An eigenvalue approach to the immigration-catastrophe process

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Abstract

Using the Jordan block decomposition the transient probabilities for the finite state immigration-catastrophe process is obtained via matrix methods. It is shown that these transition probabilities converge to the transient probabilities of the infinite state immigration-catastrophe process.

Keywords: Immigration, catastrophe, Jordan block decomposition

1. Introduction

In Swift [3] the author derives the transient solution to a simple immigration-catastrophe process with countably infinite states using standard differential equation methods to solve the Kolomogorov forward equations. This paper obtains the same solution using convergence of the respective finite state model as the number of states goes to infinity. This approach makes use of the eigenvalues and respective eigenvectors of the rate matrix and utilizes these in a matrix exponential. In order to find the “diagonalized” matrix for the matrix equation the Jordan block decomposition is used. The Jordan block decomposition of a matrix into a block diagonal matrix provides a unique decomposition of the matrix up to permutation of the Jordan blocks.

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Consider a finite state immigration-catastrophe process on \( n \)-states with the following transitions:

- Rate transition: \( \alpha \) \( i \rightarrow i + 1 \), \( i = 0, \ldots, n - 1 \)
- Rate transition: \( \gamma \) \( i \rightarrow 0 \), \( i = 1, \ldots, n \).

Writing the forward Kolmogorov equations in matrix form for this process gives:

\[
P'(t) = QP(t)
\]

where the rate matrix is:

\[
Q = \begin{pmatrix}
-\alpha & \gamma & \gamma & \cdots & \gamma & \gamma \\
\alpha & -(\alpha + \gamma) & 0 & \cdots & 0 & 0 \\
0 & \alpha & -(\alpha + \gamma) & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \cdots & \alpha & -(\alpha + \gamma) & 0 \\
0 & 0 & 0 & \cdots & \alpha & -\gamma \\
\end{pmatrix}
\]

and the transition probability matrix \( P(t) \) is:

\[
P(t) = \{P_{ij}(t)\}_{i,j=1,...,n}.
\]

Using this matrix formulation of the Kolmogorov forward equation the solution takes the form of the matrix exponential. The matrix exponential is defined as follows (c.f. Gross and Harris [1])

\[
e^{Qt} = I + Qt + \frac{Q^2t^2}{2!} + \cdots + \frac{Q^nt^n}{n!} + \cdots,
\]

where \( Q^n \) is the \( n \)th matrix product of \( Q \). Using the definition of the matrix exponential it is easy to see that

\[
\frac{d}{dt}(e^{Qt}) = Qe^{Qt}.
\]

The difficulty lies in computing the matrix exponential. Using the form above, an approximate solution can be obtained, but an exact solution is desirable. For the moment, assume the matrix \( Q \) is diagonalizable. This will occur exactly when there are \( n \) linearly independent eigenvectors corresponding to \( n \) distinct eigenvalues. Let \( \Lambda \) be the diagonal matrix of \( Q \) with the eigenvalues on the main diagonal and let \( S \) be the corresponding matrix of eigenvectors with eigenvalues in \( \Lambda \) corresponding to their...
respective eigenvectors in \( S \). Then
\[
Q = SAS^{-1}
\]
and
\[
Q^n = S\Lambda^nS^{-1}.
\]
Thus
\[
e^{Qt} = I + Qt + \frac{Q^2t^2}{2!} + \cdots + \frac{Q^nt^n}{n!} + \cdots
\]
\[
= I + SAS^{-1}t + (SAS^{-1})^2\frac{t^2}{2!} + \cdots + (SAS^{-1})^n\frac{t^n}{n!} + \cdots
\]
\[
= I + SAS^{-1}t + SA^2S^{-1}\frac{t^2}{2!} + \cdots + S\Lambda^nS^{-1}\frac{t^n}{n!} + \cdots
\]
\[
= S \left( I + \Lambda t + \Lambda^2\frac{t^2}{2} + \cdots \right) S^{-1}
\]
\[
= Se^{\Lambda t}S^{-1}.
\]

With a diagonalizable \( Q \) matrix the solution will follow with simple application of matrix multiplication. In the case when the \( Q \) matrix is not diagonalizable, as in the immigration-catastrophe case considered here, this method of finding the solution is still applicable only with a different “\( \Lambda \)” matrix. The “\( \Lambda \)” matrix needed here is obtained from the rate matrix \( Q \) by finding the Jordan block decomposition of \( Q \).

Every square matrix \( A \) has a unique, up to a permutation of the Jordan blocks, Jordan block decomposition such that
\[
J = M^{-1}AM = \begin{pmatrix} J_1 & & \\ & \ddots & \\ & & J_s \end{pmatrix}
\]
where
\[
J_i = \begin{pmatrix} \lambda_i & 1 & 0 & \cdots \\ 0 & \ddots & \ddots & \cdots \\ 0 & \ddots & \ddots & 1 \\ 0 & \ddots & 0 & \lambda_i \end{pmatrix}
\]
are the Jordan blocks and \( A \) is a set of \( s \) linearly independent eigenvectors (c.f. Nering [2]). Finding the Jordan decomposition is difficult. However,
with the use of computer algebra systems the practitioner will be able to compute many of the modestly sized finite state models. The solution for the general case will be

\[ P(t) = Me^{J}M^{-1}u_0. \]

Here \( u_0 \) is the initial condition and is included in this solution as it will be used later.

Since the method presented in this article depends upon the Jordan form of a matrix, the details for finding the Jordan form will be outlined here. Further details can be found in Nering [2]. Consider the characteristic equation,

\[ f(x) = (x - \lambda_1)^{r_1} (x - \lambda_2)^{r_2} \cdots (x - \lambda_p)^{r_p}. \]

For each eigenvalue \( \lambda_i, i = 1, \ldots, p \), this factorization has \( \lambda_i \neq \lambda_j, \) for \( i \neq j \).

