 0, \quad i = 1, 2, \ldots, m; \quad j = 1, 2, \ldots, n\) \( (8)\)

where \(\sum_{i=1}^{m} \sum_{j=1}^{n} r_{ij} x_{ij} \geq 0, \quad \sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij} x_{ij} \geq 0, \quad \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij} x_{ij} > 0\) and \(r_{ij} \geq 0\)

= amount transported from the \(i\)th origin to the \(j\)th destination,

= amount available at the \(i\)th origin, and

= demand of \(j\)th destination.

Define the index sets \(I = \{1, 2, \ldots, m\}, J = \{1, 2, \ldots, n\}\) and \(K = I \times J\).

Call \(X = \{x_{ij}/(i,j) \in K, x_{ij} \text{ satisfies the constraints } (4)-(6)\}\), a feasible solution to the problem (P2).

**Determination of local optimum criterion for problem (P2)**

An initial basic feasible solution to problem (P2) can be obtained by using any of the known methods for usual transportation problems. Denote \(I_x = \{(i,j) \in K/x_{ij} > 0, x_{ij} \in X\}\), the set of non-degenerate basic cells.

Due to constraint (7) each non-degenerate basic solution will contain \((m+n-1)\) positive components.

Now we consider the dual variables for \(u^1_i, u^2_i, u^3_i\) for \(i \in I\) and \(v^1_j, v^2_j, v^3_j\) for \(j \in J\) such that \(u^1_i + v^1_j = r_{ij}\) and \(u^2_i + v^2_j = s_{ij}\) and \(u^3_i + v^3_j = t_{ij}\) for \((i,j) \in I_x\).
Also let
\begin{align*}
    r'_{ij} &= r_{ij} - (u_1^i + v_1^j) \\
    s'_{ij} &= s_{ij} - (u_2^i + v_2^j) \\
    t'_{ij} &= t_{ij} - (u_3^i + v_3^j)
\end{align*}
for \((i, j) \in K - I_x\) \hspace{1cm} (9)

The system (9) has \((m + n - 1)\) equations and can be solved independently.

If \(x_{11}\) is a basic variable then we can arbitrarily set \(u_1^1 = 0, u_2^1 = 0, u_3^1 = 0\) and solve for other dual variables. Having determined the dual variables \(u_1^i, u_2^i, u_3^i\) or \(i \in I; v_1^j, v_2^j, v_3^j\) for \(j \in J\), we shall use these values to determine \(r'_{ij}, s'_{ij}, t'_{ij}\) for all non-basic variables.

Let \(X^* = \{x_{ij}^*\}\) be any basic feasible solution of problem (P2).

To establish the optimal criterion, we express \(Z\) in terms of non-basic variables only.

Now
\begin{align*}
    \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ij}x_{ij} &= \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ij}x_{ij} + \sum_{i=1}^{m} \left( a_i - \sum_{j \in J} x_{ij} \right) u_1^i \\
    &\quad + \sum_{j=1}^{n} \left( b_j - \sum_{i \in I} x_{ij} \right) v_1^j \\
    &= \sum_{i=1}^{m} \sum_{j=1}^{n} (r_{ij} - u_1^i - v_1^j)x_{ij} + \sum_{i=1}^{m} a_i u_1^i + \sum_{j=1}^{n} b_j v_1^j \\
    &= \sum_{(i,j) \in K - I_x} r'_{ij}x_{ij} + V_1
\end{align*}

where \(V_1 = \sum_{i=1}^{m} a_i u_1^i + \sum_{j=1}^{n} b_j v_1^j\).