Thus for each \( i = 1, \ldots, p \) the algebraic multiplicity of the eigenvalue \( \lambda_i \) is \( r_i \). The minimum polynomial of the characteristic equation \( f \) is the polynomial

\[ m(x) = (x - \lambda_1)^{s_1} (x - \lambda_2)^{s_2} \cdots (x - \lambda_p)^{s_p} \]

of minimal degree such that the equation satisfies \( m(A) = 0 \). Using the minimal polynomial the geometric multiplicity for each eigenvalue \( \lambda_i \) is defined to be \( s_i \) for \( i = 1, \ldots, n \). The Jordan form will for each eigenvalue \( \lambda_i \) have at least one block of order \( s_i \), i.e. a matrix of dimension \( s_i \times s_i \). In addition, this \( s_i \) is the maximum order possible for the Jordan block associated with the eigenvalue \( \lambda_i \), \( i = 1, \ldots, n \). If the Jordan blocks associated with \( \lambda_i \) are denoted by \( J_{n_i}, i = 1, \ldots, m \) with \( J_{n_i} \) being of order \( n_i \), then

\[ \sum_{i=1}^{m} n_i = r_i, \]

the algebraic dimension. Thus each eigenvalue has at least one Jordan block of an order as given by the geometric multiplicity, and the sum of the orders of the Jordan blocks is equal to the algebraic dimension.

A knowledge of the algebraic and geometric multiplicity sometimes suffices in the computation of the Jordan canonical form of a matrix, but many times it does not and a more careful analysis is required. It should be noted that the geometric multiplicity requires the computation of the minimal polynomial of a matrix, a task in its own right. In the general case a detailed construction of the nullspaces of the \((A - \lambda I)^k \) operators is
required. For each of the unique eigenvalues of $A$, compute the sequence of nullspaces

$$\mathcal{M}^k = \{x : (A - \lambda I)^k x = 0\}.$$  

It can be shown that $\mathcal{M}^i \subseteq \mathcal{M}^{i+1}$ and since the operator is finite dimensional there is a minimum $s$ such that $\mathcal{M}^i = \mathcal{M}^{i+s}, i = 1, 2, \ldots$.

Suppose the basis of $\mathcal{M}^s$ is $\{\alpha_1, \ldots, \alpha_m\}$. Let $\{\alpha_{m_{i-1}+1}, \ldots, \alpha_m\}$ be the set of basis elements in $\mathcal{M}^s$ which are not in $\mathcal{M}^{s-1}$. Note that this means $\{\alpha_1, \ldots, \alpha_{m_{s-1}}\}$ is a basis of $\mathcal{M}^{s-1}$. For consistency sake, replace $\{\alpha_{m_{i-1}+1}, \ldots, \alpha_m\}$ by $\beta$’s according to $\alpha_{m_{i-1}+\nu} = \beta_{m_{i-1}+\nu}$. Now set

$$(A - \lambda I)(\beta_{m_{i-1}+\nu}) = \beta_{m_{i-1}+\nu}$$

and consider the set

$$\{\alpha_1, \ldots, \alpha_{m_{i-1}}\} \cup \{\beta_{m_{i-2}+1}, \ldots, \beta_{m_{i-2}+m_{i-1}-m_{i-1}}\}.$$  

These two sets can be shown to be linearly independent. Expand this linearly independent subset of $\mathcal{M}^{i-1}$ to a basis of $\mathcal{M}^{i-1}$. Use $\beta$’s to denote these additional elements of this basis, if any elements are required. Thus the new basis of $\mathcal{M}^{i-1}$ is

$$\{\alpha_1, \ldots, \alpha_{m_{i-2}}\} \cup \{\beta_{m_{i-2}+1}, \ldots, \beta_{m_{i-1}}\}.$$  

Now set

$$(A - \lambda I)(\beta_{m_{i-2}+\nu}) = \beta_{m_{i-3}+\nu}$$

and proceed as before to obtain the new basis

$$\{\alpha_1, \ldots, \alpha_{m_{i-3}}\} \cup \{\beta_{m_{i-3}+1}, \ldots, \beta_{m_{i-2}}\}$$

of $\mathcal{M}^{i-2}$. Proceeding in this manner a new basis

$$\{\beta_1, \ldots, \beta_{m_i}\}$$

of $\mathcal{M}^i$ is derived such that

$$\{\beta_1, \ldots, \beta_{m_i}\}$$

is a basis of $\mathcal{M}^k$ and

$$(A - \lambda I)(\beta_{m_k+\nu}) = \beta_{m_{k-1}+\nu} \text{ for } k \geq 1.$$  

This relation can be rewritten in the form

$$A(\beta_{m_k+\nu}) = \lambda \beta_{m_k+\nu} + \beta_{m_{k-1}+\nu} \text{ for } k \geq 1,$$
A(βν) = λβν for ν ≤ m1.

This suggests reordering the basis vectors so that \{β1, βm1+1, ..., βm1+1\} are listed first. Next list the vectors \{β2, βm1+2, ...\}, etc. The general idea is to list each of the first elements from each section of the β’s, then each of the second elements from each section, and continue until a new ordering of the basis is obtained. Repeat this for each eigenvalue and the combined blocks will give the matrix of eigenvectors S, then the Jordan canonical form will be J = S^{-1}AS.

2. A finite state space example

To illustrate the Jordan block method, a 5-state immigration-catastrophe process will be considered. The general case will be presented in the next section. Consider the 5-state immigration-catastrophe chain with rate matrix

\[
Q = \begin{pmatrix}
-\alpha & \gamma & \gamma & \gamma & \gamma \\
\alpha & -(\alpha + \gamma) & 0 & 0 & 0 \\
0 & \alpha & -(\alpha + \gamma) & 0 & 0 \\
0 & 0 & \alpha & -(\alpha + \gamma) & 0 \\
0 & 0 & 0 & \alpha & -\gamma
\end{pmatrix}.
\]

The characteristic matrix is

\[
(Q - \lambda I) = \begin{pmatrix}
-\alpha - \lambda & \gamma & \gamma & \gamma & \gamma \\
\alpha & -(\alpha + \gamma + \lambda) & 0 & 0 & 0 \\
0 & \alpha & -(\alpha + \gamma + \lambda) & 0 & 0 \\
0 & 0 & \alpha & -(\alpha + \gamma + \lambda) & 0 \\
0 & 0 & 0 & \alpha & -\gamma - \lambda
\end{pmatrix}
\]

which can be reduced by adding each row to the one above it, starting at row 5 and working up, to

\[
(Q - \lambda I) = \begin{pmatrix}
-\gamma & -\lambda & -\lambda & -\lambda & -\lambda \\
\alpha & -(\lambda + \gamma) & -(\lambda + \gamma) & -(\lambda + \gamma) & -(\lambda + \gamma) \\
0 & \alpha & -(\lambda + \gamma) & -(\lambda + \gamma) & -(\lambda + \gamma) \\
0 & 0 & \alpha & -(\lambda + \gamma) & -(\lambda + \gamma) \\
0 & 0 & 0 & \alpha & -(\lambda + \gamma)
\end{pmatrix}.
\]

It is not hard to see what the form of the characteristic matrix for the n-state immigration-catastrophe model will be looking at the matrix above. Also, by taking the determinant along the first column it is evident
that the determinant can be written for the $n$-state chain as well. The characteristic polynomial of the $Q$ matrix is

$$C(\lambda) = -\lambda(\lambda + \gamma + \alpha)^4$$

here and in the general $n$-state case

$$C(\lambda) = -\lambda(\lambda + \gamma + \alpha)^{n-1}.$$

Hence the $Q$ matrix has two eigenvalues $0$ and $-(\alpha + \gamma)$ of order 1 and $n-1$ respectively. Considering first the eigenvalue $0$, the characteristic matrix is

$$C(0) = \begin{pmatrix}
-\alpha & \gamma & \gamma & \gamma & \gamma \\
\alpha & -(\alpha + \gamma) & 0 & 0 & 0 \\
0 & \alpha & -(\alpha + \gamma) & 0 & 0 \\
0 & 0 & \alpha & -(\alpha + \gamma) & 0 \\
0 & 0 & 0 & \alpha & -\gamma \\
\end{pmatrix}$$

with the second matrix being a reduced form of the first derived by adding subsequent rows, that is, adding row 1 to row 2, then row 2 to row 3, etc. Thus the nullspace of this matrix is one dimensional and the eigenvector is

$$\left( (\gamma + \alpha)^3 \gamma, (\gamma + \alpha)^2 \gamma, (\gamma + \alpha) \gamma, \gamma, 1 \right).$$

The Jordan block associated to this eigenvector will be $[0]$, since the algebraic multiplicity is 1. The second eigenvalue $-(\gamma + \alpha)$ has the characteristic matrix

$$C(-(\gamma + \alpha)) = \begin{pmatrix}
\gamma & \gamma & \gamma & \gamma \\
\alpha & 0 & 0 & 0 \\
0 & \alpha & 0 & 0 \\
0 & 0 & \alpha & 0 \\
0 & 0 & 0 & \alpha \\
\end{pmatrix}$$

and again the nulls pace is one dimensional. With some calculation, the basis can be shown to be

$$(0, 0, 0, 1, -1).$$
The multiplicity of the eigenvalue \(- (\gamma + \alpha)\) is four, so generalized eigenvectors must be sought. The method is to first compute the nullspaces of \((A - \lambda I)^k\), denote by \(M^k\), for \(k = 1, 2, \ldots\), until the nullspaces stop increasing in dimension. The algebraic multiplicity of this eigenvalue assures us that no more than the fourth power will be necessary in this case.

\[
(A - (\gamma - \alpha)I)^2 = \begin{pmatrix}
\gamma^2 + \gamma\alpha & \gamma^2 + \gamma\alpha & \gamma^2 + \gamma\alpha & \gamma^2 + \gamma\alpha & \gamma^2 + \gamma\alpha \\
\gamma\alpha & \gamma\alpha & \gamma\alpha & \gamma\alpha & \gamma\alpha \\
\alpha^2 & 0 & 0 & 0 & 0 \\
0 & \alpha^2 & 0 & 0 & 0 \\
0 & 0 & \alpha^2 & \alpha^2 & \alpha^2 \\
\end{pmatrix}
\]

and the basis of the nullspace is

\[M^2 = \{(0, 0, -1, 0, 1), (0, 0, 1, 0, 0)\} \]

Continuing this process it can be shown that \((A - (\gamma - \alpha)I)^3\) has the nullspace

\[M^3 = \{(0, -1, 1, 0, 0), (0, -1, 0, 0, 1), (0, -1, 0, 1, 0)\} \]

and \((A - (\gamma - \alpha)I)^4\) has nullspace

\[M^4 = \{(-1, 0, 1, 0, 0), (-1, 0, 0, 1, 0), (-1, 1, 0, 0, 0), (-1, 0, 0, 1, 0)\} \]

The task of finding the appropriate matrix \(S\) is greatly simplified due to the increase of one dimension in each of the nullspaces. To find the appropriate eigenvectors and generalized eigenvectors start with the largest nullspace \(M^4\) and find the vectors not in the nullspace for \(M^3\). There are several choices possible and knowing which will lead to a simpler matrix \(S\) is guesswork. Naturally this means this matrix \(S\) is non-unique, being it depends on the initial selection. The choice here will be the vector

\[
\left( \frac{-1}{\alpha^3}, \frac{1}{\alpha^2}, 0, 0, 0 \right) = x_4.
\]
With $\lambda = -(\gamma + \alpha)$, find a vector in $x_3 \in \mathbb{M}^3$ such that

$$(\mathbf{A} - \lambda \mathbf{I})x_4 = x_3.$$  

In other words, a generalized eigenvector $\mathbf{A}x_4 = \lambda x_4 + x_3$.