Similarly we can also write
\begin{align*}
    \sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij}x_{ij} &= \sum_{(i,j) \in K - I_x} s'_{ij}x_{ij} + V_2
\end{align*}

where \(V_2 = \sum_{i=1}^{m} a_i u_2^i + \sum_{j=1}^{n} b_j v_2^j\), and

\begin{align*}
    \sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij}x_{ij} &= \sum_{(i,j) \in K - I_x} t'_{ij}x_{ij} + V_3
\end{align*}

where \(V_3 = \sum_{i=1}^{m} a_i u_3^i + \sum_{j=1}^{n} b_j v_3^j\).
Thus the objective function $Z$ becomes

$$Z = \left( \sum_{(i,j) \in K-I_x} r'_{ij} x_{ij} + V_1 \right) + \frac{\left( \sum_{(i,j) \in K-I_x} s'_{ij} x_{ij} + V_2 \right)}{\left( \sum_{(i,j) \in K-I_x} t'_{ij} x_{ij} + V_3 \right)}.$$  

For $(i_0, j_0) \in K - I_x$, we evaluate $\frac{\partial Z}{\partial x_{i_0 j_0}}$ for any non-basic variable $x_{i_0 j_0}$ to obtain

$$\frac{\partial Z}{\partial x_{i_0 j_0}} = r'_{i_0 j_0} + \frac{s'_{i_0 j_0} \left( \sum_{(i,j) \in K-I_x} t'_{i_0 j_0} x_{i_0 j_0} + V_3 \right)}{\left( \sum_{(i,j) \in K-I_x} t'_{i_0 j_0} x_{i_0 j_0} + V_3 \right)^2} - \frac{t'_{i_0 j_0} \left( \sum_{(i,j) \in K-I_x} s'_{i_0 j_0} x_{i_0 j_0} + V_2 \right)}{\left( \sum_{(i,j) \in K-I_x} t'_{i_0 j_0} x_{i_0 j_0} + V_3 \right)^2}.$$  

$$(\frac{\partial Z}{\partial x_{i_0 j_0}})_{x^*=(x'_{ij})} = r'_{i_0 j_0} + \frac{s'_{i_0 j_0} V_3 - t'_{i_0 j_0} V_2}{(V_3)^2}.$$  

Denote $R_{ij} = r'_{ij} (V_3)^2 + s'_{ij} V_3 - t'_{ij} V_2$.

The determination of dual variables $u^1_i, u^2_i, u^3_i$ for $i \in I$; $v^1_j, v^2_j, v^3_j$ for $j \in J$ would facilitate the calculation of $R_{ij}$ for non-basic variables. The solution $X^*$ can be improved if $\exists R_{ij} < 0$ for at least one non-basic variable $x_{ij}$.

**Theorem 1.1.** A basic feasible solution $X^* = (x^*_{ij})$ is a local optimum solution of problem (P2) if $R_{ij} = r'_{ij} (V_3)^2 + s'_{ij} V_3 - t'_{ij} V_2 \geq 0 \forall$ non-basic variables $x_{ij}$.

If there exists one or more $R_{ij} < 0$, then we choose $R_{i_0 j_0} = \min \{ R_{ij} | R_{ij} < 0 \}$ and introduce the non-basic variable $x_{i_0 j_0}$ into the basis thereby improve the value of $Z$. The variable which leaves the basis and the value of the basic variable in the basis can be determined as usual.

3. **Restricted flow in a transportation problem**

If in the problem (P2) the total availability is not equal to the total demand then some of the source and/or destination constraints are satisfied as inequations. Sometimes, situations arise when one wishes to
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keep reserve stocks at the sources for emergencies, thereby restricting
the total transportation flow to a known specified level, say $P < \min (\sum a_i, \sum b_j)$. This flow constraint changes the structure of the trans-
portation problem.