In the case considered here

$$(\mathbf{A} - \lambda \mathbf{I})x_4 = \begin{pmatrix} 0, & -1, & \frac{1}{\alpha^2}, & 0, & 0 \end{pmatrix}^T = x_3,$$

$$(\mathbf{A} - \lambda \mathbf{I})x_3 = \begin{pmatrix} 0, & 0, & -1, & \frac{1}{\alpha}, & 0 \end{pmatrix}^T = x_2,$$

and

$$(\mathbf{A} - \lambda \mathbf{I})x_2 = (0, 0, -1, 1)^T = x_1.$$  

This is called a $p$-sequence of eigenvectors. The purpose in specifying a name for this sequence, in our case a 4-sequence, is that these sequences are precisely what determine the different Jordan blocks. Here there will be one Jordan block of order 4 for the eigenvalue $-(\gamma + \alpha)$ and one Jordan block of order 1 for the eigenvalue 0. Putting these together the matrix of "eigenvectors" becomes

$$\mathbf{S} = \begin{pmatrix}
\frac{\gamma(\gamma + \alpha)}{\alpha^3} & 0 & 0 & 0 & -\frac{1}{\alpha^3} \\
\frac{\gamma(\gamma + \alpha)}{\alpha^2} & 0 & 0 & -\frac{1}{\alpha} & \frac{1}{\alpha} \\
\frac{\gamma(\gamma + \alpha)}{\alpha} & 0 & -\frac{1}{\alpha} & \frac{1}{\alpha^2} \\
\frac{\gamma}{\alpha} & -1 & \frac{1}{\alpha} & 0 & 0 \\
1 & 1 & 0 & 0 & 0
\end{pmatrix}$$

and the Jordan canonical form for the matrix $\mathbf{A}$ is

$$\mathbf{J} = \mathbf{S}^{-1} \mathbf{A} \mathbf{S}$$

$$= \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & -\gamma - \alpha & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -\gamma - \alpha & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & -\gamma - \alpha & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -\gamma - \alpha & 1 & 0 & 0 & 0
\end{pmatrix}.$$
To find the transient solutions to the immigration-catastrophe process compute 

\[ P(t) = S e^{\Gamma t} S^{-1} \]

where

\[
e^{\Gamma t} =
\begin{pmatrix}
e^{0t} & 0 & 0 & 0 & 0 \\
0 & e^{-(\gamma + \alpha)t} & \frac{\alpha}{\gamma} e^{-(\gamma + \alpha)t} & \frac{\alpha^2}{\gamma^2} e^{-(\gamma + \alpha)t} \\
0 & 0 & e^{-(\gamma + \alpha)t} & \frac{\alpha}{\gamma} e^{-(\gamma + \alpha)t} \\
0 & 0 & 0 & e^{-(\gamma + \alpha)t} \\
0 & 0 & 0 & 0 & e^{-(\gamma + \alpha)t}
\end{pmatrix}
\]

but this will be considered in the general case, presented in the next section.

3. The general case

The general nth case of the Jordan canonical decomposition for the rate matrix of the immigration-catastrophe process will now be presented. Let the n-state matrix solution be denoted by

\[ P_n(t) = S_n e^{\Gamma_n t} S_n^{-1} \]

where \( S_n \) and \( \Gamma_n \) are \( n \times n \) matrices. Here the matrix of generalized eigenvectors takes the form

\[
S_n =
\begin{pmatrix}
\frac{\gamma}{\alpha} \left( \frac{\gamma + \alpha}{\alpha} \right)^{n-2} & 0 & \cdots & 0 & -1 \\
\frac{\gamma}{\alpha} \left( \frac{\gamma + \alpha}{\alpha} \right)^{n-3} & 0 & \cdots & 0 & \frac{1}{\alpha^{n-2}} \\
\vdots & \vdots & 0 & \sqrt{1} & 0 \\
\vdots & \vdots & 0 & \frac{1}{\alpha^{n-2}} & 0 \\
\frac{\gamma}{\alpha} \left( \frac{\gamma + \alpha}{\alpha} \right)^0 & -1 & \frac{1}{\alpha} & 0 & \vdots \\
1 & 1 & 0 & 0 & \cdots & 0
\end{pmatrix}_{n \times n}
\]

To simplify the presentation of the matrix product \( P_n(t) = S_n e^{\Gamma_n t} S_n^{-1} \), the following submatrices will be utilized

\[
B_{n-1} = \left( \frac{\gamma}{\alpha} \left( \frac{\gamma + \alpha}{\alpha} \right)^{n-2} & 0 & \cdots & 0 \right)_{n \times (n-1)}
\]

\[
C_{n-1} = \left[ \frac{-1}{\alpha^{n-2}} \right]_{1 \times 1}
\]
and
\[ A_{n-1} = \begin{pmatrix} 1 \alpha^{n-2} \\ 0 \\ \vdots \\ 0 \end{pmatrix}_{(n-1) \times 1}. \]

Then
\[ S_n = \begin{pmatrix} B_{n-1} & C_{n-1} \\ S_{n-1} & A_{n-1} \end{pmatrix}_{n \times n}. \]

The associated Jordan block matrix is
\[ \Lambda_n = \begin{pmatrix} J_0 & 0_{1 \times (n-1)} \\ 0_{1 \times 1} & J_{n-1} \end{pmatrix}_{n \times n}, \]

where
\[ J_0 = [0]_{1 \times 1} \]

and
\[ J_{n-1} = \begin{pmatrix} -\gamma - \alpha & 1 & 0 & \cdots & \cdots & 0 \\ 0 & -\gamma - \alpha & 1 & 0 & \cdots & 0 \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & \cdots & 0 & -\gamma - \alpha \end{pmatrix}_{(n-1) \times (n-1)}. \]

In the solution
\[ P_n(t) = S_n e^{\Lambda t} S_n^{-1} \]

the matrix \( e^{\Lambda t} \) can be written as
\[ e^{\Lambda t} = \begin{pmatrix} 1 & 0 & \cdots & \cdots & \cdots & \cdots & \cdots & \cdots & 0 \\ 0 & e^{-(\gamma + \alpha)} & te^{-(\gamma + \alpha)} & \cdots & \cdots & \cdots & \cdots & \cdots & \left(\frac{e^{-(\gamma + \alpha)}}{(n-2)!}\right) \\ \vdots & 0 & e^{-(\gamma + \alpha)} & te^{-(\gamma + \alpha)} & \cdots & \cdots & \cdots & \cdots & \left(\frac{e^{-(\gamma + \alpha)}}{(n-3)!}\right) \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & \vdots \\ \vdots & \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \ddots & 0 \\ 0 & 0 & \cdots & \cdots & 0 & e^{-(\gamma + \alpha)} & te^{-(\gamma + \alpha)} & \cdots & \left(\frac{e^{-(\gamma + \alpha)}}{(n-1)!}\right) \\ 0 & 0 & \cdots & \cdots & 0 & \cdots & \cdots & \cdots & e^{-(\gamma + \alpha)} \end{pmatrix}_{n \times n}. \]
\[
= \begin{pmatrix}
  e^\Lambda_{n-1}^t & E_{n-1} \\
  D_{n-1} & F_{n-1}
\end{pmatrix}
\]

where
\[
D_{n-1} = [0 \cdots 0]_{1 \times (n-1)}
\]
\[
F_{n-1} = [e^{-(y+\alpha)}]_{1 \times 1}
\]
\[
E_{n-1} = \begin{pmatrix}
  0 \\
  \frac{t^{n-2}}{(n-2)!} e^{-(y+\alpha)} \\
  \frac{t^{n-3}}{(n-3)!} e^{-(y+\alpha)} \\
  \vdots \\
  te^{-(y+\alpha)}
\end{pmatrix}
\]

and \(e^{\Lambda_{n-1}^t}\) is a \((n-1) \times (n-1)\) matrix defined as before for the \(n\)th case.

The inverse \(S_{n-1}^{-1}\) matrix is uniquely determined given \(S_n\). Thus with
\[
\rho_k^n = \frac{-\alpha^{n-1}}{(y + \alpha)^{n-k}}
\]
and
\[
\psi_k^n = \frac{-\alpha^{n-1}y^n}{(\alpha + y)^{n-k}} \sum_{j=1}^{n-k} \binom{n-k}{j-1} y^{n-k-j} \alpha^{j-1} \text{ for } k = 1, \ldots, n-1
\]
the inverse of \(S_n\) can be written as
\[
S_n^{-1} = \begin{pmatrix}
  -\rho_1^n & -\rho_1^n & \cdots & \cdots & -\rho_1^n & -\rho_1^n \\
  \rho_1^n & \rho_1^n & \cdots & \cdots & \rho_1^n & \psi_1^n \\
  \rho_2^n & \rho_2^n & \cdots & \cdots & \rho_2^n & \psi_2^n \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \rho_k^n & \cdots & \rho_k^n & \psi_k^n & \cdots & \psi_k^n \\
  \vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
  \rho_{n-1}^n & \psi_{n-1}^n & \cdots & \cdots & \psi_{n-1}^n
\end{pmatrix}
\]

The solution with initial condition \(S_n^{-1}u_0\) is
\[
P_n(t) = S_n e^{\Lambda_n^t} S_n^{-1} u_0
\]
where \( u_0 \) can be determined from the transient probabilities \( P_{0k}(t) \). The probability of transitioning from state 0 to any non-zero state in zero time is zero, e.g.

\[
P_{0k}(0) = 0, \quad \text{for } k = 1, 2, \ldots.
\]

Thus

\[
u_0 = \begin{pmatrix} 1 \\ 0 \\ \vdots \\ 0 \end{pmatrix},
\]

hence the task is to find the solution of the equation with this \( u_0 \).

Multiplying

\[
S_n^{-1} u_0 = \begin{pmatrix} -\rho_1^n \\ \rho_1^n \\ \vdots \\ \rho_n^n \end{pmatrix} = \begin{pmatrix} \frac{\alpha}{\gamma + \alpha} (\rho_1^{n-1}) \\ \frac{\alpha}{\gamma + \alpha} \rho_1^{n-1} \\ \vdots \\ \frac{\alpha}{\gamma + \alpha} \rho_{n-k}^{n-1} \end{pmatrix}
\]

\[
= \begin{pmatrix} \frac{\alpha}{\gamma + \alpha} S_{n-1}^{-1} u_0 \\ \rho_k^n \end{pmatrix}
\]

showing the inductive step in the product. Now multiplying the other matrices

\[
S_n e^{\mathcal{A} t} = \begin{pmatrix} B & C \\ S_{n-1} & A \end{pmatrix} \begin{pmatrix} e^{\mathcal{A}_{n-1} t} & E \\ D & F \end{pmatrix}
\]

\[
= \begin{pmatrix} Be^{\mathcal{A}_{n-1} t} + CD & CF \\ S_{n-1} e^{\mathcal{A}_{n-1} t} + AD & S_{n-1} E + AF \end{pmatrix}.
\]

Since \( D = [0 \cdots 0] \) and \( BE = 0 \)

\[
S_n e^{\mathcal{A} t} = \begin{pmatrix} Be^{\mathcal{A}_{n-1} t} & CF \\ S_{n-1} e^{\mathcal{A}_{n-1} t} & S_{n-1} E + AF \end{pmatrix}.
\]
Doing the multiplication block by block

\[ B e^{A_{n} t} = \left( \frac{\gamma (\gamma + \alpha)^{n-2}}{\alpha^{n-1}} \right) \begin{pmatrix} 0 & \cdots & 0 \end{pmatrix} = B, \]

\[ CF = \left( -\frac{1}{\alpha^{n-2}} e^{-(\gamma+\alpha)t} \right), \]

letting \( * = -(\gamma + \alpha)t \) and

\[ \eta_n(k) = \begin{pmatrix} 1 & \frac{1}{\alpha^{n-k+1}} (k - 4 - i)! & \frac{1}{\alpha^{n-k}} (k - 3 - i)! \end{pmatrix} e^{-(\gamma+\alpha)t} \]

\[ i = 0, 1, 2, \ldots \text{ and } k = 3, 4, \ldots \text{ with the convention that terms with negative} \]

powers of tare defined to be zero, it follows that

\[ S_{n-1} e^{A_{n-1} t} \]

\[ = \begin{pmatrix} \frac{\gamma}{\alpha} \left( \frac{\gamma+\alpha}{\alpha} \right)^{n-3} & \cdots & \cdots & \cdots & 0 \eta_n(3) \\ \frac{\gamma}{\alpha} \left( \frac{\gamma+\alpha}{\alpha} \right)^{n-4} & \cdots & \cdots & \cdots & 0 \eta_n(4) \eta_n(0) \\ \frac{\gamma}{\alpha} \left( \frac{\gamma+\alpha}{\alpha} \right)^{n-5} & \cdots & \cdots & \cdots & 0 \eta_n(5) \eta_n(0) \\ \vdots & \vdots & \vdots & \vdots & \vdots \end{pmatrix} e^{*} \begin{pmatrix} 0 & \cdots & \cdots & \cdots & 0 \end{pmatrix} \begin{pmatrix} 1 & e^{*} & \frac{\gamma}{2} e^{*} & \cdots & \frac{\alpha^{n-4}}{(n-4)!} e^{*} \frac{\alpha^{n-3}}{(n-3)!} e^{*} \end{pmatrix} \]