The resulting linear plus linear fractional transportation problem with
restriction on the flow is

\[
\text{(P3) Minimize } \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ij} x_{ij} + \frac{m}{\sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij} x_{ij}} \\
\text{subject to } \sum_{j=1}^{n} x_{ij} \leq a_i \quad \forall \ i \in I \\
\sum_{i=1}^{m} x_{ij} \leq b_j \quad \forall \ j \in J \\
\sum \sum x_{ij} = P \quad \text{where } P < \min (\sum a_i, \sum b_j) \\
x_{ij} \geq 0 \quad \forall \ i \in I, \ j \in J.
\]

This flow constraint in the problem (P3) implies that a total $(\sum_{i \in I} a_i - P)$
of the source reserves has to be kept at the various sources and a
total $(\sum_{j \in J} b_j - P)$ of destination slacks is to be retained at the various
destinations. Therefore an extra destination to receive the source reserves
and an extra source to fill up the destination slacks are introduced.

Hence the related linear plus linear fractional transportation problem (P4)
associated with problem (P3) is

\[
\text{(P4) Minimize } \sum_{i=1}^{m+1} \sum_{j=1}^{n+1} r_{ij} y_{ij} + \frac{m+1}{\sum_{i=1}^{m+1} \sum_{j=1}^{n+1} s_{ij} y_{ij}} \\
\text{subject to } \sum_{j=1}^{n+1} y_{ij} = a^1_i \quad \forall \ i \in I^1 = I \cup \{m + 1\} \\
\sum_{i=1}^{m+1} y_{ij} = b^1_j \quad \forall \ j \in J^1 = J \cup \{n + 1\} \\
\sum_{i \in I^1} \sum_{j \in J^1} y_{ij} = P \\
y_{ij} \geq 0 \quad \forall \ i \in I^1, \ j \in J^1
\]
where \( a^i_l = a_i, \, i \in I \), \( a^m_{l+1} = \left( \sum_{j \in J} b_j - P \right) \); \( b^j_j = b_j, \, j \in J \), \( b^n_{n+1} = \left( \sum_{i \in I} a_i - P \right) \);

\[ r^1_{i,j} = r_{i,j}, \quad s^1_{i,j} = s_{i,j}, \quad t^1_{i,j} = t_{i,j}, \quad (i,j) \in I \times J; \]

\[ r^1_{i,n+1} = s^1_{i,n+1} = t^1_{i,n+1} = 0, \quad i \in I; \quad r^1_{m+1,j} = s^1_{m+1,j} = t^1_{m+1,j} = 0, \quad j \in J; \]

\[ r^1_{m+1,n+1} = s^1_{m+1,n+1} = t^1_{m+1,n+1} = M \] where \( M \) is a large positive number.

**Definition (Corner feasible solution).** A basic feasible solution \( \{y_{ij}\} \), \( i \in I^1, \, j \in J^1 \) to problem (P4) is called a corner feasible solution (cfs) if

\[ y^i_{m+1,j+1} = a_i - \sum_{j \in J} x_{ij}, \quad i \in I; \]

\[ y^m_{j,n+1} = b_j - \sum_{i \in I} x_{ij}, \quad j \in J; \quad y^m_{m+1,n+1} = 0 \]

can be shown to be a cfs to problem (P4).

**Theorem 2.1.** Every corner feasible solution of problem (P4) provides a basic feasible solution of problem (P3) and conversely.

**Proof.** Let \( \{y_{ij}\} \) be a cfs to problem (P4).

Define \( \{x_{ij}\} = \{y_{ij}\}, \, (i,j) \in I \times J \).

\( \{x_{ij}\} \) so defined can be established to be a basic feasible solution to problem (P3).

Conversely, given \( \{x_{ij}\} \) to be a basic feasible solution to (P3) then \( \{y_{ij}\}, \, (i,j) \in I^1 \times J^1 \) defined by the transformation

\[ y_{ij} = x_{ij}, \quad (i,j) \in I \times J \]

\[ y^i_{i+n+1} = a_i - \sum_{j \in J} x_{ij}, \quad i \in I; \]

\[ y^m_{j,n+1} = b_j - \sum_{i \in I} x_{ij}, \quad j \in J; \quad y^m_{m+1,n+1} = 0 \]

can be shown to be a cfs to problem (P4).

**Remark 2.1.** The value of objective function of problem (P4) at a corner feasible solution is equal to the value of the objective function of problem (P3) at its corresponding basic feasible solution.