and

\[ S_{n-1} E = \begin{pmatrix} -\frac{1}{\alpha^{n-2}} \left( \frac{\gamma+\alpha}{\alpha} \right)^{n-3} t e^{-(\gamma+\alpha)t} \\ \frac{1}{\alpha^{n-2}} \left( \frac{\gamma+\alpha}{\alpha} \right)^{n-4} t e^{-(\gamma+\alpha)t} - \frac{1}{\alpha^{n-1}} \frac{\gamma}{2} t^2 e^{-(\gamma+\alpha)t} \\ \frac{1}{\alpha^{n-3}} \frac{\gamma}{2} t^2 e^{-(\gamma+\alpha)t} - \frac{1}{\alpha^{n-2}} \frac{\gamma}{3} t^3 e^{-(\gamma+\alpha)t} \\ \vdots \end{pmatrix} \begin{pmatrix} 1 & \frac{\gamma}{\alpha} (k - 4 - i)! e^{-(\gamma+\alpha)t} \\ \frac{1}{\alpha^{n-k+1}} (k - 4 - i)! e^{-(\gamma+\alpha)t} - \frac{1}{\alpha^{n-k}} (k - 3 - i)! e^{-(\gamma+\alpha)t} \\ \vdots \end{pmatrix} \begin{pmatrix} \frac{\gamma}{\alpha} (k - 3)! e^{-(\gamma+\alpha)t} - \frac{\gamma}{\alpha} (k - 2)! t e^{-(\gamma+\alpha)t} \\ \vdots \end{pmatrix} \]
\( \mathbf{AF} = \begin{pmatrix} \frac{1}{\alpha n} e^{-(\gamma + \alpha)t} \\ 0 \\ \vdots \\ 0 \end{pmatrix}, \)

hence

\[
\mathbf{S}_{n-1}\mathbf{E} + \mathbf{AF} = \begin{pmatrix} 
\left( \frac{1}{\alpha^2} - \frac{1}{\alpha n} \right) e^{-(\gamma + \alpha)t} \\
\left( \frac{1}{\alpha^2} - \frac{1}{\alpha^4} \right) e^{-(\gamma + \alpha)t} \\
\vdots \\
\left( \frac{1}{\alpha (n-3)!} - \frac{\mu - 2}{(n-2)!} \right) e^{-(\gamma + \alpha)t} \\
\frac{\mu - 2}{(n-2)!} e^{-(\gamma + \alpha)t} 
\end{pmatrix}.
\]

Putting all these elements together one has the matrix

\[
\mathbf{S}_n\mathbf{e}^{\Lambda_n t} = \begin{pmatrix} 
\mathbf{B} e^{\Lambda_{n-1} t} \\
\gamma (\gamma + \alpha) n^{-3} \\
\gamma (\gamma + \alpha) n^{-4} \\
\vdots \\
\gamma (\gamma + \alpha) n^{-k} \\
\gamma (\gamma + \alpha) n^{-n} \\
1 
\end{pmatrix} \begin{pmatrix} 
\mathbf{CF} \\
\mathbf{S}_{n-1}\mathbf{e}^{\Lambda_{n-1} t} \\
\mathbf{S}_{n-1}\mathbf{E} + \mathbf{AF} 
\end{pmatrix}
\]

\[
\begin{pmatrix}
0 & \cdots & \cdots & \cdots & 0 & \eta_n(\frac{2}{n-1}) \\
0 & \cdots & \cdots & \cdots & 0 & \eta_n(\frac{3}{n-1}) \\
0 & \cdots & \cdots & \cdots & 0 & \eta_n(\frac{4}{n-1}) \\
\vdots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & \cdots & \cdots & \cdots & 0 & \eta_n(\frac{k}{n-1}) \\
0 & \cdots & \cdots & \cdots & 0 & \eta_n(\frac{n}{n-1}) \\
1 & \cdots & \cdots & \cdots & 0 & \eta_n(\frac{n-1}{n-1}) \\
\end{pmatrix}
\]

Therefore the transient probabilities \( P_{0k}(t), k = 0, 1, \ldots \) with initial condition given by \( i_0 \) will be
\[ P_{0k}(t) = S_n e^{\frac{\gamma}{\alpha} - 1} S_n^{-1} u_0 \]

\[
\begin{pmatrix}
\frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-2} & 0 & \cdots & 0 & 0 & \eta_n(\frac{2}{\alpha}) \\
\frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-3} & 0 & \cdots & 0 & \eta_n(\frac{1}{\alpha}) & \eta_n(\frac{3}{\alpha}) \\
\frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-4} & 0 & \cdots & 0 & \eta_n(\frac{4}{\alpha}) & \eta_n(\frac{4}{\alpha}) \\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots \\
\frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-k} & 0 & \eta_n(k-3) & \cdots & \eta_n(k) & \eta_n(k) \\
1 & e^* & \cdots & \left(\frac{m-4}{(n-4)!}\right)^* \left(\frac{m-3}{(n-3)!}\right)^* e^* & \left(\frac{m-2}{(n-2)!}\right)^* e^* \\
\end{pmatrix}
\]

\[
\times
\begin{pmatrix}
-\rho_1^0 \\
\rho_1^0 \\
\rho_2^0 \\
\vdots \\
\rho_3^0 \\
\rho_4^0 \\
\rho_5^0 \\
\rho_6^0 \\
\rho_7^0 \\
\end{pmatrix}
\]

\[
= \begin{pmatrix}
-\rho_1^0 \frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-2} + \rho_1^0 \eta_n(\frac{2}{\alpha}) \\
-\rho_1^0 \frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-3} + \rho_1^0 \eta_n(\frac{1}{\alpha}) + \rho_1^0 \eta_n(\frac{3}{\alpha}) \\
-\rho_1^0 \frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-4} + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) \\
\vdots \\
-\rho_1^0 \frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-k} + \rho_1^0 \eta_n(k-3) + \cdots + \rho_1^0 \eta_n(k) + \rho_1^0 \eta_n(k) \\
\vdots \\
-\rho_1^0 \frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-4} + \rho_1^0 \eta_n(\frac{3}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \cdots + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) \\
-\rho_1^0 \frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-4} + \rho_1^0 \eta_n(\frac{3}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \cdots + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) \\
-\rho_1^0 \frac{\gamma}{\alpha} (\frac{y + \alpha}{\alpha})^{n-4} + \rho_1^0 \eta_n(\frac{3}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \cdots + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) + \rho_1^0 \eta_n(\frac{4}{\alpha}) \\
\end{pmatrix}
\]}
Thus the transient probabilities are

\[ P_{00}(t) = -\rho_1^0 \gamma (\frac{y + \alpha}{\alpha})^{n-2} \alpha^{n-1} + \rho_{n-1}^0 \eta_n \left( \frac{2}{-1} \right) \]

\[ = \frac{\alpha^{n-1}}{(y + \alpha)^{n-1}} \gamma (\frac{y + \alpha}{\alpha})^{n-2} \frac{1}{\alpha^{n-2}} \left( -\frac{1}{\alpha^{n-2}} \right) e^{-(y + \alpha)t} \]

\[ = \frac{\gamma}{y + \alpha} + \frac{\alpha}{\gamma + \alpha} e^{-(y + \alpha)t} \]

\[ P_{01}(t) = -\rho_1^0 \gamma \left( \frac{y + \alpha}{\alpha} \right)^{n-3} + \rho_{n-2}^0 \eta_n \left( \frac{3}{0} \right) + \rho_{n-1}^0 \eta_n \left( \frac{3}{1} \right) \]

\[ = \frac{\alpha^{n-1}}{(y + \alpha)^{n-1}} \frac{\gamma}{\alpha} \left( \frac{y + \alpha}{\alpha} \right)^{n-3} \frac{1}{\alpha^{n-2}} \left( -\frac{1}{\alpha^{n-3}} \right) e^{-(y + \alpha)t} \]

\[ = \frac{\alpha^{n-1}}{y + \alpha} \left( \frac{1}{\alpha^{n-2}} - \frac{1}{\alpha^{n-3}} \right) e^{-(y + \alpha)t} \]

\[ P_{02}(t) = -\rho_1^0 \gamma \left( \frac{y + \alpha}{\alpha} \right)^{n-4} + \rho_{n-3}^0 \eta_n \left( \frac{4}{1} \right) + \rho_{n-2}^0 \left( \frac{4}{0} \right) \]

\[ + \rho_{n-1}^0 \eta_n \left( \frac{4}{-1} \right) \]

\[ = \frac{\alpha^{n-1}}{(y + \alpha)^{n-1}} \frac{\gamma}{\alpha} \left( \frac{y + \alpha}{\alpha} \right)^{n-4} \frac{1}{\alpha^{n-3}} \left( -\frac{1}{\alpha^{n-4}} \right) e^{-(y + \alpha)t} \]

\[ = \frac{\alpha^{n-1}}{(y + \alpha)^{n-1}} \frac{\gamma}{\alpha} \left( \frac{1}{\alpha^{n-2}} - \frac{1}{\alpha^{n-3}} \right) e^{-(y + \alpha)t} \]

\[ = \frac{\alpha^{n-1}}{y + \alpha} \left( \frac{1}{\alpha^{n-3}} - \frac{1}{\alpha^{n-4}} \right) e^{-(y + \alpha)t} \]

\[ = \frac{\alpha^{2} \gamma}{(y + \alpha)^3} + \left( \frac{\alpha^2}{(\alpha + \gamma)^3} \right) \frac{\alpha^2}{(\alpha + \gamma)^2} \left( \frac{1}{\alpha^{n-2}} - \frac{1}{\alpha^{n-3}} \right) e^{-(y + \alpha)t} \]

\[ = \frac{\alpha^{3} \gamma}{(y + \alpha)^3} + \left( \frac{-\alpha^2 \gamma}{(\alpha + \gamma)^3} \right) \frac{\alpha^2}{(\alpha + \gamma)^2} \left( \frac{1}{\alpha^{n-2}} - \frac{1}{\alpha^{n-3}} \right) e^{-(y + \alpha)t} \]
\[
\begin{align*}
\frac{\alpha^2 \gamma}{(\gamma + \alpha)^3} &+ \left( \frac{\alpha^3 (\gamma + \alpha)^2 t^2 - 2\alpha^2 \gamma - 2\alpha^2 \gamma (\gamma + \alpha) t}{2(\gamma + \alpha)^3} \right) e^{-(\gamma + \alpha) t} \\
\vdots
\end{align*}
\]