**Remark 2.2.** A non-corner feasible solution to problem (P4) cannot provide a feasible solution to problem (P3).

**Theorem 2.2.** An optimal solution to problem (P4) has to be a feasible solution to problem (P3).

**Proof.** If possible, let \( \exists \) an optimal solution \( \{y'_{ij}\} \) to problem (P4) which is not a cfs. The optimal cost corresponding to \( \{y'_{ij}\} \) be \( Z' \), this contains \( M \) a large positive number. Now consider an optimal solution \( \{x'_{ij}\} \) say to problem (P3) with corresponding optimal cost \( Z^0 \). Let \( \{y'_{ij}\} \) be
the corresponding cfs to problem (P4). The cost corresponding to the cfs \(\{y_{ij}^0\}\) is also \(Z^0\). Clearly \(Z^0 < Z'\) which contradicts the fact that \(\{y_{ij}'\}\) is an optimal solution to (P4).

Hence no non-cfs to problem (P4) can be an optimal solution.

**Remark 2.3.** There is a one to one correspondence between optimal solution to problem (P3) and the optima among the corner feasible solution to problem (P4).

### 4. Enhanced flow in a transportation problem

Sometimes situations may arise when because of the extra demand in the market the total flow needs to be enhanced, compelling some of the factories to increase their productions in order to be able to meet this extra demand. The total flow from the factories in the market is now increased by the amount of extra demand.

Let \(P > (\sum a_i = \sum b_j)\) be the enhanced flow. The linear plus linear fractional transportation problem with extra flow is

\[
\begin{align*}
\text{(P5) Minimize } & \sum_{i=1}^{m} \sum_{j=1}^{n} r_{ij}x_{ij} + \frac{\sum_{i=1}^{m} \sum_{j=1}^{n} s_{ij}x_{ij}}{\sum_{i=1}^{m} \sum_{j=1}^{n} t_{ij}x_{ij}} \\
\text{subject to } & \sum_{j=1}^{n} x_{ij} \geq a_i \quad \forall \ i \in I \\
& \sum_{i=1}^{m} x_{ij} \geq b_j \quad \forall \ j \in J \\
& \sum_{i=1}^{m} \sum_{j=1}^{n} x_{ij} = P, \quad \text{where } P > (\sum a_i = \sum b_j) \\
& x_{ij} \geq 0 \quad \forall \ i \in I, \ j \in J.
\end{align*}
\]

Due to the flow constraint in the problem (P5), a fictitious factory with availability equal to \((P - \sum_{j \in J} b_j)\) and a fictitious destination with demand equal to \((P - \sum_{i \in I} a_i)\) is added.

Hence the related linear plus linear fractional transportation problem (P6) with extra flow associated with problem (P5) is

\[
\begin{align*}
\text{(P6) Minimize } & \sum_{i \in I^2} \sum_{j \in J^2} r_{ij}^2z_{ij} + \frac{\sum_{i \in I^2} \sum_{j \in J^2} s_{ij}^2z_{ij}}{\sum_{i \in I^2} \sum_{j \in J^2} t_{ij}^2z_{ij}} \\
\end{align*}
\]
subject to \[ \sum_{j \in J} z_{ij} = a_i^2 \quad \forall \, i \in I_2 = I \cup \{m + 1\} \]
\[ \sum_{i \in I} z_{ij} = b_j^2 \quad \forall \, j \in J_2 = J \cup \{n + 1\} \]
\[ z_{ij} \geq 0 \quad \forall \, i \in I_2, \, j \in J_2 \]

where \[ a_i^2 = a_i, \, i \in I, \, a_{m+1}^2 = \left( \sum_{j \in J} b_j - P \right), \quad b_j^2 = b_j, \, j \in J, \, b_{n+1}^2 = \left( \sum_{i \in I} a_i - P \right) \];
\[ r_{ij} = r_{ij}, \, (i, j) \in I \times J; \quad s_{ij}^2 = s_{ij}, \, (i, j) \in I \times J; \quad t_{ij}^2 = t_{ij}, \, (i, j) \in I \times J \]
\[ r_{i,n+1}^2 = r_{ik}, \, s_{i,n+1}^2 = s_{ik} \] and \[ t_{i,n+1}^2 = t_{ik} \] such that \[ r_{ik} + S_{ik} t_{ik} = \min_{j} \left( r_{ij} + S_{ij} t_{ij} \right) \].

Also \[ r_{m+1,i,j}^2 = r_{ij}, \, s_{m+1,i,j}^2 = s_{ij} \] and \[ t_{m+1,i,j}^2 = t_{ij} \] such that \[ r_{ij} + \frac{s_{ij}}{t_{ij}} = \min_{i} \left( r_{ij} + \frac{s_{ij}}{t_{ij}} \right) \].
\[ r_{m+1,n+1}^2 = s_{m+1,n+1}^2 = t_{m+1,n+1}^2 = M \] where \( M \) is a larger positive number.

**Theorem 3.1.** A corner feasible solution to problem (P6) gives a feasible solution to problem (P5).

**Theorem 3.2.** Every feasible solution to problem (P5) corresponds to a corner feasible to problem (P6).

**Remark 3.1.** The value of the objective function of problem (P6) yielded by cfs \( \{y_{ij}\} \) is equal to the value of the objective function of problem (P5) yielded by its corresponding feasible solution \( \{x_{ij}\} \).

**Remark 3.2.** Optimal corner feasible solution to problem (P6) provides an optimal solution to problem (P5).

**Remark 3.3.** There is a one to one correspondence between optimal solution to problem (P5) and optima among corner feasible solutions to problem (P6).

5. **Numerical example**

Consider the linear plus linear fractional transportation problem with supply = 40, demand = 28.

Let the flow be restricted to 25, i.e.

\[ P = 25 < \left[ \min \left( \sum_{i=1}^{3} a_i = 40, \sum_{j=1}^{3} b_j = 28 \right) \right] . \]
We add an additional source and a destination with supply = 3 and demand = 15, respectively.

Also, $r_{14} = s_{14} = t_{14} = 0$; $r_{24} = s_{24} = t_{24} = 0$; $r_{41} = s_{41} = t_{41} = 0$; $r_{42} = s_{42} = t_{42} = 0$; $r_{43} = s_{43} = t_{43} = 0$; $r_{44} = s_{44} = t_{44} = M$, where $M$ is a large positive number.

Its optimal basic feasible solution is given by

\[
\{ y_{12} = 4, y_{14} = 15, y_{21} = 5, y_{24} = 5, y_{32} = 4, y_{33} = 7, y_{43} = 3 \}
\]

$V_1 = 95, V_2 = 137, V_3 = 140; Z = 95 + \frac{140}{140}.$

Here $R_{ij} \geq 0 \ \forall (i,j) \notin B$.

Thus optimal value of $Z = 95 + 0.9786 = 95.9786$.

Again consider the transportation problem with same $r_{ij}, s_{ij}$ and $t_{ij}$ with supply = demand = 40.

Let the flow be enhanced to 50, i.e.,

\[
P = 50 > \left( \sum_{i=1}^{3} a_i + \sum_{j=1}^{3} b_j = 40 \right).
\]

We add an additional source and an additional destination with supply = demand = 10.

Its optimal basic feasible solution is given by

\[
\{ y_{13} = 2, y_{14} = 10, y_{21} = 15, y_{23} = 2, y_{33} = 11, y_{42} = 10, y_{43} = 0 \}.
\]

Optimal Value of $Z = 182 + \frac{295}{267} = 182 + 1.1049 = 183.104$. 

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Table I
6. Conclusion

The solution obtained for a linear plus linear fractional transportation problem with restricted or enhanced flow is a local minimum occurring at an extreme point of the feasible set.

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