\[
\begin{align*}
P_{0(k-2)}(t) &= -\rho_n \frac{\gamma}{\alpha} \frac{(\gamma + \alpha)^{n-k}}{(\gamma + \alpha)^{k-1}} + \rho_{n-k}^n \eta_n \left( \begin{array}{c} k \\ k - 3 \end{array} \right) + \cdots \\
&+ \rho_{n-2}^n \eta_n \left( \begin{array}{c} k \\ 0 \end{array} \right) + \rho_{n-1}^n \eta_n \left( \begin{array}{c} k - 1 \\ -1 \end{array} \right) \\
&= \frac{\alpha^{k-2}}{(\gamma + \alpha)^{k-1}} + \frac{[\alpha^{k-1} - \alpha^{k-2} (\gamma + \alpha)] e^{-(\gamma + \alpha) t}}{(\gamma + \alpha)^k} \\
&+ \frac{[\alpha^{k-1} - \alpha^{k-2} (\gamma + \alpha)] t e^{-(\gamma + \alpha) t}}{(\gamma + \alpha)^{k-1}} + \cdots \\
&+ \frac{[\alpha^{k-1} - \alpha^{k-2} (\gamma + \alpha)] t^{k-4}}{(\gamma + \alpha)^2} e^{-(\gamma + \alpha) t} \\
&+ \frac{[\alpha^{k-1} - \alpha^{k-2} (\gamma + \alpha)] t^{k-3}}{(\gamma + \alpha)^3} e^{-(\gamma + \alpha) t} \\
&+ \frac{\alpha^{k-1} t^{k-2}}{(\gamma + \alpha)^4} e^{-(\gamma + \alpha) t} \\
&= \frac{\alpha^{k-2}}{(\gamma + \alpha)^{k-1}} + \frac{\alpha^{k-1} t^{k-2}}{(\gamma + \alpha)^{k-1}} \left( \gamma + \alpha \right)^{k-1} - \sum_{i=0}^{k-3} \alpha^{k-2} \gamma (\gamma + \alpha)^i t^{i+1} \\
&+ \alpha^{k-5} \left( \gamma + \alpha \right)^{n-4} + \left( \gamma + \alpha \right)^{n-4} (\gamma + \alpha)^{k-4} - \sum_{i=0}^{n-6} \alpha^{n-5} \gamma (\gamma + \alpha)^i \\
&+ \alpha^{n-3} \left( \gamma + \alpha \right)^{n-3} \left( \gamma + \alpha \right)^{n-4} + \alpha^{n-4} \left( \gamma + \alpha \right)^{n-3} - \sum_{i=0}^{n-3} \alpha^{n-5} \gamma (\gamma + \alpha)^i \\
&+ \alpha^{n-4} \left( \gamma + \alpha \right)^{n-4} \left( \gamma + \alpha \right)^{n-5} + \alpha^{n-5} \left( \gamma + \alpha \right)^{n-5} \gamma (\gamma + \alpha)^i \\
&= -\rho_n \frac{\gamma}{\alpha} \frac{(\gamma + \alpha)^{n-3}}{(\gamma + \alpha)^{n-4}} + \rho_{n-3}^n \eta_n \left( \begin{array}{c} n \\ 1 \end{array} \right) + \rho_{n-2}^n \eta_n \left( \begin{array}{c} n \\ n - 3 \end{array} \right) + \cdots \\
&+ \rho_{n-3}^n \eta_n \left( \begin{array}{c} n \\ 0 \end{array} \right) + \rho_{n-1}^n \eta_n \left( \begin{array}{c} n \\ -1 \end{array} \right)
\end{align*}
\]
\[ P_{0(n-1)}(t) = -\rho_1^n e^x + \rho_2^n t e^x + \cdots + \rho_{n-3}^n \frac{\mu_{n-4}}{(n-4)!} e^x \]

\[ + \rho_{n-2}^n \frac{\mu_n}{(n-3)!} t e^x + \rho_{n-1}^n \frac{\mu_{n-2}}{(n-2)!} t^2 e^x \]

\[ = \frac{\alpha^{n-1}}{(y+\alpha)^{n-1}} - e^{-(y+\alpha)t} \left( \frac{\alpha^{n-1}}{(y+\alpha)^{n-1}} + \frac{\alpha^{n-1}}{(y+\alpha)^n} \frac{\mu_{n-4}}{(n-4)!} \frac{t^2}{2} \right) \]

\[ + \frac{\alpha^{n-1}}{(y+\alpha)^{n-1}} \frac{\mu_{n-2}}{(n-2)!} \left( 1 + (y+\alpha)t + (y+\alpha)^2 \frac{t^2}{2} + \cdots \right) \]

\[ + (y+\alpha)^{n-3} \frac{\mu_{n-3}}{(n-3)!} + (y+\alpha)^{n-2} \frac{\mu_{n-2}}{(n-2)!} \]

The transient probability \( P_{0(k-2)}(t) \) is the same transient probability from state 0 to state \( k-2 \) that Swift [3] found and is independent of \( n \). Thus the transient probabilities \( \{ P_{0k}(t) \}_{k=0,\ldots,n-2} \) of the finite state model are identical to the infinite state model. The transient probability \( P_{0(n-1)}(t) \) will converge to zero as \( n \to \infty \). To see this let

\[ R_n(x) = \sum_{i=n}^{\infty} \frac{x^i}{i!} \]

be the remainder in the exponential series. Then the transition probability

\[ P_{0(n-1)}(t) = \frac{\alpha^{n-1}}{(y+\alpha)^{n-1}} - e^{-(y+\alpha)t} \frac{\alpha^{n-1}}{(y+\alpha)^n} \frac{\mu_{n-4}}{(n-4)!} \frac{t^2}{2} \]

\[ + (y+\alpha)^{n-3} \frac{\mu_{n-3}}{(n-3)!} + (y+\alpha)^{n-2} \frac{\mu_{n-2}}{(n-2)!} \]

...
\[
\begin{align*}
&= \frac{\alpha^{n-1}}{(\gamma \alpha)^{n-1}} - e^{-(\gamma+\alpha)t} \frac{\alpha^{n-1}}{(\gamma + \alpha)^{n-1}} (\alpha^{(\gamma + \alpha)t} - R_{n-1}[\alpha^{(\gamma + \alpha)t}]) \\
&= e^{-(\gamma+\alpha)t} \frac{\alpha^{n-1}}{(\gamma + \alpha)^{n-1}} R_{n-1}[\alpha^{(\gamma + \alpha)t}].
\end{align*}
\]

Thus a necessary and sufficient condition for the probabilities to converge as \( n \to \infty \) is that

\[
e^{-(\gamma+\alpha)t} \frac{\alpha^{n-1}}{(\gamma + \alpha)^{n-1}} R_{n-1}[\alpha^{(\gamma + \alpha)t}]
\]

converges to zero. Also, since the remainder \( R_{n-1}[\alpha^{(\gamma + \alpha)t}] \to \infty \) for all \( t \), a sufficient condition is that

\[
\frac{\alpha^{n-1}}{(\gamma + \alpha)^{n-1}}
\]

be uniformly bounded in \( n \), e.g.

\[
\left| \frac{\alpha}{\gamma + \alpha} \right| \leq 1.
\]

Assuming the rates are all positive

\[
\frac{\alpha}{\gamma + \alpha} \leq 1 \Rightarrow \alpha \leq \alpha + \gamma
\]

or simply \( \gamma \geq 0 \). Hence the finite state immigration-catastrophe process converges to the infinite state model for all positive rates.

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